Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 5

Exercises

Exercise 5.1 (2+3 Points)

a) Let (X, d) be a metric space equipped with its Borel sigma algebra $\mathcal{B}(X)$, and let μ and ν be two finite measures on $\mathcal{B}(X)$. Show that if

$$\int_X f(x)\mu(\mathrm{d}x) = \int_X f(x)\nu(\mathrm{d}x)$$

for all $f \in C_b(X, \mathbb{R})$, then $\mu = \nu$.

b) Let μ and ν be two Radon measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Show that if

$$\int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x)$$

for all $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ (the infinitely differentiable functions of compact support), then $\mu = \nu$.

Hint: You may use without proof that for any $n \in \mathbb{N}$ there exists a C^∞ function φ_n with $\varphi_n|_{B[0,n]} \equiv 1$ and $\mathrm{supp}(\varphi_n) \subseteq B[0, n+1]$.

Exercise 5.2 (2+2+3+2 Points)
Let $p, q \in [1, \infty]$. Show that

a) if $(\Omega, \mathcal{F}, \mu)$ is a measure space with $\mu(\Omega) < \infty$, then $\|f\|_p \leq \mu(\Omega)^{1/p-1/q}\|f\|_q$ whenever $p \leq q$ and $f \in L^q(\Omega, \mu)$;

b) if $\Omega = \mathbb{N}$ with $\mathcal{F} = \mathcal{P}(\mathbb{N})$ and if μ is the counting measure, then $\|f\|_q \leq \|f\|_p$ whenever $p \leq q$ and $f \in L^p(\Omega, \mu)$.

c) for $p \neq q$ we have $L^p(\mathbb{R}, \lambda) \nsubseteq L^q(\mathbb{R}, \lambda)$, where λ denotes the Lebesgue measure;

d) for any measure space $(\Omega, \mathcal{F}, \mu)$ we have $\|f\|_{p_\theta} \leq \|f\|_p^{1-\theta}\|f\|_q^\theta$ whenever $1/p_\theta = (1-\theta)/p + \theta/q$ for $\theta \in [0, 1]$ and $f \in L^p(\Omega, \mu) \cap L^q(\Omega, \mu)$.

Exercise 5.3 (6 Points)
Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space and $p \in (1, \infty)$ with conjugate exponent $p' = p/(p-1)$. Let $f : \Omega \to \mathbb{R}$ be a measurable function for which

$$\sup_{\substack{g \in L^{p'}(\Omega,\mu):\\ \|g\|_{p'} \leq 1}} \int_\Omega |f(x)g(x)|\mu(\mathrm{d}x) < \infty.$$

Show that $f \in L^p(\Omega, \mu)$ and

$$\|f\|_p = \sup_{\substack{g \in L^{p'}(\Omega,\mu):\\ \|g\|_{p'} \leq 1}} \int_\Omega f(x)g(x)\mu(\mathrm{d}x).$$

(*Hint:* Start by constructing a sequence $(f_n) \subseteq L^p$ with increasing absolute value, such that f_n converges a.e. to f.)

Due date: Thursday, November 19, 2015.
(You may submit your solutions in groups of two.)