Postnikov-Stability for Complexes on Curves and Surfaces

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We present a novel notion of stable objects in the derived category of coherent sheaves on a smooth projective variety. As one application we compactify a moduli space of stable bundles using genuine complexes.

Introduction

Let X be a polarized, smooth projective variety of dimension n over an algebraically closed field k. Our aim is to introduce a stability notion for complexes, i.e. for objects of $\mathrm{D}^{\mathrm{b}}(X)$, the bounded derived category of coherent sheaves on X. The main motivation for this notion is Faltings' observation that semistability on curves can be phrased as the existence of non-trivial orthogonal sheaves [4] (similar results hold for surfaces, see [8]). In order to make this idea work, we need convolutions and Postnikov systems (the former can be seen as a generalization of total complexes, and the latter generalizes filtrations to the derived category). The details will be spelt out in the next section. As an example of our theory, we show how a classical non-complete moduli space of certain bundles can be compactified using complexes (see Section 4). Also, by construction, our notion of stability is preserved under equivalences (Fourier-Mukai transforms). Of interest to us is when classical preservation of stability conditions is a special case of our situation. A first check is done in Section 2.4. In Sections 5 and 6, we give some general facts of projective geometry from the derived point of view. In particular, Lemma 23, a generalization of the Euler sequence, is used several times.

It seems only fair to point out that the results of this article in all probability bear no connection with Bridgeland's notion of t-stability on triangulated categories (see [3]).

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His starting point about (semi)stability in the classical setting is the Harder-Narashiman filtration whereas, as mentioned above, we are interested in the possibility to capture μ-semistability in terms of Hom's in the derived category. Our approach is much closer to, but completely independent of, Inaba (see [13]).

On notation: we will denote the i-th homology of a complex a by $h^i(a)$. Functors are always derived without additional notation; e.g. for a proper map $f\colon X \to Y$ of schemes, we write $f_*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ for the triangulated (exact) functor obtained by deriving $f_*\colon \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$. For two objects a, b of a k-linear triangulated category, we write $\mathrm{Hom}^i(a,b) := \mathrm{Hom}(a, b[i])$ and $\mathrm{hom}^i(a,b) := \dim_k \mathrm{Hom}^i(a,b)$. The Hilbert polynomial of a sheaf E is denoted by $p(E)$, so that $p(E)(l) = \chi(E(l))$. Finally, by semistability for sheaves, we always mean μ-semistability.

1 P-stability

Let $\mathcal{T}$ be a k-linear triangulated category for some field k; we usually think of $\mathcal{T} = \mathrm{D^b}(X)$, the bounded derived category of a smooth, projective variety X, defined over an algebraically closed field k. A *Postnikov-datum* or just *P-datum* is a finite collection $C_d, C_{d-1}, \dots, C_{e+1}, C_e \in \mathcal{T}$ of objects together with nonnegative integers N_i^j (for $i, j \in \mathbb{Z}$) of which only a finite number are nonzero. We will write $(C_\bullet, N)$ for this.

Recall the notions of Postnikov system and convolution (see [5], [2], [18], [14]): given finitely many objects A_i (suppose $n \geq i \geq 0$) of $\mathcal{T}$ together with morphisms $d_i\colon A_{i+1} \to A_i$ such that $d^2 = 0$, a diagram of the form

$$\begin{tikzcd}
A_n \arrow[rr, "d_{n-1}"] \arrow[dr, equal] & & A_{n-1} \arrow[rr, "d_{n-2}"] \arrow[dr] & & A_{n-2} \arrow[r] \arrow[dr] & \cdots\cdots & & \cdots\cdots \arrow[r] & A_1 \arrow[rr, "d_0"] \arrow[dr] & & A_0 \arrow[dr] & \\
& T_n \arrow[ur] & & T_{n-1} \arrow[ll, "{[1]}"'] \arrow[ur] & & T_{n-2} & 5\cdots\cdots \arrow[l] & T_2 \arrow[l] \arrow[ur] & & T_1 \arrow[ll, "{[1]}"'] \arrow[ur] & & T_0 \arrow[ll, "{[1]}"'] \arrow[from=4-4, to=4-4, phantom]
\end{tikzcd}$$

(where the upper triangles are commutative and the lower ones are distinguished) is called a *Postnikov system* subordinated to the A_i and d_i. The object T_0 is called the *convolution* of the Postnikov system.

Definition. An object $A \in \mathcal{T}$ is *P-stable with respect to* $(C_\bullet, N)$ if

(i) $\mathrm{hom}^j_{\mathcal{T}}(A, C_i) = N_i^j$ for all $i = d, \dots, e$ and all j.

(ii) For $i > 0$, there are morphisms $d_i\colon C_i \to C_{i-1}$ such that $d^2 = 0$ and that the complex $(C_{\bullet \geq 0}, d_\bullet)$ admits a convolution K.

(iii) $\mathrm{Hom}^*_{\mathcal{T}}(A, K) = 0$, i.e. $K \in A^\perp$.

By the very definition of P-stability we obtain the next result.

Theorem 1. *Let $\Phi\colon \mathcal{T} \to \mathcal{S}$ be an exact equivalence of two k-linear triangulated categories $\mathcal{T}$ and $\mathcal{S}$, and $(C_\bullet, N)$ a P-datum in $\mathcal{T}$. We have for any object $A \in \mathcal{T}$*

$$A \text{ is P-stable with respect to } (C_\bullet, N) \iff \Phi(A) \text{ is P-stable for } (\Phi(C_\bullet), N).$$

The next result (see Theorem 10) shows that for algebraic surfaces X there are P-data such that the semistable vector bundles turn out to be the P-stable objects in $\mathrm{D^b}(X)$. Thus, P-stability generalizes the classical stability of vector bundles.

Theorem 2. *Let X be a smooth projective surface and H a very ample divisor on X. Given a Hilbert polynomial p, there is a P-stability datum $(C_\bullet, N)$ such that for any object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:*

(i) E is a μ-semistable vector bundle with respect to H of Hilbert polynomial p

(ii) E is P-stable with respect to $(C_\bullet, N)$.

We expect the generalization of the statement to arbitrary dimensions to be true.

Remark 3.

(a) Convolutions in general do not exist, and if they do, there is no uniqueness in general, either. There are restrictions on the $\mathrm{Hom}^j(C_a, C_b)$'s which ensure the existence of a (unique) convolution. For example, if $\mathcal{T} = \mathrm{D^b}(X)$ and all C_i are sheaves, then the unique convolution is just the complex $C_\bullet$ considered as an object of $\mathrm{D^b}(X)$.

(b) Note that the objects C_i with $i < 0$ do not take part in forming the Postnikov system. We call the conditions enforced by these objects via (i) the *passive* stability conditions. They can be used to ensure numerical constraints, like fixing the Hilbert polynomial of sheaves. We also point out that in our examples, the active parts of our data are of length two (i.e. given by a single morphism) or even absent (in which case the passive data suffice to describe P-stability).

(c) In many situations there will be trivial choices that ensure P-stability. This should be considered as a defect of the parameters (like choosing non-ample line bundles when defining μ-stability) and not as a defect of the definition.

The paper is organized as follows: we start with a section containing several examples in the curve case, which already is non-trivial. The second section contains the proof of the theorem, up to some technicalities: sheaf conditions (section 4) and a bit of general homological algebra (section 5). Section 3 contains a main application of our notion, dealing with new compactifications of classical moduli spaces.

2 Example: Stability on algebraic curves

In this section, X denotes a smooth projective curve of genus g over k. Let $r > 0$ and d be two integers and fix a line bundle L_1 on X of degree one.

2.1 Semistability conditions on curves

Our starting point is the following result. The vector bundle $F_{r,d}$ appearing in statement (iii) of the theorem below is universal, i.e. it only depends on r, d, and L_1. It is constructed in Section 6 in Remark 25. The specifications for the construction are given in the proof below. For more details on $F_{r,d}$ and its properties see the article [9], in particular Theorem 2.12.

Theorem 4. *For two fixed integers $r > 0$ and d there exists a vector bundle $F_{r,d}$ on X such that for all sheaves E of rank r and degree d the following conditions are equivalent:*

(i) E *is a semistable vector bundle.*

(ii) *There is a sheaf* $0 \neq F \in E^{\perp}$, *i.e.* $\mathrm{Hom}(F, E) = \mathrm{Ext}^1(F, E) = 0$.

(ii') *There exists a vector bundle* F *on* X *with* $\det(F) \cong L_1^{\otimes(r^2(1-g)+rd)}$ *and* $\mathrm{rk}(F) = r^2$ *such that* $\mathrm{Hom}(F, E) = \mathrm{Ext}^1(F, E) = 0$.

(iii) $\mathrm{Hom}(F_{r,d}, E) = 0$.

