Humboldt-Universität zu Berlin

Prof. Dr. Nicolas Perkowski

Functional Analysis
Winter semester 2015/16
Exercise sheet 9

Exercises

Exercise 9.1 (3+2 Points)
Let X be a reflexive Banach space. Show that

a) if Y is a linear subspace and
$$Y^\perp = \{\ell \in X^\star | \ell(x) = 0 \text{ for all } x \in Y\},$$
then $(Y^\perp)^\perp = \overline{Y}$ (after identifying X and $X^{\star\star}$);

b) for every $\ell \in X^\star$ there exists $x_0 \in X \setminus \{0\}$ such that $\ell(x_0) = \|\ell\|\|x_0\|$.

Exercise 9.2 (4 Points)
Let X be a Hilbert space and $Y \subset X$ be a closed subspace. Show that given $\ell \in Y^\star$ there exists exactly one extension $\overline{\ell} \in X^\star$ such that $\overline{\ell}|_Y = \ell$ and $\|\overline{\ell}\| = \|\ell\|$. Give an explicit formula for $\overline{\ell}$.

(*Hint:* Try $\overline{\ell} = \ell \circ A$ for a suitable $A \in \mathcal{L}(X, Y)$.)

Exercise 9.3 (4 Points)
Let $p \in [1, \infty)$ and consider the subspace $Y \subset \ell^p$,
$$Y = \{x \in \ell^p : x_1 = x_3 = x_5 = \cdots = 0\},$$
and let $f \in Y^\star \setminus \{0\}$. Show that for $p = 1$ there exists an infinite number of extensions $\overline{f} \in (\ell^p)^\star$ such that $\overline{f}|_Y = f$ and $\|\overline{f}\| = \|f\|$, but for $p > 1$ such an extension $\overline{f}$ is unique.

Exercise 9.4 (1+4+2 Points)
Let $p \in [1, \infty)$ and consider the Sobolev space in dimension one, $W^{k,p} = W^{k,p}(\mathbb{R}, \lambda)$.

a) Show that for $f \in C^1(\mathbb{R}, \mathbb{R})$ and for $x \le y \in \mathbb{R}$ we have
$$|f(y) - f(x)| \le |y - x|^{1-1/p} \Big(\int_x^y |f'(z)|^p \mathrm{d}x \Big)^{1/p}.$$

b) Deduce that there exists $C > 0$ such that for all $f \in C^1(\mathbb{R}, \mathbb{R})$
$$\|f\|_\infty + \|f\|_{C^{1-1/p}} \le C \|f\|_{W^{1,p}}, \qquad \text{where } \|f\|_{C^{1-1/p}} := \sup_{x \ne y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|^{1-1/p}};$$
then deduce that for every $f \in W^{1,p}$ there exists a continuous function $\tilde{f}$ with $\tilde{f} = f$ a.e. and such that $\|\tilde{f}\|_\infty + \|\tilde{f}\|_{C^{1-1/p}} \le C \|f\|_{W^{1,p}}$.

c) Let $x_0 \in \mathbb{R}$. Show that there exists $g \in W^{1,2}$ which satisfies for all $f \in W^{1,2}$
$$\int f g \mathrm{d}x + \int f' g' \mathrm{d}x = \tilde{f}(x_0),$$
where $\tilde{f}$ is the continuous representative of f.

Due date: Thursday, December 17, 2015.
(You may submit your solutions in groups of two.)