Humboldt-Universität zu Berlin

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Functional Analysis
Winter semester 2015/16
Exercise sheet 8

Exercises

Exercise 8.1 (4 Points)
Let X be a Banach space and let $\ell\colon X \to \mathbb{K}$ be a linear map. Show that $\ell \in X^\star$ if and only if $\ker(\ell)$ is closed.

Exercise 8.2 (2+2 Points)
Let X and Y be Banach spaces. We say a map $A \in \mathcal{L}(X,Y)$ is invertible if there exists $A^{-1} \in \mathcal{L}(Y,X)$ such that $A^{-1} \circ A = I_X$ and $A \circ A^{-1} = I_Y$, where I_X and I_Y denote the identity maps on X and Y, respectively.

a) Let $A \in \mathcal{L}(X,X)$ with $\|A\| < 1$. Show that the map $I_X - A \in \mathcal{L}(X,X)$ is invertible and that

$$\|(I_X - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

b) Let $A, B \in \mathcal{L}(X,Y)$ and assume that A is invertible and that $\|B\| < \|A^{-1}\|^{-1}$. Show that $A - B$ is invertible.

Exercise 8.3 (4 Points)
Let $p \in (1,\infty)$ and $p' = p/(p-1)$ be the conjugate exponent. Show that there exists an isometric isomorphism between $(\ell^p)^\star$ and $\ell^{p'}$.

(*Hint:* Recall Exercise 5.3.)

Exercise 8.4 (4+4 Points)
Let X be a Banach space, $A \in \mathcal{L}(X,X)$ and $x_0 \in X$.

a) Show that there exists at most one function $x \in C^1([0,\infty), X)$ which satisfies

$$\mathrm{d}x(t) = Ax(t),\ t \in \mathbb{R}, \qquad x(0) = x_0. \tag{1}$$

(*Hint:* Let y solve (1) with $y(0) = 0$ and show that

$$t_\delta = \inf\{t \geq 0 : \|y(t)\| > \|A\| \int_0^t \|y(r)\|\mathrm{d}r + \delta t\} = \infty$$

for all $\delta > 0$; then apply Gronwall's lemma.)

b) Find the explicit solution x to (1) and verify that it solves the equation.

(*Hint:* What would the solution be in the one-dimensional case $X = \mathbb{R}$?)

Due date: Thursday, December 10, 2015.
(You may submit your solutions in groups of two.)