Sheet 11 solution



11.1 Claim: The product topology is generated by the distance

$$d(x,y) = \sum_n 2^{-n} \left( d_n(x_n, y_n) \wedge 1 \right).$$

Write $\tau_p$: product

$\tau_d$: distance

$\cancel{\tau_p \subseteq \tau_d}$

$\cancel{\tau_d \subseteq \tau_p}$: For $x \in X$, $\varepsilon > 0$:

$\tau_d \subseteq \tau_p$ $\quad B(x, \varepsilon) = \{ y \in X : d(x,y) < \varepsilon \}$

If we can find $U \in \tau_p$ s.t. $U \subseteq B(x, \varepsilon)$, then every open set in $\tau_d$ is also open in $\tau_p$, so $\tau_d \subseteq \tau_p$.

But now if $N$ is large enough s.t. $\sum_{n > N} 2^{-n} < \frac{\varepsilon}{2}$, then

$$U = \bigcap_{k=1}^{N} \{ y : d_n(x_n, y_n) < \tfrac{\varepsilon}{2} \} \subseteq B(x, \varepsilon):$$

For $y \in U$,

$$d(x,y) = \sum_n 2^{-n} \left( d_n(x_n, y_n) \wedge 1 \right) < \sum_{n=1}^{N} 2^{-n} \left( \tfrac{\varepsilon}{2} \wedge 1 \right) + \sum_{n > N} 2^{-n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

$\tau_p \subseteq \tau_d$: Clearly the projections $\pi_n : X \to X_n$, $\pi_n(x) = x_n$, are continuous wrt $\tau_d$. So since $\tau_p$ is the <u>smallest</u> topo for which they are cts, $\tau_p \subseteq \tau_d$.

Now that we know $\tau_p = \tau_d$, it suffices to show that $X$ is <u>sequentially compact</u> (for non-metric spaces this would be a different statement).

So let $(x^{(m)})_m \subseteq X$. Extract subsequence

$$(x^{(m_k^1)})_k \text{ s.t. } x_1^{(m_k^1)} \xrightarrow[k\to\infty]{} x_1 \quad \text{for some } x_1 \in X_1$$

(possible since $X_1$ cpct metric space).

Inductively extract subsequences

$$(x^{(m_k^j)})_k \text{ of } (x^{(m_k^{j-1})})_k \text{ s.t.}$$

$$x_i^{(m_k^j)} \to x_i \qquad \forall i \leq j.$$

Then ~~$y_n = x^($~~ $y^{(\ell)} = x^{(m_\ell^\ell)}$ satisfies

$$y_i^{(\ell)} \to x_i \ \forall i, \quad \text{and therefore}$$

$$d(y^{(\ell)}, y) \to 0.$$

1.2

($\Rightarrow$): Let $\mathcal{F}$ have the fip. Then also

$$\mathcal{G} = \{\overline{F} : F \in \mathcal{F}\} \text{ has the f.i.p.}$$

If $\bigcap_{G \in \mathcal{G}} G = \emptyset$, then

since $X$ cpct

$$\Big(\bigcap_{G\in\mathcal{G}} G\Big)^c = \bigcup_{G\in\mathcal{G}} G^c = X, \text{ so } \exists\, G_1^c, \dots, G_n^c \in \mathcal{G}:$$

$$\bigcup_{k=1}^{n} G_k^c = X \leadsto \bigcap_{k=1}^{n} G_k = \emptyset \; ↯$$

($\Leftarrow$): Let $X = \bigcup_{i\in I} U_i$ be an open cover. Then

$\bigcap_{i \in I} U_i^c = \emptyset$, which is only possible if the

$(U_i^c)_i$ do <u>not</u> have the fip

$$\Rightarrow \exists\, U_{i_1}^c, \dots, U_{i_n}^c : \bigcap_{k=1}^{n} U_{i_k}^c = \emptyset$$

$$\Rightarrow X = \bigcup_{k=1}^{n} U_{i_k}.$$

## 11.3:

a) $x^{(n)} \rightharpoonup x \Rightarrow$ i) $x_i^{(n)} \to x_i$: take the map $l_i(x) = x_i$;

ii) $\|x^{(n)}\|$ bounded in $n$: seen in lecture

Conversely, if $x_i^{(n)} \to x_i \ \forall i$ and $\sup_n \|x^{(n)}\| = C < \infty$, then for $y \in \ell^{p'} \simeq (\ell^p)^*$: $\left| \sum_i (x_i^{(n)} - x_i) y_i \right|$

$$\leq \underbrace{\sum_{i=1}^{N} |x_i^{(n)} - x_i| \, |y_i|}_{\to 0,\ n\to\infty} + \underbrace{\left( \sum_{i=N+1}^{\infty} |x_i^{(n)} - x_i|^p \right)^{1/p}}_{\leq \|x^{(n)}\|_p + \|x\|_p} \underbrace{\left( \sum_{i=N+1}^{\infty} |y_i|^{p'} \right)^{1/p'}}_{\to 0,\ N\to\infty}$$

$$\xrightarrow[n\to\infty]{} 0.$$

We can now take $x_i^{(n)} = \delta_{i,n}$, for which $\|x^{(n)}\|_p = 1 \ \forall n$ and $x_i^{(n)} \to 0 \ \forall i$, but we do not have strong convergence to 0.

b) As in a) we see that $x^{(n)} \rightharpoonup x \Rightarrow x_i^{(n)} \to x_i \ \forall i$.

