Fourier-Mukai partners and polarised K3 surfaces

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Abstract The purpose of this note is twofold. We first review the theory of Fourier-Mukai partners together with the relevant part of Nikulin's theory of lattice embeddings via discriminants. Then we consider Fourier-Mukai partners of K3 surfaces in the presence of polarisations, in which case we prove a counting formula for the number of partners.

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The theory of FM partners has played a crucial role in algebraic geometry and its connections to string theory in the last 25 years. Here we shall concentrate on a particularly interesting aspect of this, namely the theory of FM partners of K3 surfaces. We shall survey some of the most significant results in this direction. Another aspect, and this has been discussed much less in the literature, is the question of Fourier-Mukai partners in the presence of polarisations. We shall also investigate this in some detail, and it is here that the paper contains some new results.

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To begin with we review in Section 1 the use of derived categories in algebraic geometry focusing on Fourier-Mukai partners. In Sections 2 and 3 we then give a self-contained introduction to lattices and lattice embeddings with emphasis on indefinite, even lattices. This contains a careful presentation of Nikulin's theory as well as some enhancements which will then become important for our counting formula. From Section 4 onwards we will fully concentrate on K3 surfaces. After recalling the classical as well as Orlov's derived Torelli theorem for K3 surfaces we describe the counting formula for the FM number of K3 surfaces given by Hosono, Lian, Oguiso, Yau [24]. In Section 5 we discuss polarised K3 surfaces and their moduli. The relationship between polarised K3 surfaces and FM partners was discussed by Stellari in [45] and [46]. Our main result in this direction is a counting formula given in Section 7 in the spirit of [24].

In a number of examples we will discuss the various phenomena which occur when considering Fourier-Mukai partners in the presence of polarisations.

Conventions: We work over the field $\mathbb{C}$.

We will denote bijections of sets as $A \stackrel{1:1}{=} B$. Also, all group actions will be left actions. In particular, we will denote the sets of orbits by $G\backslash A$ whenever G acts on A. However, factor groups are written G/H.

If we have group actions by G and G' on a set A which are compatible (i.e. they commute), then we consider this as a $G \times G'$-action (and not as a left-right bi-action). In particular, the total orbit set will be written as $G \times G' \backslash A$ (and not $G\backslash A/G'$).

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1 Review Fourier-Mukai partners of K3 surfaces

For more than a century algebraic geometers have looked at the classifiation of varieties up to birational equivalence. This is a weaker notion than biregular isomorphism which, however, captures a number of crucial and interesting properties.

About two decades ago, a different weakening of biregularity has emerged in algebraic geometry: derived equivalence. Roughly speaking, its popularity stems from two reasons: on the one hand, the seemingly ever increasing power of homological methods in all areas of mathematics, and on the other hand the intriguing link derived categories provide to other mathematical disciplines such as symplectic geometry and representation theory as well as to theoretical physics.

History: derived categories in algebraic geometry

Derived categories of abelian categories were introduced in the 1967 thesis of Grothendieck's student Verdier [48]. The goal was to set up the necessary homological tools for defining duality in greatest generality — which meant getting the right adjoint of the push-forward functor f_*. This adjoint cannot exist in the abelian category of coherent sheaves; if it did, f_* would be exact. Verdier's insight was to embed the abelian category into a bigger category with desirable properties, the derived category of complexes of coherent sheaves. The reader is referred to [23] for an account of this theory.

In this review, we will assume that the reader is familiar with the basic theory of derived categories [17], [51]. An exposition of the theory of derived categories in algebraic geometry can be found in two text books, namely by Huybrechts [25] and by Bartocci, Bruzzo, Hernàndez-Ruipérez [10]. We will denote by $D^b(X)$ the bounded derived category of coherent sheaves. This category is particularly well behaved if X is a smooth, projective variety. Later on we will consider K3 surfaces, but in this section, we review some general results.

We recall that two varieties X and Y are said to be *derived equivalent* (sometimes shortened to *D-equivalent*) if there is an exact equivalence of categories $D^b(X) \cong D^b(Y)$.

It should be mentioned right away that the use of the *derived* categories is crucial: a variety is uniquely determined by the abelian category of coherent sheaves, due to a theorem of Gabriel [16]. Thus, the analogous definition using abelian categories does not give rise to a new equivalence relation among varieties.

After their introduction, derived categories stayed in a niche, mainly considered as a homological bookkeeping tool. They were used to combine the classical derived functors into a single derived functor, or to put the Grothendieck spectral sequence into a more conceptual framework. The geometric use of derived categories started with the following groundbreaking result:

Theorem (Mukai, 1981 [32]). *Let A be an abelian variety with dual abelian variety $\hat{A}$. Then A and $\hat{A}$ are derived equivalent.*

Since an abelian variety and its dual are in general not isomorphic (unless they are principally polarised) and otherwise never birationally equivalent, this indicates a new phenomenon. For the proof, Mukai employs the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ and investigates the functor $D^b(A) \to D^b(\hat{A})$ mapping

$E \mapsto \mathsf{R}\hat{\pi}_*(\mathcal{P} \otimes \pi^* E)$ where $\hat{\pi}$ and π denote the projections from $A \times \hat{A}$ to $\hat{A}$ and A respectively.

Mukai's approach was not pursued for a while. Instead, derived categories were used in different ways for geometric purposes: Beilinson, Bernstein, Deligne [4] introduced perverse sheaves as certain objects in the derived category of constructible sheaves of a variety in order to study topological questions. The school around Rudakov introduced exceptional collections (of objects in the derived category), which under certain circumstances leads to an equivalence of $D^b(X)$ with the derived category of a finite-dimensional algebra [42]. It should be mentioned that around the same time, Happel introduced the use of triangulated categories in representation theory [21].

Derived categories as invariants of varieties

Bondal and Orlov started considering $D^b(X)$ as an *invariant* of X with the following highly influential result:

Theorem (Bondal, Orlov, 1997 [6]). *Let X and Y be two smooth, projective varieties with $D^b(X) \cong D^b(Y)$. If X has ample canonical or anti-canonical bundle, then $X \cong Y$.*

In other words, at the extreme ends of the curvature spectrum, the derived category determines the variety. Note the contrast with Mukai's result, which provides examples of non-isomorphic, derived equivalent varieties with zero curvature (trivial canonical bundle). This begs the natural question: which (types of) varieties can possibly be derived equivalent? The philosophy hinted at by the theorems of Mukai, Bondal and Orlov is not misleading.

Proposition. *Let X and Y be two smooth, projective, derived equivalent varieties. Then the following hold true:*

1. *X and Y have the same dimension.*
2. *The singular cohomology groups $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ are isomorphic as ungraded vector spaces; the same is true for Hochschild cohomology.*
3. *If the canonical bundle of X has finite order, then so does the canonical bundle of Y and the orders coincide; in particular, if one canonical bundle is trivial, then so is the other.*
4. *If the canonical (or anti-canonical) bundle of X is ample (or nef), the same is true for Y.*

The proposition is the result of the work of many people, see [25, §4–6]. Stating it here is ahistorical because some of the statements rely on the notion of Fourier-Mukai transform which we turn to in the next section. It should be said that our historical sketch is very much incomplete: For instance, developments like spaces of stability conditions [9] or singularity categories [11, 39] are important but will not play a role here.

Fourier-Mukai partners

Functors between geometric categories defined by a 'kernel', i.e. a sheaf on a product (as in Mukai's case) were taken up again in the study of moduli spaces: if a moduli space M of sheaves of a certain type on Y happens to possess a (quasi)universal family $\mathcal{E} \in \mathrm{Coh}(M \times Y)$, then this family gives rise to a functor $\mathrm{Coh}(M) \to \mathrm{Coh}(Y)$, mapping $A \mapsto p_{Y*}(\mathcal{E} \otimes p_M^* A)$, where p_M and p_Y are the projections from $M \times Y$ to M and Y, respectively. In particular, skyscraper sheaves of points $[E] \in M$ are sent to the corresponding sheaves E. This (generally non-exact!) functor does not possess good properties and it was soon realised that it is much better to consider its derived analogue, which we define below. Sometimes, for example, the functors between derived categories can be used to show birationality of moduli spaces.

In the following definition, we denote the canonical projections of the product $X \times Y$ to its factors by p_X and p_Y respectively.

Definition. Let X and Y be two smooth, projective varieties and let $K \in D^b(X \times Y)$. The *Fourier-Mukai functor* with *kernel* K is the composition

$$\mathsf{FM}_K \colon D^b(X) \xrightarrow{p_X^*} D^b(X \times Y) \xrightarrow{K \overset{\mathsf{L}}{\otimes}} D^b(X \times Y) \xrightarrow{\mathsf{R}p_{Y*}} D^b(Y)$$

of pullback, derived tensor product with K and derived push-forward. If FM_K is an equivalence, then it is called a *Fourier-Mukai transform*.

X and Y are said to be *Fourier-Mukai partners* if a Fourier-Mukai transform exists between their derived categories. The set of all Fourier-Mukai partners of X up to isomorphisms is denoted by $\mathsf{FM}(X)$.

Remarks. This important notion warrants a number of comments.

1. Fourier-Mukai functors should be viewed as classical correspondences, i.e. maps between cohomology or Chow groups on the level of derived categories. In particular, many formal properties of correspondences as in [15, §14] carry over verbatim: the composition of Fourier-Mukai functors is again such, with the natural 'convoluted' kernel; the (structure sheaf of the) diagonal gives the identity etc. In fact, a Fourier-Mukai transform induces correspondences on the Chow and cohomological levels, using the Chern character of the kernel.

2. Neither notation nor terminology is uniform. Some sources mean 'Fourier-Mukai transform' to be an equivalence whose kernel is a sheaf, for example. Notationally, often used is $\Phi_K^{X \to Y}$ which is inspired by Mukai's original article [33]. This notation, however, has the drawback of being lengthy without giving additional information in the important case $X = Y$.

Fourier-Mukai transforms play a very important and prominent role in the theory due to the following basic and deep result:

Theorem (Orlov, 1996 [37]). *Given an equivalence $\Phi\colon D^b(X) \xrightarrow{\sim} D^b(Y)$ (as $\mathbb{C}$-linear, triangulated categories) for two smooth, projective varieties X and Y, then there exists an object $K \in D^b(X \times Y)$ with a functor isomorphism $\Phi \cong \mathsf{FM}_K$. The kernel K is unique up to isomorphism.*

By this result, the notions 'derived equivalent' and 'Fourier-Mukai partners' are synonymous.

The situation is very simple in dimension 1: two smooth, projective curves are derived equivalent if and only if they are isomorphic. The situation is a lot more interesting in dimension 2: apart from the abelian surfaces already covered by Mukai's result, K3 and certain elliptic surfaces can have non-isomorphic FM partners. For K3 surfaces, the statement is as follows (see Section 4 for details):

Theorem (Orlov, 1996 [37]). *For two projective* K3 *surfaces X and Y, the following conditions are equivalent:*

1. *X and Y are derived equivalent.*
2. *The transcendental lattices T_X and T_Y are Hodge-isometric.*
3. *There exist an ample divisor H on X, integers $r \in \mathbb{N}$, $s \in \mathbb{Z}$ and a class $c \in H^2(X,\mathbb{Z})$ such that the moduli space of H-semistable sheaves on X of rank r, first Chern class c and second Chern class s is nonempty, fine and isomorphic to Y.*

In general, it is a conjecture that the number of FM partners is always finite. For surfaces, this has been proven by Bridgeland and Maciocia [7]. The next theorem implies finiteness for abelian varieties, using that an abelian variety has only a finite number of abelian subvarieties up to isogeny [18].