Proof. (ii') $\Longrightarrow$ (ii) is trivial and (ii) $\Longrightarrow$ (i) is well-known. Work by Popa [20] shows (i) $\Longrightarrow$ (ii'). Thus, it suffices to show (ii') $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Suppose there exists such a vector bundle F as in (ii'). It follows that F is also a semistable vector bundle. Putting $e := 3g - 1 - \lfloor \frac{d}{r} \rfloor$, the bundle $F \otimes L_1^{\otimes e}$ is then globally generated. Since X is of dimension one there exists a surjection $\mathcal{O}_X^{\oplus(r^2+1)} \xrightarrow{\pi} F \otimes L_1^{\otimes e}$. Its kernel is the line bundle $\ker(\pi) \cong \det\left(F \otimes L_1^{\otimes e}\right)^{-1} \cong L_1^{\otimes e'}$ with $e' := rd - 2gr^2 - r^2 \lfloor \frac{d}{r} \rfloor$. Eventually, we obtain a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow F \longrightarrow 0 \qquad \text{with } A = L_1^{\otimes e'-e} \text{ and } B = \left(L_1^{\otimes -e}\right)^{\oplus(r^2+1)}.$$

The semistability of E implies that $\mathrm{Ext}^1(A, E) = \mathrm{Ext}^1(B, E) = 0$. Thus, the existence of a vector bundle F with the above properties is equivalent to the existence of a morphism $\alpha \in \mathrm{Hom}(A, B)$ such that the resulting homomorphism $\mathrm{Hom}(B, E) \to \mathrm{Hom}(A, E)$ is injective. Invoking Remark 25, this is equivalent to (iii) with $F_{r,d}$ the vector bundle $h^0(S^m(\mathrm{Hom}(A, B), A, B))$ of Remark 25.

A straightforward calculation shows that $\frac{d}{r} + 1 - g < \mu(F_{r,d}) < \frac{d}{r} + 1 - g + \frac{1}{r^2}$. Suppose (iii) holds. If $E' \subset E$ were a destabilizing quotient, then we had $\mu(E') > \mu(E) + \frac{1}{r^2}$. Since $\mu(F_{r,d}) < \mu(E') - (g-1)$ we conclude $\mathrm{Hom}(F_{r,d}, E') \neq 0$ which contradicts (iii). $\square$

2.2 A first P-stability datum for algebraic curves

We consider the derived category $\mathrm{D^b}(X)$ of the smooth projective curve X. Let L be a very ample line bundle of degree D on X. As before, we fix two integers $r > 0$ and d. We assume that $d > (2g - 2 + D)r$. We have the

Proposition 5. *For an object* $E \in \mathrm{D^b}(X)$ *the following conditions are equivalent:*

(i) E *is a semistable sheaf of rank* r *and degree* d.

(ii) E *satisfies the following numerical conditions, for all* $i \neq 0$:

$$\begin{aligned} &\hom(\mathcal{O}_X, E) = d - r(g-1), && \hom(\mathcal{O}_X, E[i]) = 0, \\ &\hom(L, E) = d - r(g-1-D), && \hom(L, E[i]) = 0, \\ &\hom(L^{\otimes(r(g-1-D)-d)}, E[i]) = 0, && \hom(F_{r,d}, E) = 0. \end{aligned}$$

Proof. If E is a sheaf as in (i), then $\hom(F_{r,d}, E) = 0$ follows from Theorem 4; for the other equations in (ii), we use $\chi(E(k)) = rDk + d - r(g-1)$ and note the vanishing $H^1(E) = H^1(e \otimes L^{-1}) = 0$ due to semistability and the assumption $d > (2g-2+D)r$.

To see that (ii) implies (i), we use again that $\hom(F_{r,d}, E) = 0$ entails the stability of E, as by Lemma 16 the other five identities grant in advance that E is a sheaf of rank r and degree d. □

Remark 6. The conditions in part (ii) of Proposition 5 give a P-stability datum with stable objects the semistable vector bundles of rank r and degree d. Note that only passive stability conditions take part. Using Serre duality, the conditions can be easily brought in the form $\hom^j(E, C_i) = N_i^j$ demanded in the definition.

Remark 7. The above condition $d > (2g-2+D)r$ on the degree of our semistable vector bundles is no restriction. By twisting the vector bundles with a line bundle of sufficiently high degree this condition is always satisfied.

2.3 Another P-stability datum for algebraic curves

As before, we consider the derived category $\mathrm{D^b}(X)$ of a smooth projective curve X. Fix integers r and d. We consider the two semistable vector bundles

$$A = L_1^{\otimes(e'-e)} \text{ and } B = \left(L_1^{\otimes -e}\right)^{\oplus(r^2+1)}$$

from the proof of Theorem 4.

Proposition 8. *For an object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:*

(i) E is a semistable sheaf of rank r and degree d.

(ii) There exists a morphism $A \xrightarrow{\psi} B$ such that $\mathrm{Hom}(\mathrm{cone}(\psi), E[i]) = 0$ for all $i \in \mathbb{Z}$ and E satisfies the following conditions

$$\hom(A, E) = \left(3g - 1 + \frac{d}{r} - \left\lfloor \frac{d}{r} \right\rfloor\right)(r^3 + r), \quad \hom(A[i], E) = 0 \textit{ for } i \neq 0,$$
$$\hom(B, E) = \left(3g - 1 + \frac{d}{r} - \left\lfloor \frac{d}{r} \right\rfloor\right)(r^3 + r), \quad \hom(B[i], E) = 0 \textit{ for } i \neq 0.$$

Proof. If $E \in \mathrm{D^b}(X)$ is a semistable vector bundle of rank r and degree d, then the slope of E is bigger than $\mu(A) + 2g - 2$. Thus by Serre duality, we have $\mathrm{ext}^1(A, E) = \hom(E, A \otimes \omega_X) = 0$. Analogously we see that $\mathrm{ext}^1(B, E) = 0$. Therefore we can compute the dimensions of $\mathrm{Hom}(A, E)$ and $\mathrm{Hom}(B, E)$ using the Riemann-Roch theorem. The existence of a map $\psi \in \mathrm{Hom}(A, B)$ with the property $\mathrm{Hom}(\mathrm{cone}(\psi), E) = 0$ is shown in the proof of the implication (ii') $\implies$ (iii) in Theorem 4. Since both E and $\mathrm{cone}(\psi)$ are sheaves and we also have $\chi(\mathrm{cone}(\psi), E) = 0$, the vanishing $\mathrm{Hom}^*(\mathrm{cone}(\psi), E) = 0$ follows.

Now suppose that $E \in \mathrm{D^b}(X)$ fulfills condition (ii). It follows that $\psi \in \mathrm{Hom}(A, B)$ is not trivial. Since A is a line bundle ψ must be injective. Thus, the cone of ψ is just the cokernel F of $\psi\colon A \to B$. As X is a smooth curve, $\mathrm{D^b}(X)$ is of global dimension 1, i.e. formal (see Proposition 6.3 in[6]) and thus E is isomorphic to its cohomology: $E = \oplus_{i\in\mathbb{Z}} E_i[-i]$ with all E_i coherent sheaves. Since E is orthogonal to F, all the E_i are orthogonal to F and hence are semistable vector bundles of slope $\frac{d}{r}$ by Theorem 4. However, as in the proof of (i) $\implies$ (ii), $E_i \neq 0$ forces $\hom(A, E_i)$ to be positive. So we eventually conclude $E_i = 0$ for all $i \neq 0$. □

Remark 9. The condition (ii) in Proposition 8 gives a second P-stability datum on an algebraic curve. Here we have the additional feature that any stable object E defines a divisor Θ_E by

$$\Theta_E = \{\psi \in \mathrm{Hom}(A, B) \mid \mathrm{Hom}^*(\mathrm{cone}(\psi), E) \neq 0\}\,.$$

This divisor is invariant under the standard k^*-action on $\mathrm{Hom}(A, B)$. Thus, we obtain the Θ-divisor $\Theta_E \subset \mathbb{P}(\mathrm{Hom}(A, B)^\vee)$. A straightforward computation shows that $\deg(\Theta_E) = (3g - 1 + \frac{d}{r} - \lceil\frac{d}{r}\rceil)(r^3 + r)$.

The assignment $E \mapsto \Theta_E$ allows an identification of stable objects with points in some projective space (namely the linear system of the Θ-divisors). P-equivalence of stable objects can be defined by $E \sim_P E'$, if and only if $\Theta_E = \Theta_{E'}$. It turns out that in this case P-equivalence classes coincide with S-equivalence classes.

2.4 Preservation of semistability on an elliptic curve

The moduli space of torsion sheaves of length r

Now let X be an elliptic curve with a fixed point $P \in X(k)$ and fix a positive integer r. We propose to consider semistable vector bundles of rank r and degree zero on X.

