Autoequivalences of toric surfaces

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We study the derived categories of smooth projective toric surfaces and show that the group of autoequivalences is generated by the standard equivalences and spherical twists obtained from -2-curves. We also prove a close link between spherical objects and certain pairs of exceptional objects.

In this article, we study the derived categories of smooth, projective, toric surfaces. We follow the philosophy that varieties with zero curvature (i.e. $K_X = 0$) possess the richest autoequivalences, with the group of autoequivalences being minimal for varieties at both ends of the curvature spectrum (i.e. K_X ample or anti-ample) by the famous result of Bondal and Orlov [BO]. Our surfaces have rather positive curvature as $-K_X$ is big, i.e. a sufficiently high power of $-K_X$ gives a birational map from X to a surface in projective space (see [La1, Definition 2.2.1]). Thus we expect rather few autoequivalences beyond the standard ones. We note however, that toric surfaces can contain smooth rational curves of self-intersection -2, a simple example being the second Hirzebruch surface, and that such curves give rise to spherical twists. We prove here that these twists are essentially the only new autoequivalences which can occur.

This result has a counterpart in negative curvature, where Ishii and Uehara [IU] show the same thing for smooth projective surfaces of general type whose canonical model has at most A_n-singularities. We draw heavily upon their methods in this article.

We are also able to prove a relationship between exceptional and spherical objects on a smooth projective toric surface. It is known that such surfaces come with an abundance of exceptional objects, including, for example, all line bundles. Furthermore, Kawamata [Ka2] showed that such a surface (and more generally, a quasi-smooth, projective toric variety of any dimension) carries a full exceptional collection of sheaves. We link the rather few spherical objects to the wealth of exceptional ones, by discussing exceptional presentations of spherical objects, i.e. exact triangles $E' \to E \to S$ with E', E exceptional and S spherical.

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1. Setup and Results

Let X be a smooth, projective surface over an algebraically closed field $\mathbf{k}$. Denote its derived category by $\mathcal{D}(X) := D^b(\mathrm{Coh}(X))$, this is a $\mathbf{k}$-linear, triangulated category.

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Also, $-\otimes\omega_X[2]\colon \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(X)$ is a Serre functor, i.e.

$$\mathrm{Hom}^\bullet(A,B) \cong \mathrm{Hom}^{2-\bullet}(B, A\otimes\omega_X)^*$$

bifunctorially in $A, B \in \mathcal{D}(X)$. Here, $\mathrm{Hom}^\bullet(A,B) = \bigoplus_i \mathrm{Hom}(A, B[i])[-i]$ is a complex of $\mathbf{k}$-vector spaces with trivial differential. Note that by our assumptions on X, the total dimension of $\mathrm{Hom}^\bullet(A,B)$ is finite.

The subgroup of $\mathrm{Aut}(\mathcal{D}(X))$ containing the *standard autoequivalences* is by definition

$$A(X) := (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1],$$

where $\mathrm{Pic}(X)$ are the line bundle twists, $\mathrm{Aut}(X)$ surface automorphisms and $\mathbb{Z}[1]$ the shifts of complexes.

Sometimes, $\mathrm{Aut}(\mathcal{D}(X))$ is strictly larger than $A(X)$. For example, when X is an abelian surface there will always be the non-standard original Fourier-Mukai transform (see [Mu]). Another source for non-standard equivalances are *spherical objects* in $\mathcal{D}(X)$, i.e. objects $S \in \mathcal{D}(X)$ such that $\mathrm{Hom}^\bullet(S,S) = \mathbf{k} \oplus \mathbf{k}[-2]$ and $S \otimes \omega_X \cong S$. A crucial example is given by $S = \mathcal{O}_C$, where $C \subset X$ is a smooth, rational curve with $C^2 = -2$. We refer the reader to [ST] or [Hu] for this, as well as for the fact that there is a canonically associated autoequivalence $\mathsf{T}_S\colon \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(X)$, the *spherical twist.*

Let us introduce some notation:

$$\begin{aligned} \Delta(X) &:= \{C \subset X \text{ irreducible } (-2)\text{-curve}\}, && \text{a possibly infinite set;} \\ \mathrm{Pic}_\Delta(X) &:= \langle \mathcal{O}_X(C) \,:\, C \in \Delta \rangle, && \text{as a subgroup of } \mathrm{Pic}(X); \\ B(X) &:= \langle \mathsf{T}_S \,:\, S \in \mathcal{D}(X) \text{ spherical} \rangle, && \text{as a subgroup of } \mathrm{Aut}(\mathcal{D}(X)). \end{aligned}$$

In [IU], Ishii and Uehara prove that for a smooth projective surface of general type whose canonical model has at worst A_n-singularities, the autoequivalences are generated by $B(X)$ and the standard autoequivalences. The following theorem is a counterpart in positive curvature.

