Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 4

Exercises

Exercise 4.1 (4 Points)
Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $P\colon X \to X$ be a continuous linear map which satisfies $P^2 := P \circ P = P$. Show that there exists a closed vector subspace Y of X such that P is the orthogonal projection onto Y if and only if

$$\langle Px, x \rangle \geq 0, \qquad x \in X.$$

(*Hint:* First show that $X = P(X) \oplus \ker(P)$, where $\ker(P) = \{x \in X : Px = 0\}$.)

Exercise 4.2 (6 Points)
Let $x_0 \in \mathbb{R}$ and let $b\colon [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that there exist $T > 0$ and a solution $x \in C^1([0, T], \mathbb{R})$ to the differential equation $x(0) = x_0$,

$$\dot{x}(t) = b(t, x(t)), \qquad t \in [0, T].$$

(*Hint:* One strategy is to approximate b by Lipschitz continuous functions, uniformly on an appropriately chosen compact set.)

Exercise 4.3 (6 Points)
Consider a family of random variables $(X_k^p : p \in [0, 1], k \in \mathbb{N})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that for every fixed p the sequence $(X_k^p)_{k \in \mathbb{N}}$ consists of independent and identically distributed random variables satisfying $\mathbb{P}(X_k^p = 1) = 1 - \mathbb{P}(X_k^p = 0) = p$. (Such a family exists, but you do not need to verify this). For a given $f \in C([0, 1], \mathbb{R})$ define the sequence of functions $f_n\colon [0, 1] \to \mathbb{R}$ via

$$f_n(p) = \mathbb{E}\Big[f\Big(\frac{1}{n}\sum_{k=1}^{n} X_k^p\Big)\Big], \qquad n \in \mathbb{N}.$$

Use the (proof of the) weak law of large numbers to show that (f_n) converges uniformly to f. Conclude that the polynomials are dense in $C([0, 1], \mathbb{R})$.

Exercise 4.4 (4 Points)
Let $K \subset \mathbb{R}^d$ be the closure of a bounded open set and define for $(n, \alpha) \in \mathbb{N}_0 \times [0, 1]$ the space $C^{n,\alpha}(K, \mathbb{R})$ as in Exercise 2.4. We equip $C^{n,\alpha}(K, \mathbb{R})$ with the norm

$$\|f\|_{n,\alpha} := \sum_{k=0}^{n} \|f^{(k)}\|_\infty + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x - y|^\alpha}.$$

Show that if $0 < \alpha < \beta \leq 1$ and if A is a bounded subset of $C^{0,\beta}(K, \mathbb{R})$, then A is relatively compact in $C^{0,\alpha}(K, \mathbb{R})$.

(*Hint:* If $a, b, c \geq 0$ are such that $a \leq b$ and $a \leq c$, then for any $\lambda \in [0, 1]$ we have $a \leq b^\lambda c^{1-\lambda}$. Why?)

Due date: Thursday, November 12, 2015.
(You may submit your solutions in groups of two.)