Theorem (Orlov, Polishchuk 1996, [38], [41]). *Two abelian varieties A and B are derived equivalent if and only if $A \times \hat{A}$ and $B \times \hat{B}$ are symplectically isomorphic, i.e. there is an isomorphism $f = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\colon A \times \hat{A} \xrightarrow{\sim} B \times \hat{B}$ such that $f^{-1} = \left(\begin{smallmatrix} \hat{\delta} & -\hat{\beta} \\ -\hat{\gamma} & \hat{\alpha} \end{smallmatrix}\right)$.*

The natural question about the number of FM partners has been studied in greatest depth for K3 surfaces. The first result was shown by Oguiso [36]: a K3 surface with a single primitive ample divisor of degree $2d$ has exactly $2^{p(d)-1}$ such partners, where $p(d)$ is the number of prime divisors of d. In [24], a formula using lattice counting for general projective K3 surfaces was given. In Section 4, we will reprove this result and give a formula for polarised K3 surfaces. We want to mention that FM partners of K3 surfaces have been linked to the so-called Kähler moduli space, see Ma [29] and Hartmann [22].

Derived and birational equivalence

We started this review by motivating derived equivalence as a weakening of isomorphism, like birationality is. This naturally leads to the question whether there is an actual relationship between the two notions. At first glance, this is not the case: since birational abelian varieties are already isomorphic, Mukai's result provides examples of derived equivalent but not birationally equivalent varieties. And in the other direction, let Y be the blowing up of a smooth projective variety X of dimension at least two in a point. Then X and Y are obviously birationally equivalent but never derived equivalent by a result of Bondal and Orlov [5].

Nevertheless some relation is expected. More precisely:

Conjecture (Bondal, Orlov [5]). If X and Y are smooth, projective, birationally equivalent varieties with trivial canonical bundles, then X and Y are derived equivalent.

Kawamata suggested a generalisation using the following notion: two smooth, projective varieties X and Y are called *K-equivalent* if there is a birational correspondence $X \xleftarrow{p} Z \xrightarrow{q} Y$ with $p^*\omega_X \cong q^*\omega_Y$. He conjectures that K-equivalent varieties are D-equivalent.

The conjecture is known in some cases, for example the standard flop (Bondal, Orlov [5]), the Mukai flop (Kawamata [27], Namikawa [34]), Calabi-Yau threefolds (Bridgeland [8]) and Hilbert schemes of K3 surfaces (Ploog [40]).

2 Lattices

Since the theory of K3 surfaces is intricately linked to lattices, we provide a review of the lattice theory as needed in this note. By a lattice we always mean a free abelian group L of finite rank equipped with a non-degenerate symmetric bilinear pairing $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$. The lattice L is called *even* if $(v,v) \in 2\mathbb{Z}$ for all $v \in L$. We shall assume all our lattices to be even.

Sometimes, we denote by L_K the K-vector space $L \otimes K$, where K is a field among $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. The pairing extends to a symmetric bilinear form on L_K. The *signature* of L is defined to be that of $L_{\mathbb{R}}$.

The lattice L is called *unimodular* if the canonical homomorphism $d_L\colon L \to L^\vee = \mathrm{Hom}(L,\mathbb{Z})$ with $d_L(v) = (v,\cdot)$ is an isomorphism. Note that d_L is always injective, as we have assumed $(\cdot,\cdot)$ to be non-degenerate. This implies that for every element $f \in L^\vee$ there is a natural number $a \in \mathbb{N}$ such that af is in the image of d_L. Thus $L^\vee$ can be identified with the subset $\{w \in L \otimes \mathbb{Q} \mid (v,w) \in \mathbb{Z}\ \forall v \in L\}$ of $L \otimes \mathbb{Q}$ with its natural $\mathbb{Q}$-valued pairing.

We shall denote the *hyperbolic plane* by U. A *standard basis* of U is a basis e, f with $e^2 = f^2 = 0$ and $(e,f) = 1$. The lattice E_8 is the unique positive definite even unimodular lattice of rank 8, and we denote by $E_8(-1)$ its negative definite opposite. For an integer $n \neq 0$ we denote by $\langle n \rangle$ the rank one lattice where both generators square to n. Finally, given a lattice L, then aL denotes a direct sum of a copies of the lattice L.

Given any non-empty subset $S \subseteq L$, the *orthogonal complement* is $S^\perp := \{v \in L \mid (v,S) = 0\}$. A submodule $S \subseteq L$ is called *primitive* if the quotient group L/S is torsion free. Note the following obvious facts: $S^\perp \subseteq L$ is always a primitive submodule; we have $S \subseteq S^{\perp\perp}$; and S is primitive if and only if $S = S^{\perp\perp}$. In particular, $S^{\perp\perp}$ is the *primitive hull* of S.

A vector $v \in L$ is called *primitive* if the lattice $\mathbb{Z}v$ generated by it is primitive.

The *discriminant group* of a lattice L is the finite abelian group $D_L = L^\vee/L$. Since we have assumed L to be even it carries a natural quadratic form q_L with values in $\mathbb{Q}/2\mathbb{Z}$. By customary abuse of notation we will often speak of a quadratic form q (or q_L), suppressing the finite abelian group it lives on. Finally, for any lattice L, we denote by $l(L)$ the minimal number of generators of D_L.

Gram matrices

We make the above definitions more explicit using the matrix description. After choosing a basis, a lattice on $\mathbb{Z}^r$ is given by a symmetric $r \times r$ matrix G (often called Gram matrix), the pairing being $(v,w) = v^t G w$ for $v, w \in \mathbb{Z}^r$. To be precise, the (i,j)-entry of G is $(e_i, e_j) \in \mathbb{Z}$ where $(e_1, \dots, e_r)$ is the chosen basis.

Changing the matrix by symmetric column-and-row operations gives an isomorphic lattice; this corresponds to $G \mapsto SGS^t$ for some $S \in \mathrm{GL}(r,\mathbb{Z})$. Since our pairings are non-degenerate, G has full rank. The lattice is unimodular if the Gram matrix has determinant ± 1. It is even if and only if the diagonal entries of G are even.

The inclusion of the lattice into its dual is the map $G\colon \mathbb{Z}^r \hookrightarrow \mathbb{Z}^r$, $v \mapsto v^t G$. Considering a vector $\varphi \in \mathbb{Z}^r$ as an element of the dual lattice, there is a natural number a such that $a\varphi$ is in the image of G, i.e. $v^t G = a\varphi$ for some integral vector v. Then $(\varphi,\varphi) = (v,v)/a^2 \in \mathbb{Q}$.

The discriminant group is the finite abelian group with presentation matrix G, i.e. $D \cong \mathbb{Z}^r/\mathrm{im}(G)$. Elementary operations can be used to diagonalise it. The quadratic form on the discriminant group is computed as above, only now taking values in $\mathbb{Q}/2\mathbb{Z}$.

The *discriminant* of L is defined as the order of the discriminant group. It is the absolute value of the determinant of the Gram matrix: $\mathrm{disc}(L) := \#D_L = |\det(G_L)|$. Classically, discriminants (of quadratic forms) are defined with a factor of ± 1 or $\pm 1/4$; see Example 2.1.

Genera

Two lattices L and L' of rank r are said to be *in the same genus* if they fulfill one of the following equivalent conditions:

(1) The localisations L_p and L'_p are isomorphic for all primes p, including $\mathbb{R}$.
(2) The signatures of L and L' coincide and the discriminant forms are isomorphic: $q_L \cong q_{L'}$.
(3) The matrices representing L and L' are *rationally equivalent without essential denominators*, i.e. there is a base change in $\mathrm{GL}(r,\mathbb{Q})$ of determinant ± 1, transforming L into L' and whose denominators are prime to $2 \cdot \mathrm{disc}(L)$.

For details on localisations, see [35]. The equivalence of (1) and (2) is a deep result of Nikulin ([35, 1.9.4]). We elaborate on (2): a map $q\colon A \to \mathbb{Q}/2\mathbb{Z}$ is called a quadratic form on the finite abelian group A if $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$, $a \in A$ and if there is a symmetric bilinear form $b\colon A \times A \to \mathbb{Q}/\mathbb{Z}$ such that $q(a_1 + a_2) = q(a_1) + q(a_2) + 2b(a_1, a_2)$ for all $a_1, a_2 \in A$. It is clear that discriminant forms of even lattices satisfy this definition. Two pairs (A, q) and (A', q') are defined to be isomorphic if there is a group isomorphism $\varphi\colon A \xrightarrow{\sim} A'$ with $q(a) = q'(\varphi(a))$ for all $a \in A$.

The history of the equivalence between (1) and (3) is complicated: Using analytical methods, Siegel [44] proved that L and L' are in the same genus if and only if for every positive integer d there exists a rational base change $S_d \in \mathrm{GL}(r,\mathbb{Q})$ carrying L into L' and such that the denominators of S_d are prime to d (and he called this property rational equivalence without denominators). There are algebraic proofs of that statement, e.g. [26, Theorem 40] or [50, Theorem 50]. These references also contain (3) above, i.e. the existence of a single $S \in \mathrm{GL}(r,\mathbb{Q})$ whose denominators are prime to $2 \cdot \mathrm{disc}(L)$.

For binary forms, all of this is closely related to classical number theory. In particular, the genus can then also be treated using the ideal class group of quadratic number fields. See [13] or [52] for this. Furthermore, there is a strengthening of (3) peculiar to *field discriminants* (see [13, §3.B]):

(4) Let $L = \left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$ and $L' = \left(\begin{smallmatrix} 2a' & b' \\ b' & 2c' \end{smallmatrix}\right)$ be two binary even, indefinite lattices with $\gcd(2a, b, c) = \gcd(2a', b', c') = 1$ and of the same discriminant $D := b^2 - 4ac$ such that either $D \equiv 1 \mod 4$, D squarefree, or $D = 4k$, $k \not\equiv 1 \mod 4$, k squarefree. Then L and L' are in the same genus if and only if they are rationally equivalent, i.e. there is a base change $S \in \mathrm{GL}(2,\mathbb{Q})$ taking L to L'.

The genus of L is denoted by $\mathcal{G}(L)$ and it is a basic, but non-trivial fact that $\mathcal{G}(L)$ is a finite set. We will also have to specify genera in other ways ways, using a quadratic form $q\colon D_q \to \mathbb{Q}/2\mathbb{Z}$ on a finite abelian group D_q, as follows:

$\mathcal{G}(t_+, t_-, q)$	lattices with signature (t_+, t_-) and discriminant form q,
$\mathcal{G}(\mathrm{sgn}(K), q)$	lattices with same signature as K and discriminant form q.

Example 2.1. We consider binary forms, that is lattices of rank 2. Clearly, a symmetric bilinear form with Gram matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ is even if and only if both diagonal terms are even.