In order to do so, we first consider the following P-stability datum: an object $t \in \mathrm{D^b}(X)$ is P-stable, if and only if there exists a morphism $\alpha\colon \mathcal{O}_X(-3P) \to \mathcal{O}_X$ with $\hom(t[j], \mathrm{cone}(\alpha)) = 0$ for all $j \in \mathbb{Z}$ and such that for all $i \neq 0$ holds

$$\begin{aligned} \hom(\mathcal{O}_X(-3P), t) &= r, & \hom(\mathcal{O}_X, t) &= r, \\ \hom(\mathcal{O}_X(-3P), t[i]) &= 0, & \hom(\mathcal{O}_X, t[i]) &= 0. \end{aligned}$$

Obviously, we have $t \in \mathrm{D^b}(X)$ is P-stable $\Leftrightarrow$ t is a torsion sheaf of length r. The Θ-divisor associated to t is $\Theta_t := \{\alpha \in \mathbb{P}(H^0(\mathcal{O}_X(3P))^\vee) \mid \mathrm{Hom}(t, \mathrm{cone}(\alpha)) \neq 0\}$. It is a union of r lines, one for every point in the support of t (counted with multiplicities). Two torsion sheaves t and t' are P-equivalent if their Θ-divisors Θ_t and $\Theta_{t'}$ coincide. If the Θ-divisor is reduced, the P-equivalence class contains only isomorphic objects. However, the maximal number of isomorphism classes in a P-equivalence class is the number of partitions of r. Grothendieck's Hilbert scheme $\mathrm{Hilb}^r(X)$ (see [7]) of length r torsion quotients of $\mathcal{O}_X$ is the moduli space parameterizing the equivalence classes.

The Fourier-Mukai transform associated to the Poincaré bundle

We consider the product $X \times X$ with projections pr_1 and pr_2. Let $\Delta \subset X \times X$ be the diagonal, and $\mathcal{P} := \mathcal{O}_{X\times X}(\Delta) \otimes \mathrm{pr}_1^*\mathcal{O}_X(-P) \otimes \mathrm{pr}_2^*\mathcal{O}_X(-P)$ the Poincaré line bundle. We consider the Fourier-Mukai transform (recall that pr_{2*} is the derived push-forward)

$$\mathsf{FM}_{\mathcal{P}} \colon \mathrm{D^b}(X) \to \mathrm{D^b}(X) \qquad t \mapsto \mathrm{pr}_{2*}(\mathcal{P} \otimes \mathrm{pr}_1^* t).$$

We set $M_1 := \mathsf{FM}_{\mathcal{P}}(\mathcal{O}_X(-3P))$, and $M_0 := \mathsf{FM}_{\mathcal{P}}(\mathcal{O}_X)$. The complex M_1 is a sheaf shifted by $[-1]$, with $M_1[1]$ being locally free and $\mathrm{rk}(M_1[1]) = 3$ and $\deg(M_1[1]) = 1$. The complex M_0 is a shifted skyscraper sheaf: $M_0[1] = k(P)$.

A P-stability datum for rank r bundles of degree zero

Let $t \in \mathrm{D^b}(X)$ be P-stable with respect to the above P-datum. Then $e := \mathsf{FM}_{\mathcal{P}}(t)$ is P-stable with respect to the following P-datum:

$$\begin{gathered} \hom(M_j, e) = r, \qquad \hom(M_j, e[i]) = 0 \quad \text{for } j \in \{0,1\},\ i \neq 0, \\ \text{and there is } \alpha \colon M_1 \to M_0 \text{ with } \hom(\mathrm{cone}(\alpha), e[i]) = 0\ \forall i. \end{gathered}$$

We note that for nonzero $\alpha \in \mathrm{Hom}(M_1, M_0)$, the cone of α is a shifted semistable vector bundle F of rank 3 and degree zero: $F = \mathrm{cone}(\alpha)[1]$. Thus, as in Theorem 4 we have an orthogonal vector bundle F to any semistable e. This way we obtain a P-stability datum for rank r bundles of degree zero. Note that P-equivalence corresponds to S-equivalence of semistable vector bundles (see also Tu's article [21]). This allows a new proof of Atiyah's classification of vector bundles using the Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}}$. For more details see §14 in Polishchuk's book [19] and also [10].

3 Example: surfaces

For smooth, projective surfaces, we give a comparison theorem between μ-semistability and P-stability. A similar result is expected to hold in any dimension.

Theorem 10. *Let X be a smooth projective surface and H a very ample divisor on X. Given a Hilbert polynomial p, there is a P-stability datum $(C_\bullet, N)$ such that for any object $E \in \mathrm{D^b}(X)$ the following conditions are equivalent:*

(i) E is a μ-semistable vector bundle with respect to H of Hilbert polynomial p

(ii) E is P-stable with respect to $(C_\bullet, N)$.

Proof. Suppose that E is a μ-semistable vector bundle with given Hilbert polynomial p. As semistability implied that E appears in a bounded family, there is an integer m_0 (depending only on p) such that $E(m_0)$ is -2-regular (in the sense of Mumford, see [17]). In particular, we have

$$\begin{aligned} &H^i(E(m_0 + k)) = 0 \text{ for all } i > 0, k \geq -2, \\ &E(m_0 + k) \text{ is globally generated for } k \geq -2. \end{aligned}$$

Passing to the twist by $\mathcal{O}_X(m_0)$, we may assume that E itself is -2-regular.

Sheaf conditions. By Theorem 18, there are sheaves C_{-1}, C_{-2}, C_{-3} and integers $N_i^j := \hom^j(C_i, E)$ (for $i = -1, -2, -3$) such that any complex $a \in \mathrm{D^b}(X)$ with $\hom^j(C_i, a) = N_i^j$ is actually a -2-regular sheaf with Hilbert polynomial p. Thus, the first part of the P-datum consists of these three objects C_{-1}, C_{-2}, C_{-3}.

Torsion freeness. To avoid torsion in a we will construct sheaves C_{-4} and C_{-5} and add conditions of type $\hom^j(C_i, a) = N_i^j$ to the P-datum. We need two facts:

(1) For a semistable vector bundle E of given numerical invariants there is an integer m_1 such that $H^0(E(k)) = H^1(E(k)) = 0$ for all $k \le m_1$.

(2) If a is a sheaf on X with $H^0(a(k)) = H^0(a(k-1)) = H^1(a(k)) = H^1(a(k-1)) = 0$ for some k, then $H^0(a(k')) = H^1(a(k')) = 0$ for all $k' \le k$. (This assumes that $\mathcal{O}_X(1) = \mathcal{O}_X(H)$ is very ample and that H is general for a, i.e. does not contain the associated points of the sheaf a.)

Proof of (1): Along with E, the bundle $E^\vee \otimes \omega_X$ is semistable with certain prescribed numerics. Hence there exists m_1' with $H^l(E^\vee \otimes \omega_X(k')) = 0$ for $l = 1, 2$ and $k' \ge m_1'$. Then the statement follows from Serre duality.

Proof of (2): We use the hyperplane section sequence $0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$. Tensoring this with a and using dimensional induction shows the claim.

Using the constant m_1 from (1) above, we require

$$\begin{aligned} &h^l(a(m_1 - k)) = h^l(E(m_1 - k)) = 0, && k = 0, 1 \text{ and } l = 0, 1 \\ &h^2(a(m_1 - k)) = p(m_1 - k), && k = 0, 1. \end{aligned}$$

The second fact then implies $H^l(a(k)) = 0$ for all $k \le m_1$ and $l = 0, 1$. This in turn forces a to be torsion free. If not, consider the torsion exact sequence $0 \to T \to a \to a/T \to 0$. Then $h^0(a(k)) = 0$ implies $h^0(T(k)) = 0$, hence T is purely one-dimensional. Next we have $H^1(T(k)) \cong H^0(a/T(k))$ for all $k \le m_1$. Together with $H^0(T(k)) = 0$, this shows that the polynomial $h^0(a/T(-k))$ is eventually monotonously increasing. This is absurd for a coherent sheaf on a projective variety — contradiction.

Local freeness. The sheaf a is now automatically a vector bundle. If not, we can consider the short exact sequence $0 \to a \to a^{\vee\vee} \to Q \to 0$. As we know a to be torsion free, $a^{\vee\vee}$ is locally free and Q a sheaf of dimension 0. Hence $h^0(Q(k)) = \mathrm{length}(Q)$ for all k. But then $h^0(a^{\vee\vee}(k)) = \mathrm{length}(Q)$ for all $k \le m_1$, hence $\mathrm{length}(Q) = 0$ by construction.

Thus setting $C_{-4} := \mathcal{O}_X(1 - m_1)$ and $C_{-5} := \mathcal{O}_X(-m_1)$ we can force our objects to be vector bundles with the given numeric invariants.