Theorem 1. *Let X be a smooth, projective surface subject to the conditions*
(1) The -2-curves on X form disjoint chains of type A.
(2) The anti-canonical bundle is big.
(3) An irreducible curve C in X with $K_X.C = 0$ is a smooth, rational -2-curve.
Then, $\mathrm{Aut}(\mathcal{D}(X))$ *is generated by* $\mathrm{Pic}(X)$, $\mathrm{Aut}(X)$, $\mathbb{Z}[1]$ *and* $B(X)$*.*

Assume that X also satisfies the following condition:
(4) The inclusion $\mathrm{Pic}_\Delta(X) \hookrightarrow \mathrm{Pic}(X)$ *splits.*
Then, after choosing a complementary subgroup $\mathrm{Pic}_\Delta(X) \oplus P = \mathrm{Pic}(X)$*, there is the following decomposition of* $\mathrm{Aut}(\mathcal{D}(X))$*,*

$$\mathrm{Aut}(\mathcal{D}(X)) = B(X) \rtimes (P \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

Remark 2. On any smooth, rational surface X with $K_X \neq 0$, spherical objects are necessarily supported on curves, and then the relations $\mathrm{Pic}(X) \cap B(X) = \mathrm{Pic}_\Delta(X)$ and $\mathrm{Aut}(X) \cap B(X) = 1$ hold [IU, §4]. These hint at the semi-orthogonal decomposition for $\mathrm{Aut}(\mathcal{D}(X))$ of the theorem, but in Section 4 we give a toric example where condition (4) is violated.

Remark 3. The assumptions of the theorem are probably not optimal. We have to state (1) because we rely on [IU]. However, it is expected that this result extends to types beyond the A_n-case.

We state (2) in order to be able to use [Ka1]. While it is obvious that Kawamata's result will not extend to arbitrary surfaces — the anti-canonical bundle of $\mathbb{P}^2$ blown up in nine general points is not big — the conclusion of Theorem 1 is expected to hold for all smooth, projective surfaces with non-zero canonical class.

In the spirit of our introduction, we would like to point out that $-K_X$ big implies $\mathrm{FM}(X) = \{X\}$, i.e. there are no non-trivial Fourier-Mukai partners, as follows again from [Ka1]. Note that this is no longer true in dimension 3. However, our result shows that many surfaces have $\mathrm{Aut}(\mathcal{D}(X)) \neq A(X)$ despite $\mathrm{FM}(X) = \{X\}$.

Remark 4. Something can be said about the structure of $B(X)$. Let $\mathcal{C} = \bigcup_{C \in \Delta} C$ be the union of all -2-curves on X and $\mathcal{C} = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_r$ be its decomposition into connected components. Let $B(X)|_{\mathcal{C}_l} \subset B(X)$ be the subgroup obtained from spherical objects supported on $\mathcal{C}_l$. Then one has $B(X) = B(X)|_{\mathcal{C}_1} \oplus \cdots \oplus B(X)|_{\mathcal{C}_r}$ since spherical twists corresponding to fully orthogonal objects commute. By the work of [IUU, Corollary 37], $B(X)|_{\mathcal{C}_l}$ is isomorphic to an affine braid group on n_l strands.

Our main point is to apply the theorem to smooth projective toric surfaces:

Theorem 5. *If X is a smooth, projective, toric surface, then the conditions (1)–(3) of Theorem 1 are satisfied. Condition (4) is satisfied if there is a cone in the fan of X spanned by two rays corresponding to curves of self-intersection different from -2.*

We will prove Theorem 1 in Section 2 and the proof of Theorem 5 occupies Section 3. Explicit constructions and examples of a toric nature are given in Section 4. Particularly, an example of a smooth, toric surface not fulfilling condition (4) is given. In the remainder of this section we deal with the relationship between exceptional and spherical objects — a property peculiar to the case of positive curvature. In the final section, we mention another class of examples, rational surfaces with $\mathbf{k}^*$-action such that all isotropy groups are connected.

Exceptional and spherical objects. The wealth of exceptional objects on toric (and more generally, rational) surfaces has been investigated for some time and is now well understood (see [HiPe]). Recall, an object $E \in \mathcal{D}(X)$ is *exceptional* if $\mathrm{Hom}^\bullet(E, E) = \mathbf{k}$, i.e. it is a simple as possible from the point of view of the derived category. Spherical objects, on the other hand, are the second-most simple type of object in a certain sense (namely, they possess the smallest derived endomorphism rings among objects invariant under twisting with the canonical bundle). For our surfaces, these are much rarer than exceptional objects. Therefore, it seems natural to wonder whether spherical objects can be expressed via exceptional ones.

Before stating the theorem, we recall some standard notions: An *exceptional pair* consists of two exceptional objects $E', E \in \mathcal{D}(X)$ such that $\mathrm{Hom}^\bullet(E, E') = 0$. More generally, a sequence $(E_1, \ldots, E_n)$ is called an *exceptional collection* if all $E_i \in \mathcal{D}(X)$ are exceptional objects and $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ for $j > i$. The collection is said to be *full* if the smallest triangulated subcategory of $\mathcal{D}(X)$ containing $E_1, \dots, E_n$ is $\mathcal{D}(X)$. Finally, we introduce non-standard terminology: (E', E) is called a *special exceptional pair* if it is an exceptional pair with $\mathrm{Hom}^\bullet(E', E) = \mathbf{k} \oplus \mathbf{k}[-1]$.