Note that many classical sources use quadratic forms instead of lattices. We explain the link for binary forms $f(x,y) = ax^2+bxy+cy^2$ (where $a,b,c \in \mathbb{Z}$). The associated bilinear form has Gram matrix $G = \frac{1}{2}\left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$ — in particular, it need not be integral. An example is $f(x,y) = xy$. In fact, the bilinear form, i.e. G, is integral if and only if b is even (incidentally, Gauß always made that assumption). Note that the quadratic form $2xy$ corresponds to our hyperbolic plane $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The discriminant of f is classically defined to be $D := b^2 - 4ac$ which differs from our definition (i.e. $|\det(G)| = \#D$) by a factor of ± 4.

We proceed to give specific examples of lattices as Gram matrices. Both $A = \left(\begin{smallmatrix} 2 & 4 \\ 4 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$ are indefinite, i.e. of signature $(1,1)$, and have discriminant 16, but the discriminant groups are not isomorphic: $D_A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and $D_B = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Thus A and B are not in the same genus.

Another illuminating example is given by the forms A and $C = \left(\begin{smallmatrix} -2 & 4 \\ 4 & 0 \end{smallmatrix}\right)$. We first notice that these forms are not isomorphic: the form A represents 2, but C does not, as can be seen by looking at the possible remainders of $-2x^2+8xy$ modulo 8. The two forms have the same signature and discriminant groups, but the discriminant forms are different. To see this we note that D_A is generated by the residue classes of $t_1 = e_1/2$ and $t_2 = (2e_1 + e_2)/8$, whereas D_C is generated by the residue classes of $s_1 = e_1/2$ and $s_2 = (-2e_1 + e_2)/8$. The quadratic forms q_A and q_C are determined by $q_A(\overline{t_1}) = 1/2$, $q_A(\overline{t_2}) = 3/8$ and $q_C(\overline{s_1}) = -1/2$, $q_C(\overline{s_2}) = -3/8$. The forms cannot be isomorphic, for the subgroup of D_A of elements of order 2 consists of $\{0, t_1, 4t_2, t_1 + 4t_2\}$ (this is the Klein four group) and the values of q_A on these elements in $\mathbb{Q}/2\mathbb{Z}$ are $0, 1/2, 4^2 \cdot 3/8 = 0, 42/4 = 1/2$. Likewise, the values of q_C on the elements of order 2 in D_C are 0 and $-1/2$. Hence (D_A, q_A) and (D_C, q_C) cannot be isomorphic.

Zagier's book also contains the connection of genera to number theory and their classification using ideal class groups [52, §8]. An example from this book [52, §12] gives an instance of lattices in the same genus which are not isomorphic: the forms $D = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 12 \end{smallmatrix}\right)$ and $E = \left(\begin{smallmatrix} 4 & 1 \\ 1 & 6 \end{smallmatrix}\right)$ are positive definite of field discriminant -23. They are in the same genus (one is sent to the other by the fractional base change $-\frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ -3 & 1 \end{smallmatrix}\right)$) but not equivalent: D represents 2 as the square of $(1,0)$ whereas E does not represent 2 as $4x^2 + 2xy + 6y^2 = 3x^2 + (x+y)^2 + 5y^2 \geq 4$ if $x \neq 0$ or $y \neq 0$.

Unimodular, indefinite lattices are unique in their genus, as follows from their well known classification. A generalisation is given by [35, Cor. 1.13.3]:

Lemma 2.2. (Nikulin's criterion) *An indefinite lattice L with rk$(L) \geq 2 + l(L)$ is unique within its genus.*

Recall that $l(L)$ denotes the minimal number of generators of the finite group D_L. Since always $\mathrm{rk}(L) \geq l(L)$, Nikulin's criterion only fails to apply in two cases, namely if $l(L) = \mathrm{rk}(L)$ or $l(L) = \mathrm{rk}(L) - 1$. As a corollary of Nikulin's criterion, $L \oplus U$ is unique within its lattice for any L.

For a lattice L, we denote its group of isometries by $\mathrm{O}(L)$. An isometry of lattices, $f\colon L \xrightarrow{\sim} L'$ gives rise to $f_{\mathbb{Q}}\colon L_{\mathbb{Q}} \xrightarrow{\sim} L'_{\mathbb{Q}}$ and hence to $D_f\colon D_L \xrightarrow{\sim} D_{L'}$. In particular, there is a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(D_L)$ which is used to define the *stable isometry group* as

$$\tilde{\mathrm{O}}(L) := \ker(\mathrm{O}(L) \to \mathrm{O}(D_L)).$$

Finally, we state a well known result of Eichler, [14, §10]. It uses the notion of the *divisor* $\mathrm{div}(v)$ of a vector $v \in L$, which is the positive generator of the ideal (v, L). Note that this is the largest positive integer a such that $v = av'$ for some element $v' \in L^\vee$.

Lemma 2.3. (Eichler's criterion) *Suppose that L contains $U \oplus U$ as a direct summand. The $\mathrm{O}(L)$-orbit of a primitive vector $v \in L$ is determined by the length v^2 and the element $v/\mathrm{div}(v) \in D(L)$ of the discriminant group.*

3 Overlattices

In this section, we elaborate on Nikulin's theory of overlattices and primitive embeddings [35]; we also give some examples. Eventually, we generalise slightly to cover a setting needed for the Fourier-Mukai partner counting in the polarised case.

We fix a lattice M with discriminant form $q_M\colon D_M \to \mathbb{Q}/2\mathbb{Z}$.

By an *overlattice* of M we mean a lattice embedding $i\colon M \hookrightarrow L$ with M and L of the same rank. Note that we have inclusions

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[rrr, hook, bend right, "d_M"'] & L \arrow[r, hook, "d_L"] & L^\vee \arrow[r, hook, "i^\vee"] & M^\vee \end{tikzcd}$$

with $d_L\colon L \hookrightarrow L^\vee$ and $d_M\colon M \hookrightarrow M^\vee$ the canonical maps. (For now, we will denote these canonical embeddings just by d, and later not denote them at all.) From this we get a chain of quotients

$$L/iM \xrightarrow{d} L^\vee/diM \xrightarrow{i^\vee} M^\vee/i^\vee diM = D_M.$$

We call the image $H_i \subset D_M$ of L/iM the *classifying subgroup* of the overlattice. Note that D_M is equipped with a quadratic form, so we can also speak of the orthogonal complement $H_i^\perp$. We will consider $L^\vee/diM$ as a subgroup of D_M in the same way via $i^\vee$.

We say that two embeddings $i\colon M \hookrightarrow L$ and $i'\colon M \hookrightarrow L'$ *define the same overlattice* if there is an isometry $f\colon L \xrightarrow{\sim} L'$ such that $fi = i'$:

$$
\begin{tikzcd}
M \arrow[r, "i"] \arrow[d, equal] & L \arrow[d, "f"] \\
M \arrow[r, "i'"] & L'
\end{tikzcd}
$$

This means in particular that within each isomorphism class, we can restrict to looking at embeddings $i\colon M \hookrightarrow L$ into a *fixed* lattice L.

Lemma 3.1. [35, Proposition 1.4.1] *Let $i\colon M \hookrightarrow L$ be an overlattice. Then the subgroup H_i is isotropic in D_M, i.e. $q_M|_{H_i} = 0$. Furthermore, $H_i^\perp = L^\vee/diM$ and there is a natural identification $H_i^\perp/H_i \cong D_L$ with $q_M|_{H_i^\perp/H_i} = q_L$.*

We introduce the following sets of overlattices L of M and quotients L/M respectively, where we consider L/M as an isotropic subgroup of the discriminant group D_M:

$$
\begin{aligned}
\mathcal{O}(M) &:= \{(L,i) \mid L \text{ lattice}, i\colon M \hookrightarrow L \text{ overlattice}\} \\
\mathcal{Q}(M) &:= \{H \subset D_M \text{ isotropic}\}.
\end{aligned}
$$

We also use the notation $\mathcal{O}(M,L)$ to specify that the target lattice is isomorphic to L. With this notation we can write $\mathcal{O}(M)$ as a disjoint union

$$\mathcal{O}(M) = \coprod_L \mathcal{O}(M,L)$$

where L runs through all isomorphism classes of possible overlattices of M.

Remark 3.2. The set $\mathcal{Q}(M)$ is obviously finite. On the other hand, an overlattice $i\colon M \hookrightarrow L$ can always be modified by an isometry $f \in \mathrm{O}(M)$ to yield an overlattice $if\colon M \hookrightarrow L$. However, if $f \in \tilde{\mathrm{O}}(M)$ is a stable isometry, then it can be extended to an isometry of L and hence i and if define the same overlattice. This shows that $\mathcal{O}(M)$ is also finite.

The following lemma is well known and implicit in [35].

Lemma 3.3. *There is a bijection between $\mathcal{O}(M)$ and $\mathcal{Q}(M)$.*

Proof. We use the maps

$$
\begin{aligned}
\mathsf{H}\colon \mathcal{O}(M) \to \mathcal{Q}(M), &\quad (L,i) \mapsto H_i, \\
\mathsf{L}\colon \mathcal{Q}(M) \to \mathcal{O}(M), &\quad H \mapsto (L_H, i_H)
\end{aligned}
$$

where, for $H \in \mathcal{Q}(M)$, we define $L_H := \{\varphi \in M^\vee \mid [\varphi] \in H\} = \pi^{-1}(H)$ where $\pi\colon M^\vee \to D_M$ is the canonical projection. The canonical embedding $d\colon M \hookrightarrow M^\vee$ factors through L_H, giving an injective map $i_H\colon M \to L_H$. All of this can be summarised in a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_H"] \arrow[d, equal] & L_H \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M \arrow[r] & M^\vee \arrow[r, "\pi"] & D_M \arrow[r] & 0.
\end{tikzcd}$$

The abelian group L_H inherits a $\mathbb{Q}$-valued form from $M^\vee$. This form is actually $\mathbb{Z}$-valued because of $q_M|_H = 0$. Furthermore, the bilinear form on L_H is even since the quadratic form on D_M is $\mathbb{Q}/2\mathbb{Z}$-valued. Hence, L_H is a lattice and i_H is obviously a lattice embedding.

It is immediate that $\mathsf{HL} = \mathrm{id}_{\mathcal{Q}(M)}$. On the other hand, the overlattices $\mathsf{LH}(L, i)$ and (L, i) are identified by the embedding $L \to M^\vee$, $v \mapsto \langle v, i(\cdot)\rangle_M$ which has precisely $\mathsf{LH}(L, i)$ as image. □

We want to refine this correspondence slightly. For this we fix a quadratic form (D, q) which occurs as the discriminant of some lattice (and forget L) and set

$$\mathcal{O}(M, q) := \{(L, i) \in \mathcal{O}(M) \mid [L] \in \mathcal{G}(\mathrm{sgn}(M), q)\},$$
$$\mathcal{Q}(M, q) := \{H \in \mathcal{Q}(M) \mid q_M|_{H^\perp/H} \cong q\}.$$

The condition $q_M|_{H^\perp/H} \cong q$ here includes $H^\perp/H \cong D$.