Semistability. Using Langer's effective theorem (Theorem 5.2 in [15]), there is a constant m_2 (again depending only on the numerics of E) such that for all smooth curves $\widetilde{H} \in |m_2 H|$ the restriction $E|_{\widetilde{H}}$ is semistable. By stability on curves as in Theorem 4, there is a bundle F on $\widetilde{H}$ with $\mathrm{Hom}(E|_{\widetilde{H}}, F) = \mathrm{Ext}^1(E|_{\widetilde{H}}, F) = 0$, i.e. $F \in (E|_{\widetilde{H}})^\perp$ in $\mathrm{D^b}(\widetilde{H})$. By symmetry, F is also semistable on $\widetilde{H}$. Hence there is a short exact sequence of sheaves on X

$$0 \longrightarrow M \xrightarrow{\alpha} \mathcal{O}_{\widetilde{H}}^{\oplus r^2+1}(-m_3) \longrightarrow F \longrightarrow 0$$

i.e. $F = \mathrm{cone}(\alpha)$ is a torsion sheaf on X.

On the other hand, if $F \in (E|_{\widetilde{H}})^{\perp}$ in $\mathrm{D^b}(X)$, then $E|_{\widetilde{H}}$ is semistable, hence E is semistable with respect to the polarization $\widetilde{H}$. Thus, once we have a map $\alpha \in \mathrm{Hom}(L, \mathcal{O}_{\widetilde{H}}^{\oplus r^2+1}(-m_3))$ with $a \in \mathrm{cone}(\alpha)^{\perp}$ the sheaf a will be semistable. By setting $C_1 := M$ and $C_0 := \mathcal{O}_{\widetilde{H}}^{\oplus r^2+1}(-m_3)$ we thus complete our P-datum $(C_{-5}, \ldots, C_1, N)$. $\square$

Proposition 11. *For every $h_1 \in H^2(X, \mathbb{Z})$ and each natural number $n \in \mathbb{N}$ there exists a Postnikov-datum (C_i, N_i^j) such that for any $a \in \mathrm{D^b}(X)$ we have the equivalence*

$$\left(\begin{array}{c} \hom^j(C_i, a) = N_i^j \\ \textit{for all } j \in \mathbb{Z} \textit{ and all } i \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} a \cong A \in \mathrm{Coh}(X) \textit{ with } A \cong L \otimes \mathcal{J}_Z \\ \textit{where } L \textit{ is a line bundle with } c_1(L) = h_1 \\ \textit{and } \mathcal{J}_Z \textit{ is an ideal sheaf of colength } n. \end{array} \right)$$

Proof. This follows the line of the proof of Theorem 10. Thus, we may assume the P-datum forces a to be isomorphic to a sheaf A with the same numerical invariants as $L \otimes \mathcal{J}_Z$. We want to add conditions to our P-datum which imply that A is torsion free of the stated type. We take an integer m such that for all $k \leq m$ and all line bundles L with $c_1(L) = h_1$ we have $h^0(L(k)) = 0 = h^1(L(k))$. If we have $h^0(A(m-n-1)) = 0 = h^0(A(m))$ and $h^1(A(m-n-1)) = n = h^1(A(m))$, then it immediately follows that the torsion subsheaf $T \subset A$ is purely one-dimensional. Let $\widetilde{H}$ be a smooth divisor in $|(n+1)H|$ such that

$$0 \longrightarrow T \longrightarrow A \longrightarrow A/T \longrightarrow 0$$

remains exact when restricted to $\widetilde{H}$. The sheaf $T(m) \otimes \mathcal{O}_{\widetilde{H}}$ is of finite length $l = c_1(T).\widetilde{H} = (n+1)c_1(T).H$. Consequently, we have $T \neq 0$ implies that $l > n$. From the long exact cohomology sequence

$$0 = H^0(A(m)) \longrightarrow H^0(A(m) \otimes \mathcal{O}_{\widetilde{H}}) \longrightarrow H^1(A(m-n-1)) \cong k^n$$

we deduce $h^0(A(m) \otimes \mathcal{O}_{\widetilde{H}}) \leq n$. Since $H^0(T(m) \otimes \mathcal{O}_{\widetilde{H}}) \subset H^0(A(m) \otimes \mathcal{O}_{\widetilde{H}})$ we obtain $T = 0$. Thus, adding the above conditions forces A to be torsion free. $\square$

4 Derived compactification of moduli spaces

Let $X = \mathbb{P}^1 \times C$ be the product of $\mathbb{P}^1$ with an elliptic curve C with morphisms

$$\mathbb{P}^1 \xleftarrow{p} X \xrightarrow{q} C\,.$$

We denote a fiber of p and q by f_p and f_q respectively. Denoting the class of a point by z we have generators for the even cohomology classes

$$\begin{array}{rcccl} H^0(X,\mathbb{Z}) & = & \mathbb{Z} & = & \mathbb{Z}\langle [X] \rangle \\ H^2(X,\mathbb{Z}) & = & \mathbb{Z}^2 & = & \mathbb{Z}\langle f_q, f_p \rangle \\ H^4(X,\mathbb{Z}) & = & \mathbb{Z} & = & \mathbb{Z}\langle z \rangle\,. \end{array}$$

We take the polarization $H = f_q + 3f_p$ on X.

The moduli space $\mathcal{M}_1$. Let E be a coherent sheaf of Chern character $\mathrm{ch}(E) = 1 + 2f_q - 2z$. If E is torsion free, then we have an isomorphism $E \cong q^*L \otimes \mathcal{J}_Z$ where L is a line bundle of degree 2 on C, and Z is a closed subscheme of length 2. We obtain by straightforward computations

$$\mathrm{Hom}(E,E) = \mathbb{C}, \qquad \mathrm{Ext}^1(E,E) = \mathbb{C}^5, \qquad \mathrm{Ext}^2(E,E) = 0, \qquad \chi(E(k)) = k^2 + 7k\,.$$

Thus, the moduli space $\mathcal{M}_1$ of torsion free coherent sheaves of Chern character $\mathrm{ch}(E) = 1 + 2f_q - 2z$ is a smooth projective variety of dimension 5. Indeed, we have

$$\mathrm{Pic}^2(C) \times \mathrm{Hilb}^2(X) \xrightarrow{\sim} \mathcal{M}_1, \quad (L, Z) \mapsto q^*L \otimes \mathcal{J}_Z.$$

The relative Fourier-Mukai transform. Choosing a base point $c \in C$ we can identify C with its Picard scheme $\mathrm{Pic}^0(C)$ and, as in Section 2.4 obtain a Poincare line bundle $\mathcal{P}$ on $C \times C$ subject to the conditions $\mathcal{P}|_{\{c\}\times C} \cong \mathcal{O}_C$ and $\mathcal{P}|_{C\times\{c\}} \cong \mathcal{O}_C$. From the diagram $C \xleftarrow{\pi_1} C \times C \xrightarrow{\pi_2} C$ and the Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(C) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(C)$ with $\mathsf{FM}_{\mathcal{P}}(a) = \pi_{2*}(\mathcal{P} \otimes \pi_1^*a)$ we obtain the diagram $X \xleftarrow{\pi_{12}} \mathbb{P}^1 \times C \times C \xrightarrow{\pi_{13}} X$, the line bundle $\mathcal{P}_X = \pi_{23}^*\mathcal{P}$ on $\mathbb{P}^1 \times C \times C$, and the Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}_X} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$ which is defined by $\mathsf{FM}_{\mathcal{P}_X}(a) := \pi_{13*}(\mathcal{P}_X \otimes \pi_{12}^*a)$.

Next we study $\mathsf{FM}_{\mathcal{P}_X}$ on objects parameterized by our moduli space $\mathcal{M}_1$. We remark that any object E parameterized by $\mathcal{M}_1$ is the kernel of a surjection $q^*L \to \mathcal{O}_Z$ where L is a degree two line bundle on C and Z is a length two subscheme of X. The Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}}(L)$ is a stable vector bundle of rank two and degree -1 on the curve C (see Chapter 14 in [19]). Thus, $E_L := \mathsf{FM}_{\mathcal{P}_X}(q^*L) = q^*\mathsf{FM}_{\mathcal{P}}(L)$ is the pullback of a vector bundle from C. Suppose a line bundle $M \cong \mathcal{O}_X(n_pf_p + n_qf_q)$ is contained in E_L. Since E_L is trivial on the fibers of q we find $n_p \leq 0$. The stability of $\mathsf{FM}_{\mathcal{P}}(L)$ yields $n_q \leq -1$. Thus, we have $c_1(M).H \leq -3$.

The Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}_X}(\mathcal{O}_Z)$ of the torsion sheaf $\mathcal{O}_Z$ is a sheaf with graded object $T_1 \oplus T_2$ where the sheaves T_i are line bundles of degree zero on fibers of p extended to X. We distinguish two cases:

Case 1: $\mathcal{O}_Z$ is not contained in a fiber of p. Since any morphism of $\mathsf{FM}_{\mathcal{P}}(L)$ to a line bundle of degree zero is surjective, applying $\mathsf{FM}_{\mathcal{P}_X}$ to the short exact sequence

$$0 \to E \to q^*L \to \mathcal{O}_Z \to 0$$

remains a short exact sequence of sheaves. Thus $\mathsf{FM}_{\mathcal{P}_X}(E)$ is the kernel of the surjective sheaf morphism $E_L \to \mathsf{FM}_{\mathcal{P}_X}(\mathcal{O}_Z)$. Therefore $\mathsf{FM}_{\mathcal{P}_X}(E)$ is a rank two vector bundle with $c_1(\mathsf{FM}_{\mathcal{P}_X}(E)) = -f_q - 2f_p$, and $c_2(\mathsf{FM}_{\mathcal{P}_X}(E)) = 2z$. Consequently, $c_1(\mathsf{FM}_{\mathcal{P}_X}(E)).H = -5$. Since any line bundle M which is contained in $\mathsf{FM}_{\mathcal{P}_X}(E)$ is contained in E_L this yields together with

$$\frac{c_1(\mathsf{FM}_{\mathcal{P}_X}(E)).H}{2} = \frac{-5}{2} > -3 \geq c_1(M).H$$

the stability of $\mathsf{FM}_{\mathcal{P}_X}(E)$.

Case 2: $\mathcal{O}_Z$ is contained in a fiber $p^{-1}(x)$. In this case the morphism $E_L \to \mathsf{FM}_{\mathcal{P}_X}(\mathcal{O}_Z)$ cannot be surjective. Its cokernel is a sheaf of length one concentrated on a point of $p^{-1}(x)$. The kernel is a rank two vector bundle $h^0(\mathsf{FM}_{\mathcal{P}_X}(E))$ with numerical invariants $c_1(h^0(\mathsf{FM}_{\mathcal{P}_X}(E))) = -f_q - 2f_p$, and $c_2(h^0(\mathsf{FM}_{\mathcal{P}_X}(E))) = z$.

The moduli space $\mathcal{M}_2$. The second moduli space on X we want to consider is the moduli space $\mathcal{M}_2 := \mathcal{M}_{X,H}(2, -f_q - 2f_p, 3)$ of stable vector bundles F on X with

$$\mathrm{rk}(F) = 2, \qquad c_1(F) = -f_q - 2f_p, \qquad c_2(F) = 2z\,.$$

Suppose now that $[F] \in \mathcal{M}_2$. We consider the Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}_X}(F)$. By construction, the cohomology of $\mathsf{FM}_{\mathcal{P}_X}(F)$ lives only in degrees zero and one.

Lemma 12. *The Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}_X}(F)$ has only first cohomology. This means $\mathsf{FM}_{\mathcal{P}_X}(F) = h^1(\mathsf{FM}_{\mathcal{P}_X}(F))[-1]$.*

Proof. Suppose that $\mathsf{FM}_{\mathcal{P}_X}(F)$ has cohomology in degree zero, that is $h^0(\mathsf{FM}_{\mathcal{P}_X}(F)) \neq 0$. This implies that $\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X(-mH), \mathsf{FM}_{\mathcal{P}_X}(F)) \neq 0$ for $m \gg 0$. Therefore, we obtain

$$\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathsf{FM}_{\mathcal{P}_X}(\mathcal{O}_X(-mH)), \mathsf{FM}_{\mathcal{P}_X}\mathsf{FM}_{\mathcal{P}_X}(F)) \neq 0\,.$$

We write $\iota\colon X \xrightarrow{\sim} X$ for the involution coming from the inversion in the group law of C. Since $\mathsf{FM}_{\mathcal{P}_X}\mathsf{FM}_{\mathcal{P}_X}(F) = \iota^* F[-1]$, and the cohomology of $\mathsf{FM}_{\mathcal{P}_X}(\mathcal{O}_X(-mH))$ is exclusively in degree one, we obtain a nontrivial homomorphism $\psi\colon \iota^* h^1(\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_X(-mH))) \to F$. The restriction of $\iota^* h^1(\mathsf{FM}_{\mathcal{P}}(\mathcal{O}_X(-mH)))$ to any fiber of p is a stable vector bundle of rank m and degree one. Thus the restriction of ψ to any fiber is not surjective. Therefore, the image of ψ is of rank one. Let L be the saturation of $\mathrm{im}(\psi)$ in F. We have $L = \mathcal{O}_X(n_p f_p + n_q f_q)$. Since L contains the image of ψ we have $n_q \geq 1$. The stability of F yields $n_p \leq -3 - 3n_q$. We have a short exact sequence

$$0 \to L \to F \to \det(F) \otimes L^{-1} \otimes \mathcal{J}_Z \to 0$$

where $\mathcal{J}_Z$ is the ideal sheaf of a subscheme Z of finite length

$$\mathrm{length}(Z) = c_2(F) - c_1(L).c_1(\det(F) \otimes L^{-1}) = 2 + 2n_q + n_p + 2n_p n_q\,.$$

However, the inequalities for n_p and n_q force $\mathrm{length}(Z)$ to be negative which is impossible. □

Lemma 13. *The sheaf $h^1(\mathsf{FM}_{\mathcal{P}_X}(F))$ is torsion free.*

Proof. We consider the torsion subsheaf $T(h^1(\mathsf{FM}_{\mathcal{P}_X}(F)))$. It contains a subsheaf T which is of rank one on its support and with $\mathrm{supp}(T)$ is irreducible. If the support of T is zero-dimensional, then the morphism $T[-1] \to \mathsf{FM}_{\mathcal{P}_X}(F)$ defines via the Fourier-Mukai transform a morphism from the torsion sheaf $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T)$ to F which is impossible because F is torsion free.

Thus, we may assume that $Y = \operatorname{supp}(T)$ is of dimension one. If the induced morphism $p|_Y \colon Y \to \mathbb{P}^1$ dominates $\mathbb{P}^1$, then the the restriction of $\mathsf{FM}_{\mathcal{P}_X}(T)$ to the fibers of p is a semistable vector bundle of degree zero. Hence we obtain a morphism $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T) \to F$ which has image of rank one and conclude like in the proof of Lemma 12.

Thus, we have to discuss only the case when Y is a fiber of p, and may assume that T is torsion free on Y. We distinguish three cases depending on the degree $\deg_Y(T)$ of T on Y. In all these cases we investigate the resulting morphism $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T) \to F$.

Case $\deg_Y(T) > 0$: Here $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T)$ is a torsion sheaf on Y which is impossible because F is torsion free.

Case $\deg_Y(T) = 0$: Here $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T) = k(y)[-1]$ for a point $y \in Y$. However, for a vector bundle F we have $\operatorname{Hom}_{\mathrm{D^b}(X)}(k(y)[-1], F) = \operatorname{Ext}^1(k(y), F) = H^0(\mathcal{E}xt^1(k(y), F)) = 0$.

Case $\deg_Y(T) < 0$: The Fourier-Mukai transform $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T)$ is a stable vector bundle E_T on Y of degree one shifted by $[-1]$. The existence of a nontrivial homomorphism $\iota^*\mathsf{FM}_{\mathcal{P}_X}(T) \to F$ is equivalent to $\operatorname{Hom}(E_T, F|_Y) \neq 0$. The image of the morphism $E_T \to F|_Y$ is a line bundle on Y of positive degree. Thus, we have a surjection $F|_Y \to L$ where L is a line bundle on Y of degree $d \leq -2$. Denoting the kernel of the composition morphism $F \to F|_Y \to L$ by F', we obtain a vector bundle with invariants

$$\operatorname{rk}(F') = 2,\ c_1(F') = -f_q - 3f_p,\ c_2(F') = 3 + d,\ \Delta(F') = c_1^2(F') - 4c_2(F') = -4d - 6\,.$$

By Bogomolov's inequality (cf. Thm. 3.4.1 in [12]), F' is Bogomolov unstable. Hence there exists a destabilizing exact sequence

$$0 \to M \to F' \to \det(F') \otimes M^{-1} \otimes \mathcal{J}_Z \to 0$$

with Z of dimension zero, $(2c_1(M) - \det(F')).H > 0$, and $(2c_1(M) - \det(F'))^2 > 0$. Writing $c_1(M) = n_p f_p + n_q f_q$, we obtain the two inequalities $2n_p + 6n_q + 6 > 0$, and $(2n_p + 3)(2n_q + 1) > 0$. The first implies that not both factors in the second can be negative. Thus, $n_p \geq -1$ and $n_q \geq 0$. Now M is a subsheaf of F, too. Thus, we have $(2c_1(M) - c_1(F)).H \leq 0$ which reads $2n_p + 3n_q + 5 \leq 0$, a contradiction. $\square$