Theorem 6. *Let $E' \to E \to S$ be an exact triangle in $\mathcal{D}(X)$, where X is a smooth, projective surface.*

- *(i) If (E', E) is a special exceptional pair, then $\mathrm{Hom}^\bullet(S, S) = \mathbf{k} \oplus \mathbf{k}[-2]$. If in addition $\mathcal{D}(X)$ has a full, exceptional collection, then S is spherical.*
- *(ii) If E is exceptional, S is spherical and $\mathrm{Hom}^\bullet(E, S) = \mathbf{k}$, then (E', E) is a special exceptional pair.*
- *(iii) If S is spherical and X is a rational surface satisfying conditions (1)–(3) of Theorem 1, then (E', E) can be chosen to be a special exceptional pair.*

Remark 7. We want to point out that if X is rational surface, the category $\mathcal{D}(X)$ has a full, exceptional collection (see [HiPe]). Combined with Theorem 5, we see that toric surfaces meet the provisions required in the above theorem.

Proof. For (i) and (ii), use the following diagram in $\mathcal{D}(\mathbf{k})$:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(E', E') \arrow[r] & \mathrm{Hom}^\bullet(E', E) \arrow[r] & \mathrm{Hom}^\bullet(E', S) \\
\mathrm{Hom}^\bullet(E, E') \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(E, E) \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(E, S) \arrow[u] \\
\mathrm{Hom}^\bullet(S, E') \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(S, E) \arrow[r] \arrow[u] & \mathrm{Hom}^\bullet(S, S) \arrow[u]
\end{tikzcd}$$

This will immediately imply the first part of (i) and also (ii), using the assumption of sphericality on S to invoke Serre duality.

For the second claim of (i), note that any object $G \in \mathcal{D}(X)$ defines a functor T_G via $\mathrm{Hom}^\bullet(G, \cdot) \otimes G \to \mathrm{id} \to \mathsf{T}_G$; this can be defined as the Fourier-Mukai transform $\mathsf{FM}_{\mathcal{G}}$ with kernel $\mathcal{G} = \mathrm{cone}(G^\vee \boxtimes G \to \mathcal{O}_\Delta)$. The class $\{G\} \cup G^\perp$ spans $\mathcal{D}(X)$, in the sense that $(\{G\} \cup G^\perp)^\perp = 0$ and ${}^\perp(\{G\} \cup G^\perp) = 0$. It is trivial to see that $\mathsf{T}_G|_{G^\perp} = \mathrm{id}$. It is also easy to show $\mathsf{T}_G(G) = G[1-m]$ if $\mathrm{Hom}^\bullet(G, G) = \mathbf{k} \oplus \mathbf{k}[-m]$ for some $m \in \mathbb{Z}$. These two properties imply with little additional work that T_G is fully faithful on the spanning class (see [Hu, Proposition 8.6]), and is then bona fide fully faithful by general theory (see [Hu, Proposition 1.49]). Thus, we know that T_S is fully faithful. Now, assuming the existence of a full, exceptional collection $E_\bullet = (E_1, \dots, E_n)$ for $\mathcal{D}(X)$, the image $\mathsf{T}_S(E_\bullet) = (\mathsf{T}_S(E_1), \dots, \mathsf{T}_S(E_n))$ will obviously be an exceptional collection again. We want to show that $\mathsf{T}_S(E_\bullet)$ is full. For this, note that $\mathsf{T}_S(S) = S[-1]$ implies $S \in \langle \mathsf{T}_S(E_\bullet) \rangle$, and consider the triangle $\mathrm{Hom}^\bullet(S, E_i) \otimes S \to E_i \to \mathsf{T}_S(E_i)$. There, the first and the last term are in $\langle \mathsf{T}_S(E_\bullet) \rangle$ and hence the middle term is as well. Thus, we have $E_i \in \langle \mathsf{T}_S(E_\bullet) \rangle$ for all i, i.e. $\langle \mathsf{T}_S(E_\bullet) \rangle = \mathcal{D}(X)$ as $E_\bullet$ was supposed to be full. Hence T_S is essentially surjective and therefore an equivalence. In particular, T_S commutes with the Serre functor, one instance of which is $\mathsf{T}_S(S\otimes\omega_X)=\mathsf{T}_S(S)\otimes\omega_X=S[-1]\otimes\omega_X$. The triangle defining $\mathsf{T}_S(S\otimes\omega_X)$ then looks like

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(S,S\otimes\omega_X)\otimes S \arrow[r] \arrow[d, equal] & S\otimes\omega_X \arrow[r] \arrow[d, equal] & \mathsf{T}_S(S\otimes\omega_X) \arrow[d, equal] \\
S\oplus S[-2] \arrow[r] & S\otimes\omega_X \arrow[r] & S\otimes\omega_X[-1]
\end{tikzcd}$$

where the left equality follows from Serre duality and $\mathrm{Hom}^\bullet(S,S)=\mathbf{k}\oplus\mathbf{k}[-2]$. Since $\mathrm{Hom}(S\otimes\omega_X,S\otimes\omega_X[-1])=\mathrm{Hom}^{-1}(S,S)=0$, the triangle splits. This is enough to deduce $S\cong S\otimes\omega_X$ so that S is indeed a spherical object.