Lemma 3.4. *There is a bijection between $\mathcal{O}(M, q)$ and $\mathcal{Q}(M, q)$.*

Proof. We only have to check that the maps $\mathsf{H}\colon \mathcal{O}(M, q) \to \mathcal{Q}(M)$ and $\mathsf{L}\colon \mathcal{Q}(M, q) \to \mathcal{O}(M)$ have image in $\mathcal{Q}(M, q)$ and $\mathcal{O}(M, q)$, respectively. For H, this is part of Lemma 3.1. For L, we have $\mathrm{sgn}(L_H) = \mathrm{sgn}(M)$ and the discriminant form of L_H is $D_M|_{H^\perp/H} \cong q$, by assumption on H. □

In the course of our discussions we have to distinguish carefully between different notions equivalence of lattice embeddings. The following notion is due to Nikulin ([35, Proposition 1.4.2]):

Definition 3.5. Two embeddings $i, i'\colon M \hookrightarrow L$ define *isomorphic overlattices*, denoted $i \simeq i'$, if there exists an isometry $f \in \mathrm{O}(L)$ with $fi(M) = i'(M)$ — inducing an isometry $f|_M \in \mathrm{O}(M)$ — or, equivalently if there is a commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "i"] \arrow[d, "f|_M"] & L \arrow[d, "f"] \\
M \arrow[r, "i'"] & L
\end{tikzcd}$$

Note that this definition also makes sense if M and L do not necessarily have the same rank. Two embeddings of lattices $i, i'\colon M \hookrightarrow L$ of the same rank defining the *same* overlattice are in particular isomorphic.

Definition 3.6. Two embeddings $i, i' \colon M \hookrightarrow L$ are *stably isomorphic*, denoted $i \approx i'$, if there exists a stable isometry $f \in \tilde{\mathrm{O}}(L)$ with $fi(M) = i'(M)$, i.e. there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f|_M"] & L \arrow[d, "f \text{ stable}"] \\ M \arrow[r, "i'"] & L \end{tikzcd}$$

We note that embeddings of lattices of the same rank defining the same overlattice are not necessarily stably isomorphic.

We can put this into a broader context. For this we consider the set

$$\mathcal{E}(M, L) := \{i \colon M \hookrightarrow L\}$$

of embeddings of M into L where, for the time being, we do not assume M and L to have the same rank. The group $\mathrm{O}(M) \times \mathrm{O}(L)$ acts on this set by $(g, \tilde{g}) \colon i \mapsto \tilde{g} i g^{-1}$. Instead of the action of $\mathrm{O}(M) \times \mathrm{O}(L)$ on $\mathcal{E}(M, L)$ one can also consider the action of any subgroup and we shall see specific examples later when we discuss Fourier-Mukai partners of K3 surfaces. If M and L have the same rank, then the connection with our previously considered equivalence relations is the following:

$$\mathcal{O}(M, L) = (\{\mathrm{id}_M\} \times \mathrm{O}(L)) \backslash \mathcal{E}(M, L).$$

The set of all isomorphic overlattices of M isomorphic to L is given by $(\mathrm{O}(M) \times \mathrm{O}(L)) \backslash \mathcal{E}(M, L)$ whereas stably isomorphic embeddings are given by $(\mathrm{O}(M) \times \tilde{\mathrm{O}}(L)) \backslash \mathcal{E}(M, L)$.

We now return to our previous discussion of the connection between overlattices and isotropic subgroups.

Lemma 3.7. *Let $i, i' \colon M \hookrightarrow L$ be embeddings of lattices of the same rank. Then $i \simeq i'$ if and only if there exists an isometry $g \in \mathrm{O}(M)$ such that $D_g(H_i) = H_{i'}$.*

Proof. Given $f \in \mathrm{O}(L)$ with $fi(M) = i'(M)$, then $g := f|_M$ will have the correct property.

Given g, recall that the lattices are obtained from their classifying subgroups as $\pi^{-1}(H_i)$ and $\pi^{-1}(H_{i'})$. Then, $D_g(H_i) = H_{i'}$ implies that the map $g^\vee \colon M^\vee \xrightarrow{\sim} M^\vee$ induced from g sends L to itself, and $f = g^\vee|_L$ gives the desired isomorphism. □

Note that an isometry $g \in \mathrm{O}(M)$ with $D_g(H_i) = H_{i'}$ induces an isomorphism $H_i^\perp \xrightarrow{\sim} H_{i'}^\perp$ and hence an isomorphism of the quotients. Recall that there is a natural identification $H_i^\perp / H_i = D_L$.

Lemma 3.8. *Let $i, i' \colon M \hookrightarrow L$ be embeddings of lattices of the same rank. $i \approx i'$ if and only if there exists an isometry $g \in \mathrm{O}(M)$ such that $D_g(H_i) = H_{i'}$ and the induced map $D_L = H_i^\perp/H_i \to H_{i'}^\perp/H_{i'} = D_L$ is the identity.*

Proof. Just assuming $D_g(H_i) = H_{i'}$, we get a commutative diagram

$$\begin{tikzcd} H_i \arrow[r, hook] \arrow[d, "D_g"'] & H_i^\perp \arrow[r, two heads, "(i^\vee)^{-1}"] \arrow[d, "D_g"'] & D_L \arrow[d, "D_f"] \\ H_{i'} \arrow[r, hook] & H_{i'}^\perp \arrow[r, two heads, "(i'^\vee)^{-1}"] & D_L \end{tikzcd}$$

which, together with the proof of Lemma 3.7, shows the claim. □

Overlattices from primitive embeddings

A natural source of overlattices is $M := T \oplus T^\perp \subset L$ for any sublattice $T \subset L$. If T is moreover a primitive sublattice of L, then the theory sketched above can be refined, as we explain next. We start with an elementary lemma.

Lemma 3.9. *Let $A, B \subset L$ be two sublattices such that $i \colon A \oplus B \hookrightarrow L$ is an overlattice, i.e. A and B are mutually orthogonal and $rk(A \oplus B) = rk(L)$. Then $p_A \colon H_i \hookrightarrow D_{A \oplus B} \twoheadrightarrow D_A$ is injective if and only if B is primitive in L.*

Proof. The commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A \oplus B \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & H_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & A^\vee \arrow[r] & D_A \arrow[r] & 0 \end{tikzcd}$$

leads to the following short exact sequence of the kernels:

$$0 \to B \to B^{\perp\perp} \to \ker(p_A) \to 0$$

(note that the kernel of the map $L \to A^\vee, v \mapsto \langle v, \cdot \rangle|_A$ is the primitive hull of B). Hence p_A is injective if and only if $B = B^{\perp\perp}$, i.e. B is a primitive sublattice. □

Example 3.10. We consider the rank 2 lattice L with Gram matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$; let e_1, e_2 be an orthogonal basis, so that $e_1^2 = e_2^2 = 2$. With $T = \langle 8 \rangle$ having basis $2e_1$ and $K := T^\perp$, we get that $H_i \to D_T$ is injective whereas $H_i \to D_K$ is not.

Let $j_T \colon T \hookrightarrow L$ be a sublattice and $K := T^\perp$ its orthogonal complement with embedding $j_K \colon K \hookrightarrow L$. By Lemma 3.1, the overlattice $i := j_T \oplus j_K \colon T \oplus K \hookrightarrow L$ corresponds to the isotropic subgroup $H_i \subset D_{T \oplus K}$.

By Lemma 3.9, the map $p_T\colon H_i \hookrightarrow D_{T\oplus K} \twoheadrightarrow D_T$ is always injective, since $K \subset L$ is an orthogonal complement, hence primitive. The map $p_K\colon H_i \hookrightarrow D_{T\oplus K} \twoheadrightarrow D_K$ is injective if and only if $T \subset L$ is a primitive sublattice.

If $j_T\colon T \hookrightarrow L$ is primitive, then $\Gamma_i := p_T(H_i) \subseteq D_T$ is a subgroup such that there is a unique, injective homomorphism $\gamma_i\colon \Gamma_i \to D_K$. The image of γ_i is $p_K(H_i)$ and its graph is H_i.

For fixed lattices L, K, T we introduce the following sets

$$\mathcal{P}(T,L) := \{j_T\colon T \hookrightarrow L \text{ primitive}\},$$
$$\mathcal{P}(T,K,L) := \left\{(j_T,j_K) \;\middle|\; \begin{array}{l} j_T \in \mathcal{P}(T,L), j_K \in \mathcal{P}(K,L) \\ j_T \oplus j_K \in \mathcal{E}(T\oplus K, L) \end{array}\right\}.$$

As in the previous section we can consider various notions of equivalence on the set $\mathcal{P}(T,K,L)$ by considering the action of suitable subgroups of $\mathrm{O}(T) \times \mathrm{O}(K) \times \mathrm{O}(L)$. Since we are only interested in overlattices in this section we shall assume for the rest of this section that

Assumption 3.11. $\qquad \mathrm{rk}(T) + \mathrm{rk}(K) = \mathrm{rk}(L).$

In the previous section we said that two embeddings define the same overlattice if they differ by the action of $\{\mathrm{id}_T\} \times \{\mathrm{id}_K\} \times \mathrm{O}(L)$ and accordingly we set

$$\mathcal{O}(T,K,L) = (\{\mathrm{id}_T\} \times \{\mathrm{id}_K\} \times \mathrm{O}(L))\backslash \mathcal{P}(T,K,L).$$

We now also consider a quadratic form (D,q) which will play the role of the discriminant of the overlattice. Choosing a representative L for each element in $\mathcal{G}(\mathrm{sgn}(T\oplus K), q)$, we also introduce the equivalents of the sets of the previous section:

$$\mathcal{P}(T,K,q) := \left\{(L,j_T,j_K) \;\middle|\; \begin{array}{l} [L] \in \mathcal{G}(\mathrm{sgn}(T\oplus K), q), \\ (j_T,j_K) \in \mathcal{P}(T,K,L) \end{array}\right\},$$
$$\mathcal{Q}(T,K,q) := \{H \in \mathcal{Q}(T\oplus K, q) \mid p_T|_H \text{ and } p_K|_H \text{ are injective}\}.$$

Dividing out by the action of the overlattice we also consider $\mathcal{O}(T,K,q)$. The condition in the definition of $\mathcal{Q}(T,K,q)$ means that H is the graph of an injective group homomorphism $\gamma\colon \Gamma \hookrightarrow D_K$ with $\Gamma := p_T(H)$ and $\mathrm{im}(\gamma) = p_K(H)$. Note that $q_{T\oplus K}|_H = 0$ is equivalent to $q_K\gamma = -q_T|_\Gamma$.

Evidently, $\mathcal{P}(T,K,q)$, respectively $\mathcal{O}(T,K,q)$, is the disjoint union of $\mathcal{P}(T,K,L)$, respectively $\mathcal{O}(T,K,L)$ over representative lattices L of the genus prescribed by $\mathrm{sgn}(T\oplus K)$ and discriminant form q. The difference between $\mathcal{P}(T,K,q)$ and $\mathcal{P}(T,K,L)$ is that the former set does not specify the overlattice but just its genus and we need $\mathcal{P}(T,K,q)$ because we are interested in describing lattices by discriminant forms, but those forms only see the genus.

