Exercises

Exercise 6.1 (3+3+3 Points)
For $p \in [1, \infty]$ and $k \in \mathbb{N}_0$, a function $f \in L^p := L^p(\mathbb{R}^d, \lambda)$ (λ being the Lebesgue measure) is in the *Sobolev space* $W^{k,p} := W^{k,p}(\mathbb{R}^d, \lambda)$ if for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$ there exists an $f_\alpha \in L^p$ such that

$$\int_{\mathbb{R}^d} \varphi(x) f_\alpha(x) \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \partial^\alpha \varphi(x) f(x) \mathrm{d}x \tag{1}$$

for all $\varphi \in C_c^\infty := C_c^\infty(\mathbb{R}^d, \mathbb{R})$.

a) Show that given $f \in L^p$ and $\alpha \in \mathbb{N}_0^d$, there exists at most one $f_\alpha \in L^p$ which satisfies (1).

b) For $f \in W^{k,p}$ we set

$$\|f\|_{W^{k,p}} := \sum_{\substack{\alpha \in \mathbb{N}_0^d, \\ |\alpha| \leq k}} \|f_\alpha\|_p.$$

Show that $W^{k,p}$ is a vector space and that $\| \cdot \|_{W^{k,p}}$ is a norm under which $W^{k,p}$ is complete.

c) Show that $C_c^\infty \subseteq W^{k,p}$ and that for $\psi \in C_c^\infty$ and $\alpha \in \mathbb{N}_0^d$ we have $\psi_\alpha = \partial^\alpha \psi$.

Exercise 6.2 (3+3 Points)
Let $p \in [1, \infty]$ and let $\varphi \in C_c^\infty$ be such that $\varphi|_{[-1,1]} \equiv 1$ and $f = g\varphi$ for

$$g(x) = \begin{cases} 0, & x \leq 0, \\ x^\gamma, & x \in (0, 1], \\ 1, & x > 1. \end{cases}$$

Show that

a) for $\gamma > (p-1)/p$ we have $f \in W^{1,p}(\mathbb{R}, \lambda)$;

b) for $\gamma = 0$ we have $f \notin W^{1,p}(\mathbb{R}, \lambda)$.

Exercise 6.3 (2+3 Points)
Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. For two sets $A, B \in \mathcal{F}$ we define $d(A, B) := \mu(A \Delta B)$.

a) Show that d is a pseudometric on $\mathcal{F}$ (i.e. d satisfies all conditions in the definition of a metric, except that $d(A, B) = 0$ does not necessarily mean $A = B$). In particular, if we write $A \sim B$ for $A, B \in \mathcal{F}$ with $d(A, B) = 0$, then $\sim$ defines an equivalence relation on $\mathcal{F}$ and d is a metric on $\mathcal{F}^\star := \mathcal{F}/\sim$.

b) Let $p \in [1, \infty)$. Show that if $L^p(\Omega, \mu)$ is separable, then $(\mathcal{F}^\star, d)$ is separable as well.

Due date: Thursday, November 26, 2015.
(You may submit your solutions in groups of two.)