Claim (iii) follows from Theorem 1, coupled with [IU, Proposition 1.6] which states that the spherical twists of objects supported on a chain act transitively on these spherical objects: given a spherical object $S\in\mathcal{D}(X)$, there is an $\varphi\in\mathrm{Aut}(\mathcal{D}(X))$ such that $\varphi(S)\cong\mathcal{O}_C(a)$ for some $C\in\Delta(X)$ and $a\in\mathbb{Z}$. (As in Remark 4, knowing this property for a single A_n-chain is enough in order to apply it to X.)

Since X is assumed to be rational, line bundles are exceptional objects and we get the exceptional presentation $\mathcal{O}_X(-C)\to\mathcal{O}_X\to\mathcal{O}_C$ for the sheaf $\mathcal{O}_C$, we get another one for S as $\varphi^{-1}(\mathcal{O}_X(-C+aH))\to\varphi^{-1}(\mathcal{O}_X(aH))\to S$, where we just temporarily denote by H a polarisation such that $H.C=1$.

Finally, the assertions $\mathrm{Hom}^\bullet(E',E)=\mathbf{k}\oplus\mathbf{k}[-1]$ and $\mathrm{Hom}^\bullet(E,E')=0$ follow at once from $\mathrm{Hom}^\bullet(\mathcal{O}_X,\mathcal{O}_X(-C))=H^\bullet(\mathcal{O}_X(-C))=0$ together with $\mathrm{Hom}^\bullet(\mathcal{O}_X(-C),\mathcal{O}_X)=H^\bullet(\mathcal{O}_X(C))=\mathbf{k}\oplus\mathbf{k}[-1]$. □

Remark 8. In the proof of (i) we have used that the functor T_G is fully faithful for an object G with $\mathrm{Hom}^\bullet(G,G)=\mathbf{k}\oplus\mathbf{k}[-m]$. See [HePl, §2.4] for examples of such objects which are not spherical.

We have also shown a very particular case of the following statement: given a full, exceptional collection $E_\bullet$ and another exceptional collection $F_\bullet$ of same length, then $F_\bullet$ is full as well. Whether this holds in general is an open question (beyond the hereditary case, like smooth, projective curves).

2. Proof of Theorem 1

Let $\mathcal{C}=\bigcup_{C\in\Delta}C=\mathcal{C}_1\cup\cdots\cup\mathcal{C}_r$ be the decomposition of all -2-curves into disjoint chains. For a given autoequivalence $\varphi\in\mathrm{Aut}(\mathcal{D}(X))$ we start by finding an isomorphism q of X and an integer i such that skyscraper sheaves $k(x)$ at points $x\in X\setminus\mathcal{C}$ are fixed by $q^*\circ\varphi[-i]$.

By a well-known result of Orlov [O] there is a unique Fourier-Mukai kernel $P\in\mathcal{D}(X\times X)$, so that $\varphi\cong\mathsf{FM}_P$. As the anti-canonical sheaf is big by assumption, the conditions required for [Ka1, Theorem 2.3(2)] hold. In particular, looking at the proof of this theorem we see that there exists an irreducible component $Z\subset\mathrm{supp}(P)\subset X\times X$ such that the restrictions to Z of the natural projections $\pi_1,\pi_2\colon X\times X\to X$ are surjective and birational. Following [IU], we set

$$q:=\pi_2|_Z\circ\pi_1|_Z^{-1}\colon X\to X.$$

As X is a smooth surface, using [Ka1, Lemma 4.2], we note that this birational map is a genuine isomorphism (any birational map between smooth surfaces is a sequence of blow ups and blow downs but Kawamata's lemma shows that the biration map in question is an isomorphism in codimension 1).