Lemma 3.12. *For T, K and q as above, the sets $\mathcal{O}(T,K,q)$ and $\mathcal{Q}(T,K,q)$ are in bijection.*

Proof. The main idea is that the restrictions of H and L to the newly introduced sets factor as follows

$$\begin{tikzcd} \mathcal{O}(T,K,q) \arrow[r, hook] \arrow[d, dashed, shift left] & \mathcal{O}(T\oplus K,q) \arrow[d, shift left, "\mathsf{H}"] \\ \mathcal{Q}(T,K,q) \arrow[r, hook] \arrow[u, dashed, shift left] & \mathcal{Q}(T\oplus K,q) \arrow[u, shift left, "\mathsf{L}"] \end{tikzcd} \qquad \begin{matrix} (L,j_T,j_K)\mapsto (L,j_T\oplus j_K) \\ \\ H\mapsto H \end{matrix}$$

Indeed, the map $\mathsf{H}|_{\mathcal{P}(T,K,q)}$ factors via $\mathcal{Q}(T,K,q)$ in view of Lemma 3.9.

In order to see that $\mathsf{L}|_{\mathcal{Q}(T,K,q)}$ factors over $\mathcal{O}(T,K,q)$, we take an isotropic subgroup $H\subset D_{T\oplus K}$. Then we can form the overlattice $L_H=\pi^{-1}(H)$ of $T\oplus K$. Obviously, this gives embeddings $j_T\colon T\hookrightarrow L_H$ and $j_K\colon K\hookrightarrow L_H$. These are primitive since the projections $H\to p_T(H)$ and $H\to p_K(H)$ are isomorphisms. Next, the sublattices are orthogonal to each other: $j_T\colon T\to T^\vee\oplus K^\vee, v\mapsto(\langle v,\cdot\rangle,0)$ and $j_K\colon K\to T^\vee\oplus K^\vee, w\mapsto(0,\langle w,\cdot\rangle)$. Finally, they obviously span L_H over $\mathbb{Q}$. □

Fix a subgroup $G_T\subseteq \mathrm{O}(T)$. Two pairs $(L,i,j),(L',i',j')\in\mathcal{P}(T,K,q)$ are called *G_T-equivalent* if there is an isometry $\varphi\colon L\cong L'$ such that $\varphi(iT)=i'T$ and $\varphi_T:=(i')^{-1}\circ\varphi|_{i(T)}\circ i\in G_T$ for the induced isometry of T.

Lemma 3.13. [35, 1.15.1] *Let $H,H'\in\mathcal{Q}(T,K,q)$. Then $\mathsf{L}(H)$ and $\mathsf{L}(H')$ are G_T-equivalent if and only if there is $\psi\in G_T\times\mathrm{O}(K)$ such that $D_\psi(H)=H'$.*

Proof. First note that the condition $D_\psi(H)=H'$ is equivalent to the one in [35]: there are $\psi_T\in G_T$ and $\psi_K\in\mathrm{O}(K)$ such that $D_{\psi_T}(\varGamma)=\varGamma'$ and $D_{\psi_K}\gamma=\gamma' D_{\psi_T}$ where H and H' are the graphs of $\gamma\colon\varGamma\to D_K$ and $\varGamma'\colon H'\to D_K$, respectively.

Suppose that (L,i,j) and (L',i',j') are G_T-equivalent. Thus there is an isometry $\varphi\colon L\xrightarrow{\sim}L'$ with $\varphi(iT)=i'T$. In particular, $\varphi(i(T)^\perp_L)=i'(T)^\perp_{L'}$; using the isomorphisms j and j' we get an induced isometry $\varphi_K\in\mathrm{O}(K)$. We have established the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & iT\oplus jK \arrow[r] \arrow[d, "\varphi_T\oplus\varphi_K"] & L \arrow[r] \arrow[d, "\varphi"] & L/(iT\oplus jK) \arrow[r] \arrow[d, dotted, "D_\varphi"] & 0 \\ 0 \arrow[r] & i'T\oplus j'K \arrow[r] & L' \arrow[r] & L'/(i'T\oplus j'K) \arrow[r] & 0 \end{tikzcd}$$

Put $\psi:=(\varphi_T,\varphi_K)\in G_T\times\mathrm{O}(K)$. Using the identification of $L/(iT\oplus jK)$ with $H\subset D_{T\oplus K}$ obtained from i and j (and analogously for H'), the isomorphism D_φ on discriminants turns into the isomorphism $D_\psi\colon H\xrightarrow{\sim}H'$. Note that by construction $\psi^\vee|_L=\varphi$.

Given $\psi\in G_T\times\mathrm{O}(K)$, consider the induced isomorphism on the dual $\psi^\vee\colon(T\oplus K)^\vee\xrightarrow{\sim}(T\oplus K)^\vee$. By the assumption $D_\psi(H)=H'$, this isomorphism restricts to $\varphi:=\psi_T^\vee\oplus\psi_K^\vee|_{L_H}\colon L_H\xrightarrow{\sim}L_{H'}$. Finally, under the embeddings $i_H, j_H, i_{H'}, j_{H'}$ the induced isometries of φ combine to $(\varphi_T, \varphi_K) = \psi$. $\square$

Assumption 3.14. From now on we suppose that the embedding lattice is uniquely determined by the signature (derived from $T \oplus K$) and the discriminant form q. In other words, we postulate that there is a single lattice L in that genus, i.e. $\mathcal{P}(T,K,L) = \mathcal{P}(T,K,q_L)$.

We say that two primitive embeddings $(i,j),(i',j') \in \mathcal{P}(T,K,L)$ are G_L-*equivalent* if there is an isometry $f \in G_L$ such that $f(iT \oplus jK) = i'T \oplus j'K$.

Combining the two isometry subgroups $G_T \subset \mathrm{O}(T)$ and $G_L \subset \mathrm{O}(L)$, we say that $(i,j),(i',j') \in \mathcal{P}(T,K,L)$ are *equivalent up to G_L and G_T* if there is an isometry $f \in G_L$ such that $f(iT \oplus jK) = i'T \oplus j'K$ and $f(iT) = i'T$ and $f_T \in G_T$ for the induced isometry.

For later use, we now present a version of Lemma 3.13 in the presence of a subgroup G_L but with Assumption 3.14.

Lemma 3.15. *Assume that L is an overlattice of $T \oplus K$ which is unique within its genus. Let $H, H' \in \mathcal{Q}(T,K,q_L)$.*

Then $\mathsf{L}(H)$ and $\mathsf{L}(H')$ are equivalent in $\mathcal{P}(T,K,L)$ up to G_L and G_T if and only if there is an isometry $\psi \in G_T \times \mathrm{O}(K)$ such that $D_\psi(H) = H'$ and $\psi^\vee|_L \in G_L$.

Proof. Note that the G_L-action is well defined by Assumption 3.14. The proof of the lemma is the same as the one of Lemma 3.13, taking into account the additional assumption. $\square$

Lemma 3.16 ([30, Lemma 23]). *Let L be an overlattice of $T \oplus K$ such that L is unique in its genus and let K' be a lattice in the genus of K. Then there is a bijection $\mathcal{O}(T,K,L) \overset{1:1}{=} \mathcal{O}(T,K',L)$.*

In particular, there is a primitive embedding $K' \hookrightarrow L$ such that L becomes an overlattice of $T \oplus K'$.

Proof. We observe that the set $\mathcal{Q}(T,K,q) = \mathcal{Q}(T,K,L)$ does not really depend on K, but rather just on the discriminant form q_K. Hence from Lemma 3.12 and using Assumption 3.14 we get a chain of bijections

$$\mathcal{O}(T,K,L) \overset{1:1}{=} \mathcal{Q}(T,K,L) \overset{1:1}{=} \mathcal{Q}(T,K',L) \overset{1:1}{=} \mathcal{O}(T,K',L)$$

and hence the claim. $\square$

The situation is particularly nice for indefinite unimodular overlattices where we recover a result proved by Hosono et al.:

Corollary 3.17 ([24, Theorem 1.4]). *Let $T \oplus K$ be indefinite. Then there is a bijection $G_T \backslash \mathcal{Q}(T,K,0) \overset{1:1}{=} G_T \times \mathrm{O}(K) \backslash \mathrm{O}(D_K)$, where G_T acts on D_K via $G_T \hookrightarrow \mathrm{O}(T) \to \mathrm{O}(D_T) \xrightarrow{\sim} \mathrm{O}(D_K)$.*

Proof. We have $D_T \cong D_K$ by the following standard argument: the map $L = L^\vee \to T^\vee$ is surjective with kernel K, hence $L \cong T^\vee \oplus K$, and $T^\vee/T \cong (T^\vee \oplus K)/(T \oplus K) \cong L/(T \oplus K)$; similarly for $K^\vee/K$ by symmetry. Also, the forms on D_T and D_K coincide up to sign: $q_T \cong -q_K$. This also shows that subgroups H of Lemmas 3.13 and 3.15 are graphs of isomorphisms.

Therefore, primitive embeddings $T \hookrightarrow L$ are determined by anti-isometries $\gamma\colon D_T \xrightarrow{\sim} D_K$. If there exists such an embedding (and hence such an anti-isometry), this set is bijective to $\mathrm{O}(D_T)$. We deduce the claim from Lemma 3.15. □

Remark 3.18. Note that in the unimodular case ($q = 0$), the prescription of T and of the genus of the overlattice (i.e. just the signature in this case) already settles the genus of K by $q_K = -q_T$ and the signature of K is obviously fixed. This statement is wrong in the non-unimodular case: It can happen that a sublattice has two embeddings with orthogonal complements of different discriminant (so in particular of different genus) as in the following example.

Example 3.19. Let $T := \langle 2 \rangle$ with generator t and $L := U \oplus \langle 2 \rangle$ with generators $e, f \in U$, $x \in \langle 2 \rangle$. Consider the embeddings $\iota_1, \iota_2 \colon T \hookrightarrow L$ given by $\iota_1(t) = e + f$ and $\iota_2(t) = x$. Then, bases for the orthogonal complements are $\{e - f, x\} \subset \iota_1(T)^\perp$ and $\{e, f\} \subset \iota_2(T)^\perp$. Hence $\mathrm{disc}(\iota_1(T)^\perp) = 4$ but $\mathrm{disc}(\iota_1(T)^\perp) = 1$.

4 K3 surfaces

In this text, a K3 *surface* will mean a smooth compact complex surface which is simply connected and carries a nowhere vanishing 2-form. By classical surface theory, the latter two conditions are equivalent to zero irregularity ($H^1(X, \mathcal{O}_X) = 0$) and trivial canonical bundle ($\Omega^2_X \cong \mathcal{O}_X$). See [2, VIII] or [3] for details.

We denote the *Picard rank* of a K3 surface X by ρ_X. It is the number of independent line bundles on X. If X is projective, then ρ_X is also the number of independent divisor classes and always positive but not vice versa. The cohomology groups listed below carry lattice structures coming from the cup product on the second cohomology:

$H^2_X = H^2(X, \mathbb{Z})$	full second cohomology,	$\mathrm{sgn}(H^2_X) = (3, 19)$
T_X	transcendental lattice,	$\mathrm{sgn}(T_X) = (2, 20 - \rho_X)$
NS_X	Néron-Severi lattice,	$\mathrm{sgn}(NS_X) = (1, \rho_X - 1)$

where the signatures in the second and third cases are valid only for X projective. Following usage in algebraic geometry, we will often write $\alpha.\beta = (\alpha, \beta)$ for the pairing. Likewise, we will use the familiar shorthand $L.M$ for the pairing of the first Chern classes $c_1(L).c_1(M)$ of two line bundles L and M.