Putting together our results we have obtained the

Corollary 14. *The Fourier-Mukai transform identifies the open subset U of $\mathcal{M}_1$ which parameterizes twisted ideal sheaves of two points in different fibers of $p \colon X \to \mathbb{P}^1$ with the moduli space $\mathcal{M}_2$.*

Two compactifications of the moduli space $\mathcal{M}_2$. At this point it seems natural to compactify $\mathcal{M}_2$ by adding the objects $\mathsf{FM}_{\mathcal{P}_X}(E)$ with $[E] \in \mathcal{M}_1 \setminus U$. This way we have an isomorphism $\mathsf{FM}_{\mathcal{P}_X} \colon \mathcal{M}_1 \to \overline{\mathcal{M}}_2$. Since the dimensions of $\operatorname{Ext}^i(a, a)$ are invariant under $\mathsf{FM}_{\mathcal{P}_X}$, we obtain a smooth moduli space $\overline{\mathcal{M}}_2$ from the smooth moduli space $\mathcal{M}_1$. We are compactifying with simple objects having two cohomology sheaves.

The classical construction of moduli spaces compactifies $\mathcal{M}_2$ with coherent sheaves E with one singular point. That is, the morphism $E \to E^{\vee\vee}$ has cokernel of length one. The Fourier-Mukai transform $\mathsf{FM}_{\mathcal{P}_X}$ of these objects does not yield torsion free sheaves. This can be seen be applying $\mathsf{FM}_{\mathcal{P}_X}$ to the distinguished triangle containing the morphism $E \to E^{\vee\vee}$. We hope this illustrates that the compactification $\overline{\mathcal{M}}_2$ by derived objects is natural and important.

5 Sheaf conditions for objects in $\mathrm{D^b}(X)$

Notation: Let X be a projective variety of dimension n with a very ample polarization $\mathcal{O}_X(1)$. For an object $a \in \mathrm{D^b}(X)$, we denote the ith cohomology of the complex a by $a^i := h^i(a)$. The object a can be represented by a sheaf concentrated in zero if and only if $a^i = 0$ for all integers $i \neq 0$. Abbreviating, we call such an object a a sheaf in $\mathrm{D^b}(X)$. For an object $a \in \mathrm{D^b}(X)$, we define the cohomology group $H^i(a(k))$ to be the vector space $\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X(-k)[-i], a)$.

To compute the cohomology groups $H^i(a)$, we use the Eilenberg-Moore cohomology spectral sequence (see Example 2.70 (i) in [11]):

$$E_2^{pq} = \mathrm{Ext}^p(\mathcal{O}_X(-k), a^q) = H^p(a^q(k)) \Rightarrow H^{p+q}(a(k)) .$$

If the dimension of X is zero this spectral sequence degenerates and we conclude the

Lemma 15. *If X is of dimension zero, then $a \in \mathrm{D^b}(X)$ is a sheaf if and only if $H^i(a) = 0$ for all $i \neq 0$.* $\square$

Lemma 16. *Let X be of dimension one, $p\colon \mathbb{Z} \to \mathbb{Z}$ be a polynomial of degree less than two. If $a \in \mathrm{D^b}(X)$ is an object satisfying*

(i) $H^i(a(k)) = 0$ *for all pairs* (i,k) *with* $k \in \{-1, 0, p(-1)\}$, *and* $i \neq 0$,
(ii) $\dim(H^0(a(k))) = p(k)$ *for* $k \in \{-1, 0\}$,

then a is a sheaf of Hilbert polynomial p.

Proof. Put $m := p(-1)$. By choosing a two-dimensional vector subspace $V \subset H^0(\mathcal{O}_X(1))$ such that $V \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$ is surjective, we get a morphism $V \otimes H^0(a(-1)) \to H^0(a)$. The object $S := S^{m-1}(V, \mathcal{O}_X, \mathcal{O}_X(1))$ obtained in Construction 6.1 has, by Lemma 23, the following property: the resulting morphism $\varrho_v\colon H^0(a(-1)) \to H^0(a)$ is injective for a general $v \in V$ if and only if $\mathrm{Hom}(S, a[-1]) = 0$. As noted in Remark 24, S is a vector bundle of rank one and determinant $\mathcal{O}_X(-m)$, i.e. $S \cong \mathcal{O}_X(-m)$. Thus, by assumption (i) we get $0 = \mathrm{Hom}(S, a[-1]) = H^{-1}(a(m))$. Now consider a general divisor H in the linear system $\mathbb{P}(V^\vee)$ giving a distinguished triangle

$$a(-1) \xrightarrow{v} a \to a \otimes \mathcal{O}_H \to a(-1)[1] .$$

By general divisor we mean: That the morphism $H^0(a(-1)) \to H^0(a)$ is injective and that $-\otimes\mathcal{O}_H$ commutes with the cohomology of the complex, i.e. $h^i(a\otimes\mathcal{O}_H) = h^i(a)\otimes\mathcal{O}_H$. We derive that $a \otimes \mathcal{O}_H$ fulfills the assumption of Lemma 15 and is a sheaf. Thus for $i \neq 0$ the cohomology sheaves a^i are skyscraper sheaves concentrated in points outside H. Hence, the above spectral sequence fulfills $E_2^{pq} = 0$ unless p or q are zero. Since $H^i(a) = 0$ for all $i < 0$, we have $a^i = 0$ for all $i < 0$. We consider now the distinguished triangle corresponding to the natural t-structure

$$\tau_{\leq 0} a \to a \to \tau_{\geq 1} a \to \tau_{\leq 0} a[1] .$$

We have that $\tau_{\leq 0}a$ is a sheaf, which implies $H^i(\tau_{\leq 0}a) = 0$ for $i \notin \{0,1\}$. By assumption (i) we have $H^i(a) = 0$ for all $i \neq 0$. Thus, applying the functor $\mathrm{Hom}(\mathcal{O}_X, \_\,)$ to the filtration triangle yields $H^i(\tau_{\geq 1}a) = 0$ for all integers $i \geq 1$. Since all cohomology sheaves a^i of the complex $\tau_{\geq 1}a$ are zero-dimensional, we deduce again from the Eilenberg-Moore spectral sequence that $H^0(a^i) = 0$ for $i > 0$. Therefore we have $a^i = 0$ for all $i > 0$ which implies $\tau_{\geq 1}a = 0$. So a is isomorphic to the sheaf $\tau_{\leq 0}a$. □

Lemma 17. *Let X be of dimension two, $V \subset H^0(\mathcal{O}_X(1))$ a subspace such that the evaluation morphism $V \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$ is surjective, and $p\colon \mathbb{Z} \to \mathbb{Z}$ be a polynomial of degree less than three. Its derivative is the polynomial $p'(k) := p(k) - p(k-1)$. Put $m := (\dim(V) - 1)(\max\{p(0), p(-1), p'(-1)\} - 1)$. Any object $a \in \mathrm{D}^{\mathrm{b}}(X)$ with*

(i)	$H^i(a(k)) = 0$ *for all pairs* (i,k) *with* $k \in \{-2,-1,0\}$, *and* $i \neq 0$,
(ii)	$\dim(H^0(a(k))) = p(k)$ *for* $k \in \{-2,-1,0\}$,
(iii$_1$)	$\mathrm{Hom}_{\mathrm{D^b}(X)}(S^m(V, \mathcal{O}_X, \mathcal{O}_X(1)), a[i]) = 0$ *for all* $i \neq 0$,
(iii$_2$)	$\mathrm{Hom}_{\mathrm{D^b}(X)}(S^m(V, \mathcal{O}_X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X(1), a[i]) = 0$ *for all* $i \neq 0$,
(iii$_3$)	$\mathrm{Hom}_{\mathrm{D^b}(X)}(S^m(V, \mathcal{O}_X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X(-p'(-1)), a[i]) = 0$ *for all* $i \neq 0$,
(iii$_4$)	$\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X(-p'(-1)), a[i]) = 0$ *for all* $i \neq 0$, *and*
(iii$_5$)	$\mathrm{Hom}_{\mathrm{D^b}(X)}(\mathcal{O}_X(1 - p'(-1)), a[i]) = 0$ *for all* $i \neq 0$

is a sheaf of Hilbert polynomial p.