Now we show that for any point $x \in X$, the support of $\varphi(k(x))$ is either the point $q(x)$, or is a connected subset of $\mathcal{C}$. Note that $\mathrm{supp}(\varphi(k(x))$ must be connected, as the map $\mathrm{Hom}_{\mathcal{D}(X)}(k(x), k(x)) \to \mathrm{Hom}_{\mathcal{D}(X)}(\varphi(k(x)), \varphi(k(x)))$ is bijective. It is a general property of equivalences to commute with Serre functors, in particular $\varphi(k(x)) = \varphi(k(x) \otimes \omega_X) \cong \varphi(k(x)) \otimes \omega_X$, for any point x. As ω_X is a non-trivial line bundle, this forces $\varphi(k(x))$ to have proper support, i.e. $\dim \varphi(k(x)) < 2$. Therefore $\varphi(k(x))$ is either supported at a point, or it is a union of curves. Suppose $C \subset \mathrm{supp}(\varphi(k(x)))$ is any irreducible curve contained in the support. Since $\omega_X|_C \otimes \varphi(k(x))|_C = (\omega_X \otimes \varphi(k(x)))|_C = \varphi(k(x))|_C$ and $\varphi(k(x))$ is supported on the curve C, we get $\omega_X|_C = \mathcal{O}_C$. Hence, $C \subset X$ is a curve with $K_X.C = 0$. By assumption (3), we deduce that C is a smooth, rational curve with $C^2 = -2$. Now looking at the FM transform at the level of its support, we observe that

$$q(x) = \pi_2(Z \cap (\{x\} \times X)) \subseteq \pi_2(\mathrm{supp}(P) \cap (\{x\} \times X)) = \mathrm{supp}(\varphi(k(x))).$$

If $\varphi(k(x))$ is supported at a point then this point must be $q(x)$. Otherwise we have shown that all components of $\mathrm{supp}(\varphi(k(x)))$ are -2-curves and so $q(x)$ is contained in some -2-curve C. As q is a surface automorphism, we find $x \in q^{-1}(C)$, another -2-curve. In particular this implies that if $x \in X \setminus \mathcal{C}$ then $\varphi(k(x))$ is supported at the point $q(x)$ and is therefore a shifted skyscraper sheaf (of length 1)

$$\varphi(k(x)) = k(q(x))[j] = q_*(k(x))[j].$$

(The integer j is fixed and cannot jump: for an equivalence between derived categories of smooth, projective schemes, mapping a skyscraper sheaf to a skyscraper sheaf is an open property; see [Hu, Corollary 6.14].) Hence, $\psi := q^* \circ \varphi[-j]$ is an autoequivalence of $\mathcal{D}(X)$ which fixes all skyscraper sheaves $k(x)$ for $x \in X \setminus \mathcal{C}$. This is enough to deduce that ψ preserves $\mathcal{C}$, i.e. induces an autoequivalence of $\mathcal{D}_{\mathcal{C}}(X)$. Here, $\mathcal{D}_{\mathcal{C}}(X)$ is the full subcategory of $\mathcal{D}(X)$ consisting of objects whose support (of all cohomology sheaves) is contained in $\mathcal{C}$. (If $A \in \mathcal{D}_{\mathcal{C}}(X)$, we need to show that $\mathrm{supp}(\psi(A)) \subseteq \mathcal{C}$. If there was $y \in \mathrm{supp}(\psi(A))$, $y \notin \mathcal{C}$, there would be a non-zero morphism $\psi(A) \to k(y)$. However, this would imply a non-zero map $A \to \psi^{-1}(k(y)) = k(y)$, in contradiction to the assumption $\mathrm{supp}(A) \subset \mathcal{C}$.)

In fact we can see that ψ preserves each connected component $\mathcal{C}_i$. For this, consider a curve B (not contained in $\mathcal{C}$). If B does not meet the component $\mathcal{C}_i$, then the same is true for the transform, i.e. $\mathrm{supp}(\psi(\mathcal{O}_B))$ does not intersect $\mathcal{C}_i$, using same reasoning as in the previous paragraph. More generally, if B does not meet several of the components, then the same will be true for the transform. So if we can find enough curves B to separate the components of $\mathcal{C}$, then ψ has to preserve each of them. See Lemma 9 for a proof of this fact in the general case and Lemma 13 for a self-contained proof in the toric case.

Therefore we are in a position to use the 'Key Proposition' from Ishii and Uehara [IU] repeatedly on each chain of -2-curves: there exists an integer i and an autoequivalence

$\Psi \in B(X)$ such that $\Psi \circ \psi$ sends every skyscraper sheaf $k(x)$ for $x \in \mathcal{C}$ to $k(y)[i]$ for some $y \in \mathcal{C}$. (In [IU], only globally defined autoequivalences are used, so that the presence of several chains does not pose an obstacle.)

A well-known lemma of Bridgeland and Maciocia (see [Hu, Corollary 5.23]) states that an autoequivalence permutating skyscraper sheaves (of length 1) must be in $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$. Thus we get

$$\Psi \circ \psi = \Psi \circ q^* \circ \varphi[-i] \in \mathrm{Pic}(X) \rtimes \mathrm{Aut}(X).$$

This shows that $\mathrm{Aut}(\mathcal{D}(X))$ is indeed generated by $\mathrm{Aut}(X)$, $\mathrm{Pic}(X)$, $B(X)$ and $\mathbb{Z}[1]$.

The relations $\mathrm{Aut}(X) \cap B(X) = 1$ and $\mathrm{Pic}(X) \cap B(X) = \mathrm{Pic}_\Delta(X)$ are proved in [IU] (See Lemma 4.14 and Proposition 4.18 respectively and note that we can treat each chain individually using Remark 4).