By Poincaré duality, H^2_X is a unimodular lattice; it follows from Wu's formula that the pairing is even. Indefinite, even, unimodular lattices are uniquely determined by their signature; we get that H^2_X is isomorphic to the so-called K3 *lattice* made up from three copies of the hyperbolic plane U and two copies of the negative E_8 lattice:

$$L_{\mathrm{K3}} = 3U \oplus 2E_8(-1).$$

The Néron-Severi and transcendental lattices are mutually orthogonal primitive sublattices of H^2_X. In particular, H^2_X is an overlattice of $T_X \oplus NS_X$.

We denote by ω_X the canonical form on X. It has type $(2,0)$ and is unique up to scalars, since $H^0(X,\Omega^2_X) = \mathbb{C}$ for a K3 surface. By abuse of notation, we also write ω_X for its cohomology class, so that $\omega_X \in T_X \otimes \mathbb{C}$. In fact, T_X is the smallest primitive submodule of H^2_X whose complexification contains ω_X.

As X is a complex Kähler manifold, the second cohomology H^2_X comes equipped with a pure Hodge structure of weight 2: $H^2_X \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$. Note that $H^{1,1}(X) = (\mathbb{C}\omega_X + \mathbb{C}\overline{\omega}_X)^\perp$. The transcendental lattice T_X is an irreducible Hodge substructure with unchanged $(2,0)$ and $(0,2)$ components.

A *Hodge isometry* of H^2_X (or T_X) is an isometry that maps each Hodge summand to itself. As the $(2,0)$-component is one-dimensional, Hodge isometries are just isometries $\varphi\colon H^2_X \xrightarrow{\sim} H^2_X$ with $\varphi_{\mathbb{C}}(\omega_X) = c\omega_X$ for some $c \in \mathbb{C}^*$. (Analogous for Hodge isometries of T_X.) If L is a lattice with Hodge structure, we denote the group of Hodge isometries by $\mathrm{O_H}(L)$.

The following two Torelli theorems are basic for all subsequent work. They say that essentially everything about a K3 surface is encoded in its second cohomology group, considered as a lattice with Hodge structure — for both the classical and derived point of view. (We repeat Orlov's result about equivalent surfaces up to derived equivalence.)

Classical Torelli Theorem for K3 surfaces. *Two* K3 *surfaces X and Y are isomorphic if and only if there is a Hodge isometry between their second cohomology lattices H^2_X and H^2_Y.*

Derived Torelli Theorem for K3 surfaces (Orlov). *Two projective* K3 *surfaces X and Y are derived equivalent if and only if there is a Hodge isometry between the transcendental lattices T_X and T_Y.*

See [3] or [2, §VIII] for the classical case (the latter reference gives an account of the lengthy history of this result), and [37] or [25, §10.2] for the derived version.

A *marking* of X is the choice of an isometry $\lambda_X\colon H^2_X \xrightarrow{\sim} L_{\mathrm{K3}}$. The period domain for K3 surfaces is the following open subset of the projectivised K3 lattice:

$$\Omega_{L_{\mathrm{K3}}} = \{\omega \in \mathbb{P}(L_{\mathrm{K3}} \otimes \mathbb{C}) \mid \omega.\omega = 0, \omega.\overline{\omega} > 0\}.$$

Since L_{K3} has signature $(3,k)$ with $k>2$, this set is connected. By the surjectivity of the period map [2, VIII.14], each point of $\Omega_{L_{\mathrm{K3}}}$ is obtained from a marked K3 surface. Forgetting the choice of marking by dividing out the isometries of the K3 lattice, we obtain a space $\mathcal{F}=\mathrm{O}(L_{\mathrm{K3}})\backslash\Omega_{L_{\mathrm{K3}}}$ parametrising all (unmarked) K3 surfaces. As is well known, $\mathcal{F}$ is a 20-dimensional, non-Hausdorff space. In particular, it is not a moduli space in the algebro-geometric sense.

Denote by $\mathrm{K3}_{\mathsf{FM}}$ the set of all K3 surfaces up to derived equivalence — two K3 surfaces get identified if and only if they are Fourier-Mukai partners, i.e. if and only if their transcendental lattices are Hodge-isometric. Its elements are the sets $\mathsf{FM}(X)$ of Fourier-Mukai partners of K3 surfaces X. One cannot expect this set to have a good analytic structure: the fibres of the map $\mathcal{F}\to \mathrm{K3}_{\mathsf{FM}}$ can become arbitrarily large (see [36]). On the other hand, any K3 surface has only finitely many FM partners ([7]), so that the fibres are finite at least.

Since the transcendental lattices determine D-equivalence by Orlov's derived Torelli theorem, the Fourier-Mukai partners of a K3 surface X are given by embeddings $T_X\subseteq L_{\mathrm{K3}}$, modulo automorphisms of T_X. This can be turned into a precise count:

Theorem (Hosono, Lian, Oguiso, Yau [24, Theorem 2.3]). *The set of Fourier-Mukai partners of a* K3 *surface X has the following partition*

$$\mathsf{FM}(X)=\coprod_{S\in\mathcal{G}(NS_X)}\mathrm{O}_{\mathsf{H}}(T_X)\times\mathrm{O}(S)\backslash\,\mathrm{O}(D_S)$$

with $\mathrm{O}(S)$ and $\mathrm{O}_{\mathsf{H}}(T_X)$ acting on $\mathrm{O}(D_S)$ as in Corollary 3.17 above.

The special case of a generic projective K3 surface, $\mathrm{rk}(NS_X)=1$, was treated before, leading to a remarkable formula reminiscent of classical genus theory for quadratic number fields (and proved along these lines):

Theorem (Oguiso [36]). *Let X be a projective* K3 *surface with* $\mathrm{Pic}(X)$ *generated by an ample line bundle of self-intersection $2d$. Then X has $2^{p(d)-1}$ FM partners, where $p(d)$ is the number of distinct prime factors of d, and $p(1)=1$.*

Oguiso's theorem can also be interpreted as a result about polarised K3 surfaces, which we turn to next. In particular, the number $2^{p(d)-1}$ is the order of $\mathrm{O}(D_{L_{2d}})/\langle\pm1\rangle$, where$L_{2d}$ is the replacement of the lattice L_{K3} in the polarised case.

5 Polarised K3 surfaces

A *semi-polarised* K3 *surface* of degree $d>0$ is a pair (X,h_X) of a K3 surface X together with a class $h_X\in NS_X$ of a nef divisor with $h_X^2=2d>0$. A nef divisor of positive degree is also called *pseudo-ample*. We recall that an effective divisor is nef if and only if it intersects all -2-curves nonnegatively [2, §VIII.3]. We will also assume that h_X is primitive, i.e. not a non-trivial integer multiple of another class.

We speak of a *polarised* K3 *surface* (X, h_X) if h_X is the class of an ample divisor. However, we call h_X the *polarisation*, even if it is just nef and not necessarily ample. For details, see [2, §VIII.22]. The relevant geometric lattice is the complement of the polarisation

$$H_X = (h_X)^{\perp}_{H^2_X} \qquad \text{non-unimodular of signature } (2,19).$$

which inherits lattice and Hodge structures from H^2_X.
On the side of abstract lattices, recall that $L_{\mathrm{K3}} \cong 3U \oplus 2E_8(-1)$; we denote the three orthogonal copies of U in L_{K3} by $U^{(1)}$, $U^{(2)}$, and $U^{(3)}$. Basis vectors e_i, f_i of $U^{(i)}$, defined by $e_i^2 = f_i^2 = 0$ and $e_i.f_i = 1$, always refer to such a choice. For $h \in L_{\mathrm{K3}}$ with $h^2 > 0$, set

$$\begin{aligned} L_h &= h^{\perp}_{L_{\mathrm{K3}}} && \text{non-unimodular of signature } (2,19), \\ L_{2d} &= 2U \oplus \langle -2d \rangle \oplus 2E_8(-1) && \text{the special case } h = e_3 + df_3. \end{aligned}$$

Since all primitive vectors of fixed length appear in a single $\mathrm{O}(L_{\mathrm{K3}})$-orbit by Eichler's criterion (Lemma 2.3), we can assume $h = e_3 + df_3$. Note that $H_X \cong L_{2d}$ as lattices. Obviously, $D_{L_{2d}}$ is the cyclic group of order $2d$. The non-unimodular summand $\langle -2d \rangle$ of L_{2d} is generated by $e_3 - df_3$; thus $D_{L_{2d}}$ is generated by the integer-valued functional $\frac{1}{2d}(e_3 - df_3, \cdot)$. The quadratic form $D_{L_{2d}} \to \mathbb{Q}/2\mathbb{Z}$ is then given by mapping this generator to the class of $\frac{-2d}{4d^2} = \frac{-1}{2d}$.

There are two relevant groups in this situation: the full isometry group $\mathrm{O}(L_{2d})$ and the subgroup $\tilde{\mathrm{O}}(L_{2d})$ of *stable* isometries which by definition act trivially on the discriminant $D_{L_{2d}}$. The next lemma gives another description of stable isometries.

Lemma 5.1. *The stable isometry group coincides with the group of L_{K3}-isometries stabilising h, i.e. $\tilde{\mathrm{O}}(L_{2d}) = \{g \in \mathrm{O}(L_{\mathrm{K3}}) \mid g(h) = h\}$.*

Proof. Given $g \in \mathrm{O}(L_{\mathrm{K3}})$ with $g(h) = h$, we make use of the fact that the discriminant groups of $h^\perp = L_{2d}$ and $\langle 2d \rangle$ (the latter generated by h) are isomorphic and their quadratic forms differ by a sign. This is true because these are complementary lattices in the unimodular L_{K3}; see the proof of Corollary 3.17. The induced maps on discriminants, $D_{g,h^\perp} \colon h^\perp \xrightarrow{\sim} h^\perp$ and $D_{g,h} \colon \langle 2d \rangle \xrightarrow{\sim} \langle 2d \rangle$, are the same under the above identification. Since $D_{g,h}$ is the identity by assumption, $D_{g,h^\perp}$ is, too. Hence, $g|_{L_{2d}}$ is stable.

On the other hand, any $f \in \mathrm{O}(L_{2d})$ allows defining an isometry $\tilde{f}$ of the lattice $L_{2d} \oplus \mathbb{Z}h$, by mapping h to itself. Note that $L_{2d} \oplus \mathbb{Z}h \subset L_{\mathrm{K3}}$ is an overlattice. If f is a stable isometry, i.e. $D_f = \mathrm{id}$, then $\tilde{f}$ extends to an isometry of L_{K3}. (This reasoning has already been alluded to in Remark 3.2.) $\square$

An isomorphism of semi-polarised K3 surfaces is an isomorphism of the surfaces respecting the polarisations. Here we recall two Torelli theorems which are essential for the construction of moduli spaces of K3 surfaces.

Strong Torelli Theorem for polarised K3 surfaces. *Given two properly polarised* K3 *surfaces (X, h_X) and (Y, h_Y), i.e. h_X and h_Y are ample classes, and a Hodge isometry $\varphi\colon H^2_X \xrightarrow{\sim} H^2_Y$ with $\varphi(h_X) = h_Y$, there is an isomorphism $f\colon Y \xrightarrow{\sim} X$ such that $\varphi = f^*$.*

This result only holds for polarised K3 surfaces. For semi-polarised K3 surfaces we have a different result, where we say that (X, h_X) and (Y, h_Y) are *isomorphic* if there is an isomorphism $f\colon X \xrightarrow{\sim} Y$ such that $f^*(h_Y) = h_X$.

