Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 7

Exercises

Exercise 7.1 (3+2+3 Points)
For $p \in [1, \infty)$ and $k \in \mathbb{N}_0$ let $W^{k,p} = W^{k,p}(\mathbb{R}^d, \mathbb{R})$ be the Sobolev space of Exercise 7.1. Show that

a) $C^\infty \cap W^{k,p}$ is dense in $W^{k,p}$;

b) for $\alpha \in \mathbb{N}_0^d$ and $k \geq |\alpha|$, the map

$$\partial^\alpha \colon C^\infty \cap W^{k,p} \to C^\infty \cap W^{k,p}$$

can be uniquely extended to a bounded linear operator $A_\alpha \colon W^{k,p} \to W^{k-|\alpha|,p}$.

c) Let $\ell \in \mathbb{N}_0$ and $K \in W^{\ell,1}$. Show that for all $k \in \mathbb{N}_0$ the following map B is a bounded linear operator:

$$B \colon W^{k,p} \to W^{k+\ell,p}, \qquad Bf = K * f.$$

Exercise 7.2 (3 Points)
Calculate and sketch the convolution $f * g$ of the following two functions $f, g \colon \mathbb{R} \to \mathbb{R}$:

$$f(x) = \mathbf{1}_{[0,1]}(x), \qquad g(x) = \mathbf{1}_{[1,3]}(x).$$

Exercise 7.3 (3 Points)
Let $K \subset \mathbb{R}^d$ be compact and let $\varepsilon > 0$. Show that there exists $\varphi \in C_c^\infty$ such that $\|\varphi\|_\infty \leq 1$, $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$, $\varphi|_K \equiv 1$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^d$ with $d(x, K) > \varepsilon$.

Exercise 7.4 (2+2+2 Points)
Let $p \in (1, \infty)$ and consider the space $(C([0,1], \mathbb{R}), \|\cdot\|_p)$ with

$$\|f\|_p = \Big(\int_0^1 |f(x)|^p \mathrm{d}x \Big)^{1/p}.$$

Let $(b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}} \subset [0,1]$ with $b_n < c_n$ for all n, and let $(a_n)_{n \in \mathbb{N}} \subset [0, \infty)$. We define

$$A_n \colon C([0,1], \mathbb{R}) \to \mathbb{R}, \qquad A_n f = a_n \int_{b_n}^{c_n} f(x) \mathrm{d}x.$$

a) Show that $\sup_n |A_n f| < \infty$ for all $f \in C([0,1], \mathbb{R})$ if and only if $\sup_n a_n (c_n - b_n) < \infty$.

b) Show that $\sup_n \|A_n\|_{\mathcal{L}(C([0,1],\mathbb{R}),\mathbb{R})} < \infty$ if and only if $\sup_n a_n (c_n - b_n)^{1-1/p} < \infty$.

c) Define sequences $(a_n), (b_n), (c_n)$ such that (A_n) is pointwise bounded but not norm bounded. Why is this no contradiction to the uniform boundedness principle?

Due date: Thursday, December 3, 2015.
(You may submit your solutions in groups of two.)