If the embedding $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$ splits, we have the direct sum decomposition $\mathrm{Pic}(X) = P \oplus \mathrm{Pic}_\Delta(X)$ for some subgroup P. Since $\mathrm{Pic}_\Delta(X)$ is fixed by the $\mathrm{Aut}(X)$-action, we get another decomposition

$$\begin{aligned} A(X) &= \mathbb{Z}[1] \times (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \\ &\cong \mathbb{Z}[1] \times \big((\mathrm{Pic}_\Delta(X) \rtimes \mathrm{Aut}(X)) \oplus (P \rtimes \mathrm{Aut}(X))\big) \end{aligned}$$

Then we arrive at the situation of having two subgroups of $\mathrm{Aut}(\mathcal{D}(X))$, namely $\mathbb{Z}[1] \times (P \rtimes \mathrm{Aut}(X))$ and the normal divisor $B(X)$ which together generate $\mathrm{Aut}(\mathcal{D}(X))$ and whose intersection is trivial. Hence we get the desired semi-direct product decomposition, and the proof of Theorem 1 is finished, apart from the following lemma.

Lemma 9. *Let X be a projective surface such that all -2-curves appear in disjoint ADE-chains. Then for any two such chains, there is a curve meeting one transversally while being disjoint to the other.*

Proof. Fix two different chains $\mathcal{C}$, $\mathcal{C}'$ of -2-curves. By assumption, these are disjoint. We contract $\mathcal{C}$ and $\mathcal{C}'$ to obtain a surface Y with two rational singularities y, y'. This is possible, i.e. Y is algebraic, since we are dealing with chains of -2-curves of type ADE (for the main theorem, only type A is relevant); see [Ar, Theorem 2.7].

In fact, Y is projective since X was. Choosing an ample divisor of sufficiently large degree, we find a curve $B \subset Y$ going through y but missing y'. Its preimage under the contraction $X \to Y$ then has the desired property. $\square$

3. Proof of Theorem 5

In this section, we will work with a smooth, projective toric surface X. We start by fixing some notation and gathering a few well-known properties of toric surfaces that we will use later. For more detailed discussion and proofs, we refer the reader to [Fu].

Let N be a rank 2 lattice and define $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. A toric surface X is specified by a fan Σ of (strongly convex rational polyhedral) cones in $N_\mathbb{R}$. We denote by $\Sigma(1)$ the set of rays (one dimensional cones) in Σ, by $\{v_i\}_{i \in \Sigma(1)} \subset N$ the set of primitive generators of the rays and by $\{D_i\}_{i\in\Sigma(1)}$ the set of torus invariant divisors corresponding to the rays. We assume that the fan is complete (the support of Σ is $N_{\mathbb{R}}$) which (in the surface case) is equivalent to the property that X is projective. The variety X is smooth and this is equivalent to the condition on the fan, that for each two-dimensional cone σ, the generators of the rays of σ form a basis for N. Ordering the generators cyclicly, it follows that

$$\alpha_i v_i = v_{i-1} + v_{i+1} \qquad \forall i = 1, \dots, |\Sigma(1)|$$

for some integers α_i. It can be shown that $-\alpha_i$ is the self-intersection number of D_i for each $i \in \Sigma(1)$.

Since X is smooth, there is an exact sequence [Fu, §3.4]

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \operatorname{Pic}(X) \to 0, \tag{1}$$

where we denote by $M := N^{\vee}$ the dual lattice of N. $\operatorname{Pic}(X)$ is a free abelian group so $\operatorname{Pic}_{\Delta}(X)$ is the free abelian subgroup generated by $\Delta(X)$.

We start by looking at the intersection properties of -2-curves on a smooth projective toric surface.

Lemma 10. *$\Delta(X)$ consists of a finite number of disjoint chains of type A.*

Proof. Let C be a curve in $\Delta(X)$. Using the exact sequence (1) we observe that C is linearly equivalent to a sum $\sum_{i\in\Sigma(1)} a_i D_i$ of torus invariant divisors (irreducible torus invariant curves in X) indexed by the rays in the fan Σ of X. Since C is effective, we may choose this Weil divisor in such a way that it is also effective, so $a_i \geq 0$ for each $i \in \Sigma(1)$. Then

$$-2 = C.C = C.\Big(\sum_i a_i D_i\Big) = \sum_i a_i (C.D_i)$$

so there exists some $i \in \Sigma(1)$ such that $C.D_i < 0$. Since C and D_i are both irreducible curves, we conclude that $C = D_i$. Thus all curves in $\Delta(X)$ are torus invariant curves corresponding to rays of Σ. Such curves intersect if and only if the corresponding rays span a cone (see for example [Fu, §5.1]). By looking at the fan Σ which is supported on $N_{\mathbb{R}} \cong \mathbb{R}^2$ we see that the only possible configurations are a finite number of disjoint chains of type A or a single closed chain of type $\tilde{A}_{|\Sigma(1)|}$. It is easy to see that this final possibility doesn't occur. □