Torelli Theorem for semi-polarised K3 surfaces. *Two semi-polarised* K3 *surfaces (X, h_X) and (Y, h_Y) are isomorphic if and only if there is a Hodge isometry $\varphi\colon H^2_X \xrightarrow{\sim} H^2_Y$ with $\varphi(h_X) = h_Y$.*

Proof. Let $\varphi\colon H^2_X \xrightarrow{\sim} H^2_Y$ be a Hodge isometry with $\varphi(h_X) = h_Y$. Since h_X is not ample we cannot immediately invoke the strong Torelli theorem. Following [3, p.151], let Γ be the subgroup of the Weyl group of Y generated by those roots of $H^2(Y, \mathbb{Z})$ which are orthogonal to h_Y. Then Γ acts transitively on the chambers of the positive cone, whose closure contains h_Y. Hence we can find an element $w \in \Gamma$ such that $w(h_Y) = h_Y$ and $w \circ \varphi$ maps the ample cone of X to the ample cone of Y. By the strong Torelli theorem $w \circ \varphi$ is now induced by an isomorphism $f\colon Y \xrightarrow{\sim} X$. This gives the claim. $\square$

The counterpart of the unpolarised period domain is the open subset

$$\Omega^{\pm}_{L_{2d}} = \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) = \mathbb{P}^{20} \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\} = \Omega_{L_{\mathrm{K3}}} \cap h^{\perp}$$

where we abuse notation to also write $h^{\perp}$ for the projectivised hyperplane. Obviously, both $\mathrm{O}(L_{2d})$ and its subgroup $\tilde{\mathrm{O}}(L_{2d})$ act on $\Omega^{\pm}_{L_{2d}}$. Since the signature of L_{2d} is $(2, 19)$, the action is properly discontinuous.

Furthermore, signature $(2, 19)$ also implies that $\Omega^{\pm}_{L_{2d}}$ has two connected components. These are interchanged by the (stable) involution induced by $\mathrm{id}_{U^{(1)}} \oplus (-\mathrm{id}_{U^{(2)}}) \oplus \mathrm{id}_{U^{(3)} \oplus 2E_8(-1)}$. Denote by $\Omega^{+}_{L_{2d}}$ one connected component; this is a type IV domain. Also, let $\mathrm{O}^{+}_{L_{2d}}$ and $\tilde{\mathrm{O}}^{+}_{L_{2d}}$ be the subgroups of the (stable) isometry group of L_{2d} fixing the component. They are both arithmetic groups, as they have finite index in $\mathrm{O}(L_{2d})$.

Next, let $\Delta \subset L_{2d}$ be the subset of all (-2)-classes, and for $\delta \in \Delta$ denote by $\delta^{\perp} \subset L_{2d} \otimes \mathbb{C}$ the associated hyperplane ('wall'). In analogy to the unpolarised case we define a parameter space as the quotient by the group action — however, there are certain differences to be explained below: let

$$\mathcal{F}_{2d} = \tilde{\mathrm{O}}(L_{2d})\backslash\Omega^{\pm}_{L_{2d}} = \tilde{\mathrm{O}}^{+}_{L_{2d}}\backslash\Omega^{+}_{L_{2d}}.$$

This space has an analytic structure as the quotient of a type IV domain by a group acting properly discontinuously. Furthermore, $\mathcal{F}_{2d}$ is actually quasi-projective by Baily-Borel [1]. Note that the group actions preserve the collection of walls $\delta^{\perp}$, which by abuse of notation are given the same symbol in the quotient. Hence, the group action also preserves the complement

$$\mathcal{F}^{\circ}_{2d} = \mathcal{F}_{2d} \setminus \bigcup_{\delta\in\Delta} \delta^{\perp}.$$

The definition of $\mathcal{F}^{\circ}_{2d}$ means that -2-classes orthogonal to the polarisation are transcendental. In other words, the polarisation is ample, as it is nef and non-zero on all -2-curves.

The subspace $\mathcal{F}^{\circ}_{2d}$ is the moduli space of pairs (X, h_X) consisting of (isomorphism classes of) a K3 surface X and the class h_X of an ample, primitive line bundle with $h_X^2 = 2d$: given such a pair, we choose a *marking*, i.e. an isometry $\lambda_X\colon H^2_X \xrightarrow{\sim} L_{\mathrm{K3}}$ such that $\lambda_X(h_X) = h$. This induces $\lambda_X|_{H_X}\colon H_X \xrightarrow{\sim} L_{2d}$ and gives the period point $\lambda_X(\omega_X) \in \Omega^{\pm}_{2d}$. Since h_X is ample, the period point avoids the walls.

Conversely, given an $\tilde{\mathrm{O}}(L_{2d})$-orbit of a point $[\omega] \in \Omega^{\pm}_{2d}$ not on any wall, we get a pair (X, h_X) by considering $[\omega]$ as period point for the full K3 lattice: this uses the surjectivity of the period map. Now our assumptions on ω imply $h_X^2 = 2d$ and that h_X is ample as ω avoids the walls. Then, the strong Torelli theorem says that both the K3 surface X and the polarisation h_X are unique (up to isomorphism).

Finally, using again the surjectivity of the period map, one can find for every element $[\omega] \in \mathcal{F}_{2d}\backslash\mathcal{F}^{\circ}_{2d}$ a semi-polarised K3 surface (X, h_X) and a marking $\varphi\colon H^2(X,\mathbb{Z}) \to L_{\mathrm{K3}}$ with $\varphi(h_X) = h$, $\varphi([\omega_X]) = \omega$. The fact that the points contained in $\mathcal{F}_{2d}$ correspond to isomorphism classes of semi-polarised K3 surfaces of degree $2d$ now follows from the Torelli theorem for semi-polarised K3 surfaces.

Example 5.2. For $d = 1$, the smallest example of a proper semi-polarisation (i.e. nef, not ample) occurs for a generic elliptic K3 surface X with section. Its Néron-Severi lattice will be generated by the section s and a fibre f. The intersection form on $NS(X)$ is $\left(\begin{smallmatrix}-2 & 1\\ 1 & 0\end{smallmatrix}\right)$ and we set $D := s + 2f$. This effective divisor is primitive and nef as $D.s = 0$, $D.f = 1$, $D^2 = 2$.

We remark that the lattice L_{2d} is more difficult to work with than L_{K3} as it is not unimodular anymore. On the other hand, the moduli space $\mathcal{F}_{2d}$ of $2d$-polarised K3 surfaces is a quasi-projective variety which is a huge improvement over $\mathcal{F} = \mathrm{O}(L_{\mathrm{K3}})\backslash\Omega_{L_{\mathrm{K3}}}$.

In the polarised case, another natural quotient appears, taking the full isometry group of the lattice L_{2d}:

$$\hat{\mathcal{F}}_{2d} = \mathrm{O}(L_{2d})\backslash\Omega^{\pm}_{2d}.$$

(The unpolarised setting has $D_{L_{\mathrm{K3}}} = 0$ and hence there is only one natural group to quotient by.)

There is an immediate quotient map $\pi\colon \mathcal{F}_{2d} \to \hat{\mathcal{F}}_{2d}$. It has finite fibres and was investigated by Stellari:

Lemma 5.3 ([45, Lemma 2.3]). *The degree of* π *is* $\deg(\pi) = 2^{p(d)-1}$ *where* $p(d)$ *is the number of distinct primes dividing* d.

Proof. The degree is given by the index $[\mathrm{O}(L_{2d}) : \tilde{\mathrm{O}}(L_{2d})]$ up to the action of the non-stable isometry $-\mathrm{id}$ which permutes the two components.

We use the exact sequence $0 \to \tilde{\mathrm{O}}(L_{2d}) \to \mathrm{O}(L_{2d}) \to \mathrm{O}(D_{L_{2d}}) \to 0$ where the right-hand zero follows from [35, Theorem 1.14.2]. The index thus equals the order of the finite group $\mathrm{O}(D_{L_{2d}})$. Now $D_{L_{2d}}$ is the cyclic group of order $2d$ and decomposes into the product of various p-groups. Automorphisms of $D_{L_{2d}}$ factorise into automorphisms of the p-groups. However, the only automorphisms of $\mathbb{Z}/p^l$ respecting the quadratic (discriminant) form are those induced by $1 \mapsto \pm 1$. Hence $|\,\mathrm{O}(D_{L_{2d}})| = 2^{p(d)}$. The degree is then $2^{p(d)-1}$, taking the non-stable isometry $-\mathrm{id}$ of L_{2d} into account.

In case $d = 1$, we have $\tilde{\mathrm{O}}(L_{2d}) = \mathrm{O}(L_{2d})$, fitting with $p(1) = 1$. $\square$

Points of $\hat{\mathcal{F}}_{2d}$ or rather the fibres of π have the following property:

Lemma 5.4. *Given semi-polarised* K3 *surfaces* (X, h_X) *and* (Y, h_Y) *with* $\pi(X, h_X) = \pi(Y, h_Y)$*, there are Hodge isometries* $H_X \cong H_Y$ *and* $T_X \cong T_Y$*. In particular,* X *and* Y *are FM partners.*

Proof. Fix markings $\lambda_X\colon H^2_X \xrightarrow{\sim} L_{\mathrm{K3}}$ and $\lambda_Y\colon H^2_Y \xrightarrow{\sim} L_{\mathrm{K3}}$ with $\lambda_X(h_X) = \lambda_Y(h_Y) = h$ and $\lambda_X(\omega_X) = \omega$, $\lambda_Y(\omega_Y) = \omega'$. By $\pi(X, h_X) = \pi(Y, h_Y)$, there is some $g \in \mathrm{O}(L_{2d})$ such that $g(\lambda_X(\omega_X)) = \lambda_Y(\omega_Y)$. In particular, the primitive lattices generated by ω_X and ω_Y (which are the transcendental lattices T_X and T_Y) get mapped into each other by $\lambda_Y^{-1} \circ g \circ \lambda_X$. Thus, the latter isometry respects the Hodge structures, and induces Hodge isometries $T_X \xrightarrow{\sim} T_Y$ and $H_X = h_X^\perp \xrightarrow{\sim} H_Y = h_Y^\perp$. $\square$

6 Polarisation of FM partners

In this section, we want to consider the relationship between polarisations and FM partners. A priori these concepts are very different: the condition that two K3 surfaces are derived equivalent is a property of their transcendental lattices, whereas the existence of polarisations concerns the Néron-Severi group. Indeed, we shall see that there are FM partners where one K3 surface carries a polarisation of given degree but the other does not. On the opposite side, we shall see in the next Section 7 that one can count the number of FM partners among polarised K3 surfaces of a given degree.

Introduce the set

$$\mathrm{K3}^{2d}_{\mathsf{FM}} := \left\{ X \;\middle|\; \begin{array}{l}\text{K3 surface admitting a primitive}\\ \text{nef line bundle } L \text{ with } L^2 = 2d\end{array} \right\} / \sim$$

where $X \sim Y$ if and only if $D^b(X) \cong D^b(Y)$.