Proof. For $v \in V$ we have a short exact sequence $0 \to \mathcal{O}_X(-1) \xrightarrow{v} \mathcal{O}_X \to \mathcal{O}_{H_v} \to 0$. We obtain long exact cohomology sequences

$$H^{-1}(a(k) \otimes \mathcal{O}_{H_v}) \to H^0(a(k-1)) \xrightarrow{H(v)} H^0(a(k)) \to H^0(a(k) \otimes \mathcal{O}_{H_v}) \to H^1(a(k-1)).$$

We will restrict ourselves to those $v \in V$ which are general in the sense that they commute with forming cohomology, that is

$$h^i(a \otimes \mathcal{O}_{H_v}) = h^i(a) \otimes \mathcal{O}_{H_v} \qquad \text{and} \qquad a \otimes \mathcal{O}_{H_v} = a \overset{\mathrm{L}}{\otimes} \mathcal{O}_{H_v}\,.$$

By assumption (i) and (iii$_4$) the left-most arrow is injective for $k \in \{-1, 0, p'(-1)\}$. For these values of k the right-most term vanish by assumptions (i) and (iii$_5$). We deduce from Lemma 23 and (iii$_3$) the injectivity of $H(v)$ for $k = 0$ and $v \in V$ general. Analogously it follows from (iii$_4$) and (iii$_5$) that for a general v the morphism $H(v)$ is injective for $k = -1$ and $k = p'(-1)$. Thus, for a general $v \in V$ the tensor product $a \otimes \mathcal{O}_{H_v}$ fulfills the requirements (i) and (ii) of Lemma 16 with p replaced by p'. From now on we suppose that $v \in V$ is general in the above sense. Therefore $a \otimes \mathcal{O}_{H_v}$ is a sheaf concentrated on H_v. As in the proof of Lemma 16, we obtain that for $i \neq 0$ the cohomology sheaves a^i have supports disjoint from the ample divisor H_v. Thus, they are supported in closed points disjoint from H_v. As before, the Eilenberg-Moore spectral sequence yields $a^i = 0$ for all $i < 0$. By Mumford's regularity criterion (see chapter 14 in [17]) $a^0 \otimes \mathcal{O}_{H_v}$ is 0-regular which gives the vanishing $H^i(a^0 \otimes \mathcal{O}_{H_v}(k))$ for all $k \geq -1$ and $i > 0$. Thus, from the long exact sequence we obtain $H^2(a^0(k-1)) \cong H^2(a^0(k))$

for all $k \geq 0$. Therefore we conclude $H^2(a^0(k)) = 0$ for all $k \geq -1$. As in the proof of Lemma 16 we consider the filtration triangle

$$\tau_{\leq 0} a \to a \to \tau_{\geq 1} a \to \tau_{\leq 0} a[1]\,,$$

and remark that $\tau_{\leq 0} a = a^0$ is a sheaf. Since $H^i(\tau_{\leq 0} a(k)) = 0$ for all $i > 2$ we deduce from the associated long exact sequence $H^1(\tau_{\geq 1} a(k)) \cong H^2(\tau_{\leq 0} a(k))$ and $H^i(\tau_{\geq 1} a(k)) = 0$ for $i \geq 2$. Since $H^2(\tau_{\leq 0} a(k))$ vanishes for $k \gg 0$ and the dimension $h^1(\tau_{\geq 1} a(k)) = h^0(a^1(k))$ is independent of k, we deduce that $H^1(\tau_{\geq 1} a(k)) = 0$. Now we conclude as in the proof of Lemma 16 that $\tau_{\geq 1} a = 0$. $\square$

Theorem 18. *Let X be a projective variety of dimension $n \leq 2$ and $p\colon \mathbb{Z} \to \mathbb{Z}$ be a polynomial of degree at most n. There exists a sheaf $b \in \mathrm{D^b}(X)$ such that any object $a \in \mathrm{D^b}(X)$ with*

- *(i)* $H^i(a(k)) = 0$ *for all pairs* (i, k) *with* $k \in \{-n, \ldots, 0\}$, *and* $i \neq 0$,
- *(ii)* $\dim(H^0(a(k))) = p(k)$ *for* $k \in \{-n, \ldots, 0\}$, *and*
- *(iii)* $\mathrm{Hom}_{\mathrm{D^b}(X)}(b, a[i]) = 0$ *for all* $i \neq 0$

is a sheaf of Hilbert polynomial p. Furthermore, any sheaf $a \in \mathrm{D^b}(X)$ satisfying conditions (i) and (ii) fulfills (iii).

Proof. For $\dim(X) = 0$ we can set $b = 0$ and are done by Lemma 15.

In case $\dim(X) = 1$ we set $b = \mathcal{O}_X(-p(-1))$. Lemma 16 tells us that conditions (i)–(iii) force a to be a sheaf. To see that a sheaf $a \in \mathrm{D^b}(X)$ which satisfies (i) also satisfies (iii), we remark that (i) implies the 0-regularity of the sheaf a. Thus, condition (iii) holds because $p(-1)$ being the dimension of a vector space can not be negative.

If $\dim(X) = 2$, then we set $b = b_1 \oplus b_2 \oplus b_3 \oplus b_4 \oplus b_5$ with b_i the sheaf of condition (iii$_i$) of Lemma 17. (For example: $b_2 = S^m(V, \mathcal{O}_X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X(1)$.) Again Lemma 17 tells us that conditions (i)–(iii) for $a \in \mathrm{D^b}(X)$ imply that a is a sheaf. Suppose now that a is a sheaf fulfilling (i) and (ii). Since for a general $v \in V$ the resulting morphisms $H^0(v)\colon H^0(a(k-1)) \to H^0(a)$ are injective we obtain by Lemma 23 that conditions (iii$_1$), (iii$_2$), and (iii$_3$) of Lemma 17 hold. Again the the Mumford-Castelnuovo regularity of a yields that conditions (iii$_4$), and (iii$_5$) hold, too. $\square$

Remark 19. (1) Considered as an element of the Grothendieck group $K(X)$ the sheaf b of the above theorem is in the subgroup spanned by the elements $\mathcal{O}_X(k)$ with $k = 0, \ldots, n$. Thus, for an object a which fulfills the conditions of the theorem the dimension of $\mathrm{Hom}_{\mathrm{D^b}(X)}(b, a)$ is given.

(2) The object b depends on the Hilbert polynomial p of a. This can be seen best in Lemma 16.

(3) If $a \in \mathrm{D^b}(X)$ is a sheaf with Hilbert polynomial p, then conditions (i) and (ii) of Theorem 18 do in general not hold. However, after a suitable twist these conditions hold.

6 The Euler triangle

Lemma 20. *Let U and W be k-vector spaces of finite dimension. Suppose that the morphism $U \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\rho} W \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ is not injective. Then for any integer $m \geq (\dim(U) - 1)n$ we have $H^0(\ker(\rho)(m)) \neq 0$.*

Proof. From the morphism ρ we obtain two short exact sequences

$$0 \to \ker(\rho) \to U \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathrm{im}(\rho) \to 0, \qquad 0 \to \mathrm{im}(\rho) \to W \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to \mathrm{coker}(\rho) \to 0\,.$$

The resulting long cohomology sequences yield two inequalities for all integers k

$$\begin{array}{rcl} h^0(\ker(\rho)(k)) & \geq & h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(k)) - h^0(\mathrm{im}(\rho)(k)) \\ h^0(\mathrm{im}(\rho)(k)) & \leq & h^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(1))\,. \end{array}$$

First we assume that $\dim(W) \leq \dim(U) - 1$. This implies $h^0(\mathrm{im}(\rho)(m)) \leq (\dim(U) - 1)\binom{n+1+m}{n}$. Since $h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = \dim(U)\binom{n+m}{n}$, this yields $h^0(U \otimes \mathcal{O}_{\mathbb{P}^n}(m)) > h^0(\mathrm{im}(\rho)(m))$ for all $m \geq (\dim(U) - 1)n$. Thus, we obtain $h^0(\ker(\rho)(m)) > 0$ for $m \geq (\dim(U) - 1)n$.