Lemma 11. *If X is a smooth, projective, toric variety (not necessarily a surface), then $-K_X$ is big.*

Proof. It is well-known (see [Fu, §4.3]) that $-K_X$ is linearly equivalent the sum $\sum_{i\in\Sigma(1)} D_i$ of all torus invariant divisors. In particular, the divisor class of $-K_X$ lies in the interior of the pseudoeffective cone (since this is generated as a cone by the classes of the torus invariant divisors). It follows (see for example [La1, Theorem 2.2.26]) that $-K_X$ is big. □

Lemma 12. *If X is a smooth, projective, toric surface and $C \subset X$ is an irreducible curve with $K_X.C = 0$, then C is a smooth, rational curve with $C^2 = -2$.*

Proof. Using again the fact that $-K_X$ can be written as the sum of all torus invariant divisors, we have $\sum_{i\in\Sigma(1)} D_i.C = 0$. Since the D_i generate the pseudoeffective cone (which contains an ample divisor) there is at least one value of i for which $D_i.C \neq 0$. Therefore there exists $j \in \Sigma(1)$ for which $D_j.C < 0$ and since D_j and C are both irreducible it follows that $D_j = C$. All the torus invariant divisors are isomorphic to $\mathbb{P}^1$ and so it follows from the adjunction formula that C is a -2-curve. □

Lemma 13. *If X is a smooth, projective, toric surface then for each pair of chains of -2-curves $\mathcal{C}_1$ and $\mathcal{C}_2$, there exists a curve (not contained in $\Delta(X)$) which intersects $\mathcal{C}_1$ but not $\mathcal{C}_2$.*

Proof. We saw in the proof of Lemma 10 that all -2-curves on a smooth projective toric surface are torus invariant curves corresponding to rays of the fan. Such curves intersect if and only the corresponding rays span a cone in the fan. Therefore, ordering the rays cyclically, each chain of -2-curves corresponds to a sequence of rays with generators $v_r, \ldots, v_s$ satisfying the property that $2v_i = v_{i-1} + v_{i+1}$ for all $i = r, \ldots, s$ and such that D_{r-1} and D_{s+1} are not -2-curves. Given a chain, it is enough to show that D_{r-1} and D_{s+1} do not both intersect any other chain. If they did, it follows that there are precisely two chains, and all but two rays in the fan correspond to -2-curves. Recall that $\{v_i, v_{i+1}\}$ is a basis of N for all i. By composing the basis change matrices going around the complete fan, we get an equation of the form

$$\begin{pmatrix} 0 & -1 \\ 1 & \alpha_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^{\ell_1} \begin{pmatrix} 0 & -1 \\ 1 & \alpha_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^{\ell_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $-\alpha_1, -\alpha_1 \neq -2$ are the self-intersection numbers of the other two torus invariant curves, and ℓ_1, ℓ_2 are the lengths of the two chains of -2-curves. It is basic linear algebra to check that this equation has no solutions. □

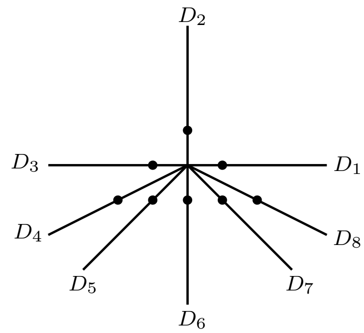
Lemma 14. *If X is a smooth, projective toric surface containing two divisors D_i, D_{i+1} corresponding to adjacent rays $i, i+1 \in \Sigma(1)$ with $D_i^2 \neq -2$ and $D_{i+1}^2 \neq -2$, then the group embedding $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$ splits.*

Proof. We use the standard exact sequence $0 \to M \xrightarrow{\iota} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} \mathrm{Pic}(X) \to 0$. Since X is smooth, the generators v_i and v_{i+1} of the rays $i, i+1 \in \Sigma(1)$ form a basis of $N \cong \mathbb{Z}^2$. Using the dual basis for M and considering the map ι, it is easy to see that the free abelian group $\mathrm{Pic}(X)$ has a basis $\{\pi(D_j) | j \in \Sigma, j \neq i, i{+}1\}$. Furthermore, since $D_i^2 \neq -2$ and $D_{i+1}^2 \neq -2$, the subgroup spanned by classes of -2-curves is generated by elements of this basis, and so is primitive in $\mathrm{Pic}(X)$. Hence, the quotient $\mathrm{Pic}(X)/\mathrm{Pic}_\Delta(X)$ is free and there exists a splitting. □

4. Toric surfaces

We start with an explicit example of a smooth, toric surface violating condition (5) of Theorem 1, that is a surface where the embedding $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$ of abelian groups does not split.

