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a) i) $\langle u,v\rangle_{H^s}$ defines an inner product:

~~Linearity in first component is inherited by~~

- Linearity in first component inherited from linearity of $u \mapsto \hat{u}(k)$.

- $\langle u,v\rangle_{H^s} = \overline{\langle v,u\rangle_{H^s}}$ ✓

- $\langle u,u\rangle_{H^s} = \sum_k (1+|k|^2)^s |\hat{u}(k)|^2 \geq 0$,

$$\text{if} = 0 \Rightarrow \hat{u}(k) = 0 \;\; \forall k \overset{\text{Parseval}}{\Rightarrow} u = 0.$$

ii) Completeness: (as always)

$$\|u_n - u_m\|_{H^s} \geq \|u_n - u_m\|_{L^2}$$

$\Rightarrow$ If (u_n) is Cauchy in H^s, then Cauchy in L^2

$\Rightarrow \exists u \in L^2 : \|u_n - u\|_{L^2} \to 0$.

Now by Fatou:

$$\|u\|_{H^s}^2 = \sum_k (1+|k|^2)^s |\hat{u}(k)|^2 = \sum_k (1+|k|^2)^s \lim_{n\to\infty} |\hat{u}_n(k)|^2$$

$$\leq \liminf_n \|u_n\|_{H^s}^2 \Rightarrow u \in H^s \text{ and similarly}$$

$$\|u-u_n\|_{H^s}^2 = \sum_k (1+|k|^2)^s \lim_m |\hat{u}_m(k) - \hat{u}_n(k)|^2$$

$$\leq \liminf_m \|u_m - u_n\|_{H^s}^2 \leq \sup_{m\geq n} \|u_m - u_n\|_{H^s}^2 \to 0, \; n\to\infty.$$

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b) If $u \in W^{n,2}$ (i.e. the weak derivatives exist), then by definition of $\partial^\alpha u$ $\forall\, |\alpha| \le n$:

$$\widehat{\partial^\alpha u}(k) = \int_{\mathbb{T}^d} e^{-2\pi i k\cdot x}\, \partial^\alpha u(x)\,dx$$

~~split $e^{-2\pi i k\cdot x}$ into real and imaginary part~~

$$\overset{!}{=} (-1)^{|\alpha|} \int_{\mathbb{T}^d} \partial^\alpha e^{-2\pi i k\cdot x}\, u(x)\,dx$$

$$= (2\pi i k)^\alpha \underbrace{\int_{\mathbb{T}^d} e^{-2\pi i k\cdot x}\, u(x)\,dx}_{=\hat{u}(k)}.$$

So

$$\|u\|_{W^{n,2}} = \sum_{|\alpha|\le n} \|\partial^\alpha u\|_{L^2} \overset{\text{Parseval}}{=} \sum_{|\alpha|\le n} \Big(\sum_{k\in\mathbb{Z}^d} |\widehat{\partial^\alpha u}(k)|^2 \Big)^{1/2}$$

$$= \sum_{|\alpha|\le n} \Big(\sum_{k} |(2\pi k)^\alpha|^2\, |\hat{u}(k)|^2 \Big)^{1/2}$$

We are not interested in optimal constants, so let's be brutal:

$$|(2\pi k)^\alpha|^2 \le (2\pi)^{2n} (1+|k|^2)^n.$$

Since there are $<\infty$ many $\alpha \in \mathbb{N}_0^d$ w/ $|\alpha| \le n$, we get

So for $k \in \mathbb{Z}^d$

$$(1+|k|^2)^n |\hat{u}(k)|^2 = \sum_{m=0}^{n} \binom{n}{m} |k|^{2m} |\hat{u}(k)|^2$$

$$= \sum_{m=0}^{n} \binom{n}{m} \left(\sum_{j=1}^{d} |k_j|^2 \right)^m |\hat{u}(k)|^2$$

$$= (2\pi)^{-2m} \sum_{m=0}^{n} \binom{n}{m} \sum_{j_1,\dots,j_m=1}^{d} |(2\pi i k_{j_1}) \cdot \ldots \cdot (2\pi i k_{j_m}) \hat{u}(k)|^2$$

e.g. $j_1=2, j_2=j_3=1$, $j_2=2, j_1=j_3=1$ both give $(2,1)$.

$$\le C_1 \sum_{m=0}^{n} \binom{n}{m} \sum_{|\alpha|=m} |\widehat{\partial^\alpha u}(k)|^2$$

$$\le C_2 \sum_{m=0}^{n} \sum_{|\alpha|=m} |\widehat{\partial^\alpha u}(k)|^2$$

$$= C_2 \sum_{|\alpha|\le n} |\widehat{\partial^\alpha u}(k)|^2$$

And $\sum_k$ + Parseval gives

$$\|u\|_{H^n} \le \left(C_2 \sum_{|\alpha|\le n} \|\partial^\alpha u\|_{L^2}^2 \right)^{1/2} \overset{\text{Ex. 5.2}}{\le} \sqrt{C_2} \sum_{|\alpha|\le n} \|\partial^\alpha u\|_{L^2} .$$