Now we assume that $\dim(W) \geq \dim(U)$. The cokernel $\mathrm{coker}(\rho)$ has rank at least $\dim(W) - \dim(U) + 1$. Therefore there exists a subspace $W' \subset W$ of dimension $\dim(W) - \dim(U) + 1$ such that the resulting morphism $W' \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to \mathrm{coker}(\rho)$ is injective in the generic point, and eventually injective. Thus, the image of the injective morphism $H^0(\mathrm{im}(\rho)(k)) \to H^0(W \otimes \mathcal{O}_{\mathbb{P}^n}(k+1))$ is transversal to $H^0(W' \otimes \mathcal{O}_{\mathbb{P}^n}(k+1))$. This implies $h^0(\mathrm{im}(\rho)(m)) \leq (\dim(U) - 1)\binom{n+1+m}{n}$ as before. □

6.1 Construction: The Euler triangle and objects $S^m(V, a, b)$

For any two objects a, b of a k-linear triangulated category $\mathcal{T}$ and some subspace $V \subset \mathrm{Hom}(a, b)$ of finite dimension we get a distinguished (Euler) triangle

$$S^m(V, a, b)[-1] \longrightarrow \mathrm{Sym}^{m+1}(V) \otimes a \xrightarrow{\theta} \mathrm{Sym}^m(V) \otimes b \longrightarrow S^m(V, a, b)$$

where tensor products of vector spaces and objects are just finite direct sums, and θ is induced by the natural map

$$\mathrm{Sym}^{m+1}(V) \to \mathrm{Sym}^m(V) \otimes \mathrm{Hom}(a, b), \quad f_0 \vee \cdots \vee f_m \mapsto \sum_i (f_0 \vee \cdots \hat{f_i} \cdots \vee f_m) \otimes f_i.$$

Remark 21. In the special case where $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\mathbb{P}^n_k)$ is the bounded derived category of the projective space $\mathbb{P}^n_k$ over k and $a = \mathcal{O}_{\mathbb{P}^n}$, $b = \mathcal{O}_{\mathbb{P}^n}(1)$, $V = \mathrm{Hom}(a, b) = H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and $m = 0$, the above triangle is induced by the classical Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

Remark 22. For any $c \in \mathcal{T}$, the triangle defining $S^m(V,a,b)$ yields a long exact sequence

$$\begin{tikzcd} \ldots \to \mathrm{Hom}^{k-1}(a,c)\otimes \mathrm{Sym}^{m+1}(V^\vee) \arrow[r] & \mathrm{Hom}^k(S^m(V,a,b),c) \arrow[dl] \\ \mathrm{Hom}^k(b,c)\otimes \mathrm{Sym}^m(V^\vee) \to \arrow[r] & \mathrm{Hom}^k(a,c)\otimes \mathrm{Sym}^{m+1}(V^\vee) \to \cdots \end{tikzcd}$$

Lemma 23. *Let $\mathcal{T}$ be a triangulated k-linear category with finite-dimensional Hom's, $a, b, c \in \mathcal{T}$ objects with $\mathrm{Hom}^{-1}(a,c) = 0$ and let $V \subset \mathrm{Hom}(a,b)$ be a subspace. Then the following conditions are equivalent:*

(i) The morphism $\varrho_v\colon \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,c)$ is injective for $v \in V$ general.

(ii) $\mathrm{Hom}^0(S^m(V,a,b),c) = 0$ holds for some $m \geq (\dim(V)-1)(\hom(b,c)-1)$.

Proof. We consider the morphism $\mathrm{Hom}(b,c) \to V^\vee \otimes \mathrm{Hom}(a,c)$. Together with the natural surjection $V^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \to \mathcal{O}_{\mathbb{P}(V^\vee)}(1)$, this gives a morphism

$$\varrho\colon \mathrm{Hom}(b,c)\otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \to \mathrm{Hom}(a,c)\otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \quad \text{on} \quad \mathbb{P}(V^\vee)\,.$$

The injectivity of ϱ is equivalent to the injectivity of the maps $\varrho_v : \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,c)$ for generic (or just one) $v \in V$. By Lemma 20 this is equivalent to the injectivity of

$$H^0(\varrho\otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m))\colon H^0(\mathrm{Hom}(b,c)\otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m)) \to H^0(\mathrm{Hom}(a,c)\otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m+1))$$

for $m = (\dim(V)-1)(\hom(b,c)-1)$. Since $\mathrm{Hom}^{-1}(a,c) = 0$, the long exact cohomology sequence of the triangle from Construction 6.1 gives that the kernel of $H^0(\varrho\otimes\mathcal{O}_{\mathbb{P}(V^\vee)}(m))$ is $\mathrm{Hom}^0(S^m(V,a,b),c)$. □

Remark 24. Suppose that L is a base point free line bundle on a smooth projective variety X and $V \subset H^0(L)$ is a subspace such that $V \otimes \mathcal{O}_X \to L$ is surjective. Then the object $S^m(V,\mathcal{O}_X,L)$ is a vector bundle concentrated in degree -1 with invariants $\mathrm{rk}(h^{-1}(S^m(V,\mathcal{O}_X,L))) = \binom{m+\dim(V)-1}{m+1}$, and determinant given by $\det(h^{-1}(S^m(V,\mathcal{O}_X,L))) = L^{-\otimes\binom{m+\dim(V)-1}{m}}$.

Remark 25. Let A and $\bar{B}$ be two line bundles on a smooth projective curve X such $\deg(\bar{B}) - \deg(A) > 2g-2$. For an integer $R > 0$ we set $B = (\bar{B})^{\oplus R}$. Setting $V = \mathrm{Hom}(A,B)$ we deduce from the triangle $S^m(V,A,B)[-1] \to \mathrm{Sym}^{m+1}(V)\otimes A \to \mathrm{Sym}^m(V)\otimes B \to S^m(V,A,B)$ that $S^m(V,A,B)$ is a complex with nontrivial cohomology sheaves only in degree zero and -1. Suppose now that E is a sheaf on X with $\hom(A,E) = \hom(B,E)$, and $\hom^1(A,E) = \hom^1(B,E) = 0$. Lemma 23 implies that for $m \geq (\dim(V)-1)(\dim(B,E)-1)$ we have an equivalence

$$\mathrm{Hom}(S^m(V,A,B),E) = 0 \iff \mathrm{cone}(v) \text{ is orthogonal to } E \text{ for a general } v \in V.$$

However, for any non-zero $v \in V$ the morphism $A \xrightarrow{v} B$ is injective, and its cone is a sheaf concentrated in degree one. Thus, we have an equivalence

$$\mathrm{Hom}(S^m(V,A,B),E) = 0 \iff \left\{ \begin{array}{c} \text{there exists a short exact sequence} \\ 0 \to A \to B \to F \to 0 \text{ such that} \\ \mathrm{Hom}^*(F,E) = 0\,. \end{array} \right\}$$

Since $S^m(V,A,B)$ is a complex concentrated in degree zero and -1 the vanishing condition $\mathrm{Hom}(S^m(V,A,B),E)=0$ is equivalent to the vanishing of the two direct summands $\mathrm{Hom}(h^{-1}(S^m(V,A,B))[1],E)$ and $\mathrm{Hom}(h^0(S^m(V,A,B))[0],E)$. However the first direct summand is $\mathrm{Ext}^{-1}(h^{-1}(S^m(V,A,B)),E)$ which is zero. Thus $\mathrm{Hom}(S^m(V,A,B),E)=0$ is equivalent to the vanishing condition $\mathrm{Hom}(h^0(S^m(V,A,B)),E)=0$ on sheaves.

Remark 26. Concerning the functoriality of the objects $S^m(V,a,b)$, let us first introduce the relevant category $\mathcal{C}^m(\mathcal{T})$: its objects are triples (V,a,b) consisting of two objects $a,b\in\mathcal{T}$ and a subspace $V\subset\mathrm{Hom}(a,b)$. A morphism $(V,a,b)\to(V',a',b')$ in $\mathcal{C}^m(\mathcal{T})$ is given by two maps $\alpha\colon \mathrm{Sym}^{m+1}(V)\otimes a\to\mathrm{Sym}^{m+1}(V')\otimes a'$ and $\beta\colon\mathrm{Sym}^m(V)\otimes b\to\mathrm{Sym}^m(V)\otimes b'$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sym}^{m+1}(V)\otimes a \arrow[r, "\theta"] \arrow[d, "\alpha"'] & \mathrm{Sym}^m(V)\otimes b \arrow[d, "\beta"] \\
\mathrm{Sym}^{m+1}(V')\otimes a' \arrow[r, "\theta'"] & \mathrm{Sym}^m(V')\otimes b'
\end{tikzcd}$$

This way, $\mathcal{C}^m(\mathcal{T})$ is a k-linear category. With $S^m(V,a,b)=\mathrm{cone}(\theta)$ and $S^m(V',a',b')=\mathrm{cone}(\theta')$ in the above diagram, the nuisance of non-functoriality of cones in triangulated categories prevents S^m from being a functor $\mathcal{C}^m(\mathcal{T})\to\mathcal{T}$. However, if $\mathcal{T}$ is algebraic in the sense of Keller, e.g. the homotopy category of a dg category, then cones become functorial. This is the case for $\mathcal{T}=\mathrm{D^b}(X)$, for example.

Thus, we obtain functors $S^m\colon\mathcal{C}^m(\mathrm{D^b}(X))\to\mathrm{D^b}(X)$ for all $m\in\mathbb{N}$. Note that even if we set the subspace V to be the full homomorphism space and fix either a or b, the resulting functor $\mathrm{D^b}(X)\to\mathrm{D^b}(X)$ cannot be triangulated, except in the case $m=0$.

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