Example 15. We consider the toric surface given by the fan in the following picture:



It can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ four times. The self-intersection numbers are $D_2^2 = 0$, $D_4^2 = D_8^2 = -1$, and $D_1^2 = D_3^2 = D_5^2 = D_6^2 = D_7^2 = -2$. Choosing v_1 and v_2 as a basis of N, we see that the map $M \to \mathbb{Z}^{\Sigma(1)}$ is given by the transpose of the matrix $\left(\begin{smallmatrix} 1 & 0 & -1 & -2 & -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \end{smallmatrix}\right)$. In particular, the classes of $D_3, \dots, D_8$ form a basis of $\operatorname{Pic}(X)$. All of these are -2-classes except for D_4 and D_8. Writing the -2-curve D_1 in terms of this basis, we have $D_1 = 2(D_8 - D_4) + D_3 + D_5 + D_7$ in $\operatorname{Pic}(X)$. Therefore $0 = 2(D_8 - D_4)$ in $\operatorname{Pic}(X)/\operatorname{Pic}_\Delta(X)$, so there is torsion. This implies that the embedding of $\operatorname{Pic}_\Delta(X)$ into $\operatorname{Pic}(X)$ is not primitive.

We conclude with a few general observations about, and on the construction of, some classes of examples: As a straight forward consequence of Theorem 1, we see that any smooth projective toric surface without -2-curves has no autoequivalences beyond the standard ones, i.e. $\operatorname{Aut}(\mathcal{D}(X)) = A(X)$. We note that there are infinitely many examples of such surfaces including, for example, all Hirzebruch surfaces F_n for $n > 2$. It is not hard to check that $-K_X$ is ample if and only if there are no torus invariant curves of self-intersection -2 or lower. In fact, there are famously just five smooth toric Fano surfaces. Therefore, there are infinitely many smooth projective toric surfaces where $-K_X$ is not ample (and so are not covered by the theorem of Bondal and Orlov [BO]) but for which $\operatorname{Aut}(\mathcal{D}(X)) = A(X)$.

On the other hand, it is easy to construct examples with more interesting groups of autoequivalences. If v_0, v_1 form a basis for a rank two lattice N, then we can define inductively $v_{s+1} = 2v_s - v_{s-1}$ for $s = 1, \dots, \ell$. Taking these as generators of rays of a fan, we can produce a complete smooth fan by adding extra rays (with generators $v_{\ell+2}, \dots, v_{d-1}$) making sure we do not subdivide any of the existing cones. This doesn't affect the self-intersection numbers of $D_1, \dots, D_\ell$, which are by construction -2. Indeed by making an appropriate choice of $v_{\ell+2}, \dots, v_{d-1}$, we can ensure that D_0 and $D_{\ell+1}$ do not have self-intersection number -2. Therefore it is possible to construct a smooth projective toric surface with a chain of -2-curves of arbitary length. Blowing up the intersection point of two torus invariant curves D_s and D_{s+1} (which corresponds to subdividing the cone spanned by v_s and v_{s+1}) has the effect of reducing the self-intersection numbers of the strict transforms $\widetilde{D}_s$ and $\widetilde{D}_{s+1}$ by 1, and inserting an exceptional -1-curve. In this way we can split up a chain of -2-curves into pieces. In fact, we can produce any number of chains of -2-curves of any length.

5. Surfaces with k^*-action of connected isotropy

Let X be a smooth, projective surface having an effective $\mathbf{k}^*$-action. Such surfaces have been classified in [OW] using the intersection graph of all negative curves. Note that irreducible curves of negative self-intersection are either pointwise fixed or orbit closures.

Denote by F^+ the locus of fixed points without negative eigenspace for the $\mathbf{k}^*$-action on the tangent space, and similarly by F^- the locus of fixed points without positive eigenspace. Both F^+ and F^- are nonempty and irreducible. From now on we assume that F^+ and F^- are curves; then they are in fact isomorphic.

The graph of Orlik and Wagreich has as vertices all negative curves on X, together with F^+ and F^-, and two vertices are joined by an edge if the corresponding curves meet (which they then do transversely). The self-intersection numbers have to fulfill certain numerical conditions expressed with continued fractions in [OW]. All graphs consist of a number of linear graphs, the ends of each of which are connected to F^+ and F^-.

At any point x of X, the isotropy is a subgroup of $\mathbf{k}^*$. Here we will only look at such surfaces where isotropy groups are either $\mathbf{k}^*$ (in fixed points) or 1. In other words, we exclude non-trivial cyclic isotropy groups. This property translates to the self-intersection of all negative curves to be -2 and $(F^+)^2 = -1$. In particular, property (i) of Theorem 1 is satisfied.

Furthermore, we assume that X is rational. This is the case if and only if F^+ has genus 0 (see the proof of [OW, Theorem 2.5]). For properties (ii) and (iii) we need the following formula for K_X: denote the curves indexed by the i-th arm as $B^i_1, \dots, B^i_{m_i}$, we have $-K_X = F^+ + 2(B^i_1 + \cdots + B^i_{m_i}) + F^-$.

This immediately gives property (iii).

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