Subtracting $x$: wlog $x = 0$.

Assume $x^{(n)} \not\to 0$, so $\exists\, \varepsilon > 0$ subsequence $(n_k)_k$, s.t.

$$\|x^{(n_k)}\|_1 > \varepsilon \quad \forall k.$$

We will construct an element $y \in \ell^\infty$ s.t. ~~$\sum_i x^{(n}$ for a furth~~

$\sum_i x_i^{(n_k)} y_i \not\to 0$, along further subsequence

Define $M_1$ s.t. $\quad (M_0 = 0)$

$$\sum_{i>M_1} |x_i^{(n_1)}| < \frac{\varepsilon}{6}.$$

Find $k_2$ large enough s.t.

$$\sum_{i\le M_1} |x_i^{(n_{k_2})}| < \frac{\varepsilon}{6} \qquad (\text{ok since } x_i^{(n)} \to 0),$$

and then $M_2 > M_1$ s.t.

$$\sum_{i > M_2} |x_i^{(n_{k_2})}| < \frac{\varepsilon}{6}.$$

Continue in this way to generate $(n_{k_\ell}, M_\ell)$, s.t.

$$\sum_{i\le M_{\ell-1}} |x_i^{(n_{k_\ell})}| < \frac{\varepsilon}{6}, \quad \sum_{i>M_\ell} |x_i^{(n_{k_\ell})}| < \frac{\varepsilon}{6},$$

so since $\|x^{(n_\ell)}\|_1 > \varepsilon$: $\quad \sum_{i\in(M_{\ell-1},M_\ell]} |x_i^{(n_\ell)}| > \frac{2}{3}\varepsilon$.

Define now $\quad y_i = \cancel{\mathrm{sign}(x_i^{(n_{k_\ell})}}\ \mathrm{sign}(x_i^{(n_{k_\ell})}), \quad i\in(M_{\ell-1},M_\ell]$.

Then $\|y\|_\infty = 1$, and

$$\sum_i y_i x^{(n_{k_\ell})} > \sum_{i\in(M_{\ell-1},M_\ell]} |x_i^{(n_\ell)}| - \frac{\varepsilon}{3} > \frac{\varepsilon}{3} \ \lightning$$

c) ~~Take again~~ ~~Let~~ ~~$e_1^{(n)} = \delta_{1,n}$, with~~

Let $f^{(n)}(x) = x_n$, so $\|f^{(n)}\| \underset{(=)}{\leq} 1 \ \forall n$.

Let $(n_k)$ be an arbitrary subsequence. Then $f^{(n_k)}$ does not converge:

Define $x \in \ell^\infty$,

$$x_{n_k} = (-1)^k. \quad \text{Then}$$

$$f^{(n_k)}(x_{n_k}) = (-1)^k \text{ which does not converge.}$$

This is no contradiction to Banach-Alaoglu since compactness $\neq$ seq. compactness.

<u>11.4:</u> Problem: $X$ may not be separable, so we cannot simply apply Thm. 5.37. But

$$Y = \overline{\operatorname{span}(x_n : n \in \mathbb{N})}$$

is separable (take $\operatorname{span}_{\mathbb{Q}}(x_n : n \in \mathbb{N})$). So Thm. 5.37 shows that $\exists$ subsequence $(x_{n_k})$ and $x \in Y$ s.t. $\forall y \in Y$:

$$\langle y, x_{n_k} \rangle \longrightarrow \langle y, x \rangle$$

Let now $z \in X$ be arbitrary and $P$ be the orth. proj. on $Y$.

$$\langle z, x_{n_k} \rangle = \langle Pz, x_{n_k} \rangle + \langle z - Pz, \underbrace{x_{n_k}}_{\in Y} \rangle = \langle Pz, x_{n_k} \rangle$$

$$\longrightarrow \langle Pz, x \rangle \overset{x \in Y}{=} \langle z, x \rangle.$$

11.5:

a)

$$|\ell_n(x_n)-\ell(x)|$$

$$\leq \underbrace{|\ell_n(x_n-x)|}_{\leq \|\ell_n\|\,\underbrace{\|x_n-x\|}_{\to 0}} + \underbrace{|\ell_n(x)-\ell(x)|}_{\to 0}$$

So we need to show $\|\ell_n\|$ is bdd. But since $X$ is reflexive, weak-$*$ convergence in $X^*$ is the same as weak convergence in $X^*$, and for weakly convergent sequences we know from the lecture that they are norm bounded.

b) Assume $A$ is unbounded. Then $\exists (x_n) \subseteq X$: $x_n \to 0$, but $\|Ax_n\| \geq n$ $\forall n$. Now $x_n \to 0 \Rightarrow x_n \rightharpoonup 0$, so by assumption $Ax_n \rightharpoonup 0$. Let now $\ell \in Y^*$. ~~Then~~

$$\ell(Ax_n) \to 0,$$

and therefore $\sup_n |\ell(Ax_n)| < \infty$.

So for $J_Y : Y \to Y^{**}$ : $\sup_n |(J_Y(Ax_n))(\ell)| < \infty$ $\forall \ell \in Y^*$

By the uniform boundedness principle ($Y^*$ is always Banach) we get $\sup_n \|J_Y(Ax_n)\|_{Y^{**}} \overset{J_Y \text{ is isometry}}{=} \sup_n \|Ax_n\|_Y < \infty$, ↯ to