Conversely, let ~~$u \in W^{n,2}$~~ $u \in H^n$. We define for $|\alpha| \le n$:

$$\partial^\alpha u = \sum_k e^{2\pi i k \cdot} (2\pi i k)^\alpha \hat{u}(k).$$

Since

$$\sum_{k \in \mathbb{Z}^d} |(2\pi i k)^\alpha|^2 \, |\hat{u}(k)|^2$$

$$\leq (2\pi)^{2n} (1+|k|^2)^n \, |\hat{u}(k)|^2,$$

we get $\partial^\alpha u \in L^2$ by Parseval, and

$$\sum_{|\alpha| \leq n} \|\partial^\alpha u\|_{L^2} \leq (2\pi)^{2n} \sum_{|\alpha| \leq n} \|u\|_{H^n}$$

$$= C_3 \|u\|_{H^n}.$$

To obtain the claimed equivalence between $\|\cdot\|_{W^{n,2}}$ and $\|\cdot\|_{H^n}$, simply set $C = \sqrt{C_2 \cancel{(C_3)^n}} \vee C_3$.

Remains to show that $\partial^\alpha u$ is really ~~aw~~ the weak derivative of order α. For $\varphi \in C^\infty$:

$$(-1)^{|\alpha|} \int \partial^\alpha \varphi(x) \, u(x) \, dx = (-1)^{|\alpha|} \int \partial^\alpha \varphi(x) \, \overline{\overline{u(x)}} \, dx$$

$$\overset{\text{Parseval}}{=} (-1)^{|\alpha|} \sum_k \underbrace{\widehat{\partial^\alpha \varphi}(k)}_{\overset{\text{Prop. 6.5}}{=} (2\pi i k)^\alpha \hat{\varphi}(k)} \, \underbrace{\overline{\hat{\bar{u}}(k)}}_{= \hat{u}(-k)}$$

$$= \sum_k \hat{\varphi}(k) \, \widehat{\partial^\alpha u}(-k) = \ldots \overset{\text{Parseval}}{=} \int \varphi(x) \, \partial^\alpha u(x) \, dx$$

b. Set up a Picard iteration $\Gamma : L^2 \to L^2$,

$$\Gamma(u) = (a-\Delta)^{-1} F(u),$$

where we define

$$\widehat{(a-\Delta)^{-1} u}(k) := \frac{\hat{u}(k)}{a+|k|^2},$$

so $\|(a-\Delta)^{-1} u\|_{L^2}^2 = \sum_k \frac{|\hat{u}(k)|^2}{(a+|k|^2)^2} \le \frac{1}{a^2} \|u\|_{L^2}^2$.

Moreover,

$$\|F(u)\|_{L^2}^2 = \left(\int_{\mathbb{T}^d} |F(u)|^2 dx\right)^{1/2} \le |F(0)|^2 + \left(\int_{\mathbb{T}^d} \|F'\|_\infty^2 |u|^2 dx\right)^{1/2}$$

$$= |F(0)| + \|F'\|_\infty \|u\|_{L^2},$$

so $\Gamma : L^2 \to L^2$ indeed, and $\|\Gamma(u)\|_{L^2} \le \frac{1}{a}\left(|F(0)| + \|F'\|_\infty \|u\|_{L^2}\right)$.

Moreover,

$$\|\Gamma(u) - \Gamma(v)\|_{L^2} \le \frac{1}{a} \|F(u) - F(v)\|_{L^2} \le \frac{\|F'\|_\infty}{a} \|u-v\|_{L^2}.$$

So for $\frac{\|F'\|_\infty}{a} < 1$: contraction, and then by Picard iteration:

$\exists!\ u \in L^2$ s.t. $u = (a-\Delta)^{-1} F(u)$.

But then $\|u\|_{H^2}^2 = \sum_k (1+|k|^2)^2 \frac{1}{(a+|k|^2)^2} \|\widehat{F(u)}(k)\|_{L^2}^2 \le C(a) \|F(u)\|_{L^2}^2 \to$

and thus $u \in \cancel{H} H^2$. Moreover,

$$\widehat{\Delta u}(k \quad \widehat{(a-\Delta)u}(k) = (a+|k|^2)\,\hat{u}(k) = \widehat{F(u)}(k) \quad \forall k$$

$$\Rightarrow (a-\Delta)u = F(u).$$