We shall first discuss two examples which shed light on the relationship between FM partnership and existence of polarisations.

Example 6.1. Derived equivalence does not respect the existence of polarisations of a given degree. To give an example we use the rank 2 Néron-Severi lattices defined by $\left(\begin{smallmatrix} 0 & -7 \\ -7 & -2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -7 \\ -7 & 10 \end{smallmatrix}\right)$ which are related by the rational base change $\frac{1}{3}\left(\begin{smallmatrix} 1 & 0 \\ -2 & 9 \end{smallmatrix}\right)$. The former obviously represents -2 whereas the latter does not. Furthermore, the latter primitively represents 10 via the vector $(0,1)$ and 6 via the vector $(2,3)$ whereas the former does not. For example, if we had $10 = -14xy - 2y^2$, then y would have to be one of $1,2,5,10$ up to sign and neither of these eight cases works.

The orthogonal lattices in L_{K3} are isomorphic, as follows from Nikulin's criterion. Denote the common orthogonal complement by T. As in the previous example, we choose a general vector $\omega \in T_{\mathbb{C}}$ with $(\omega,\omega) = 0$, $(\omega,\overline{\omega}) > 0$. We see that T admits primitive embeddings $\iota, \iota' \colon T \hookrightarrow L_{\mathrm{K3}}$ such that $\iota(T)^\perp$ does not contain any vectors of square $2d$ whereas $\iota'(T)^\perp$ does, for $d = 3$ or $d = 5$. Furthermore, $\iota'(T)^\perp$ does not contain any -2-classes. The surface X' corresponding to $\iota'(\omega) \in T_{\mathbb{C}}$ can actually be $2d$-polarised since $\iota'(\omega) \in L_h^\perp \cong L_{2d}$ and we have $\mathcal{F}^\circ_{2d} = \mathcal{F}_{2d}$ by the absence of -2-classes. On the other hand, the surface X corresponding to $\iota(\omega)$ has no $2d$-(semi)polarisations — there are not even classes in $NS(X)$ of these degrees.

Example 6.2. Let $d > 1$ be an integer, not divisible by 3 such that $2d$ can be represented primitively by the positive definite root lattice A_2 (e.g. $d = 7$). Let $T = 2U \oplus E_8(-1) \oplus E_6(-1) \oplus \langle -2d \rangle$. Following an idea of M. Schütt we construct two primitive embeddings $\iota, \iota' \colon T \hookrightarrow L_{2d}$. Both of them are the identity on $2U \oplus E_8(-1)$. On the $E_6(-1) \oplus \langle -2d \rangle$ part of T, we use

$$\begin{array}{lll} \iota\colon & E_6(-1) \hookrightarrow E_8(-1), \langle -2d \rangle \xrightarrow{\sim} \langle -2d \rangle & \text{with } \iota(T)^\perp_{L_{2d}} \cong A_2(-1) \\ \iota'\colon & E_6(-1) \oplus \langle -2d \rangle \hookrightarrow E_8(-1) & \text{with } \iota'(T)^\perp_{L_{2d}} \cong \langle -6d \rangle \oplus \langle -2d \rangle. \end{array}$$

We choose a general point $\omega \in T_{\mathbb{C}}$ with $(\omega,\omega) = 0$ and $(\omega,\bar{\omega}) > 0$. Then the points $\iota(\omega)$ and $\iota'(\omega)$ in $\mathcal{F}_{2d}$ represent (semi-)polarised K3 surfaces (X, h_X) and $(X', h_{X'})$. In the first case h_X is only semi-polarised, as $A_2(-1)$ contains (-2)-vectors, in other words $\iota(\omega) \in \mathcal{F}_{2d} \setminus \mathcal{F}^\circ_{2d}$. In the second case $\iota(\omega) \in \mathcal{F}^\circ_{2d}$ since the orthogonal complement of $h_{X'}$ in $NS(X')$ equals $\langle -2d \rangle \oplus \langle -6d \rangle$ which does not contain a (-2)-class. This shows that there are examples of polarised and semi-polarised K3 surfaces of the same degree which have the same FM partner.

Incidentally we notice that $NS(X) \cong NS(X') \cong A_2(-1) \oplus \langle 2d \rangle$. This follows from Nikulin's criterion since both lattices have rank 3 and length 1 since we have assumed that $(3,d) = 1$.

We want to construct a map

$$\tau\colon \mathcal{F}_{2d} \to \mathrm{K3}^{2d}_{\mathsf{FM}}.$$

By Lemma 5.4, we have a map $\pi\colon \hat{\mathcal{F}}_{2d} \to \mathrm{K3}^{2d}_{\mathsf{FM}}$; combining it with $\sigma\colon \mathcal{F}_{2d} \to \hat{\mathcal{F}}_{2d}$, we obtain a commutative triangle

$$\begin{tikzcd} \mathcal{F}_{2d} \arrow[rr, "\sigma"] \arrow[dr, "\tau"'] & & \hat{\mathcal{F}}_{2d} \arrow[dl, "\pi"] \\ & \mathrm{K3}^{2d}_{\mathsf{FM}} & \end{tikzcd}$$

By the counting results of Proposition 7.1, the fibres are finite. Here we give a geometric argument for that fact, following Stellari [46, Lemma 2.5], where we pay special attention to the 'boundary points' of $\mathcal{F}_{2d}$.

Proposition 6.3. *Given a $2d$-(semi-)polarised* K3 *surface (X, h_X), there are only finitely many $2d$-(semi-)polarised* K3 *surfaces (Y, h_Y) up to isomorphism with $D^b(X) \cong D^b(Y)$.*

Proof. Disregarding polarisations, there are only finitely many FM partners of X, as X is a smooth projective surface [7]. Given such an FM partner Y, consider the set $A_{Y,2d} = \{c \in \mathcal{C}(Y) \mid c^2 = 2d\}$ of elements of length $2d$ in the positive cone of Y. If the divisor c is ample, then $3c$ is very ample, by Saint-Donat's result [43]. By Bertini, there are irreducible divisors $D \in |3c|$. The set $B_{Y,18d} = \{\mathcal{O}_X(D) \mid D^2 = 18d, D \text{ irreducible}\}$ of divisor classes of irreducible divisors of length $18d$ is finite up to automorphisms of Y by Sterk [47]. As $A_{Y,2d} \to B_{Y,18d}$, $c \mapsto |3c|$ is injective (this uses $H^1(\mathcal{O}_Y) = 0$), this shows that the number of non-isomorphic $2d$-polarisations on Y is finite.

However, there are points (Y, h_Y) where h_Y is only pseudo-ample. Denote the set of pseudo-ample divisors of degree $2d$ by $\overline{A}_{Y,2d} = \{c \in \overline{\mathcal{C}}(Y) \mid c^2 = 2d\}$. If we have a non-ample polarisation h_Y, then contracting the finitely many -2-curves which are orthogonal to h_Y produces a projective surface Y' with only ADE singularities, trivial canonical bundle and $H^1(\mathcal{O}_{Y'}) = 0$. Morrison shows that Saint-Donat's result is also true for this surface [31, §6.1], i.e. $3c$ is again very ample. We can then proceed as above, as the generic divisor in $|3c|$ will be irreducible and avoid the finitely many singularities. Sterk's result on finiteness of $B_{Y',18d}/\mathrm{Aut}(Y')$ still applies as he simply assumes that a linear system is given whose generic member is irreducible. □

7 Counting FM partners of polarised K3 surfaces in lattice terms

Taking our cue from the fact that the fibres of $\mathcal{F} \to \mathcal{F}/\mathsf{FM}$ are just given by FM partners (the unpolarised case), and the latter can be counted in lattice terms, we study the following general setup: let L be an indefinite, even lattice, let T be another lattice, occurring as a sublattice of L, and let finally $G_T \subseteq \mathrm{O}(T)$ and $G_L \subseteq \mathrm{O}(L)$ be two subgroups, the latter normal. As in Section 3, we consider the set $\mathcal{P}(T,L)$ of all primitive embeddings $\iota\colon T \hookrightarrow L$. This set is partitioned into $\mathcal{P}(T,K,L)$, containing all primitive embeddings $\iota\colon T \hookrightarrow L$ with $\iota(T)^\perp_L \cong K$.

In the application to geometry, we will have $L = L_{2d} = h^\perp$ the perpendicular lattice of the polarisation inside the K3 lattice, $T = T_X$ the transcendental lattice of a K3 surface X and $K = NS(X)$ the Néron-Severi lattice of X. By Nikulin's criterion (Lemma 2.2), L_{2d} is unique in its genus, thus fulfilling Assumption 3.14. As to the groups, $G_T = \mathrm{O}_\mathsf{H}(T)$ is the group of Hodge isometries of T_X and G_L is either the full or the stable isometry group of L_{2d}.

We recall when two embeddings $\iota_1, \iota_2\colon T \hookrightarrow L$ are equivalent with respect to G_T and G_L (see page 18): if there are isometries $g \in G_T$ and $\tilde{g} \in G_L$ such that

$$\begin{tikzcd} T \arrow[r, "\iota_1"] \arrow[d, "g"] & L \arrow[d, "\tilde{g}"] \\ T \arrow[r, "\iota_2"] & L \end{tikzcd}$$

This corresponds to orbits of the action $G_T \times G_L \times \mathcal{P}(T,L) \to \mathcal{P}(T,L)$, $(g,\tilde{g}) \cdot \iota = \tilde{g}\iota g^{-1}$.

All of this is essentially the setting of [24] — the novelty is the subgroup G_L, which always was the full orthogonal group in loc. cit.

Proposition 7.1. *For a* $2d$*-polarised* K3 *surface* (X, h_X)*, there are bijections*

$$\sigma^{-1}([X,h_X]) \overset{1:1}{=} \mathrm{O}_\mathsf{H}(T_X) \times \mathrm{O}(H_X) \backslash \mathcal{P}(T_X, H_X),$$
$$\tau^{-1}([X,h_X]) \overset{1:1}{=} \mathrm{O}_\mathsf{H}(T_X) \times \tilde{\mathrm{O}}(H_X) \backslash \mathcal{P}(T_X, H_X).$$

Remark 7.2. The unpolarised analogue of the proposition was given in Theorem 2.4 of [24], stating $\mathsf{FM}(X) = \mathrm{O}_\mathsf{H}(T_X) \times \mathrm{O}(H^2_X) \backslash \mathcal{P}(T_X, H^2_X)$.

Proof. The proof proceeds along the lines of [24, Theorem 2.4]. Fix a marking $\lambda_X\colon H_X \xrightarrow{\sim} L_{2d}$ for X. Set $T := \lambda_X(T_X)$. This yields a primitive embedding

$$\iota_0\colon\ T \xrightarrow[\lambda_X^{-1}]{\sim} T_X \hookrightarrow H_X \xrightarrow[\lambda_X]{\sim} L_{2d}\ .$$

This embedding (or rather the equivalence class of $\iota_0\lambda_X(\omega_X)$) gives a point in $\mathcal{F}_{2d}$. By definition of $\mathcal{F}_{2d}$, this period point does not depend on the choice of marking.

If (Y,h_Y) belongs to (a period point given by an embedding in) $\mathcal{P}(T,L_{2d})$, then — as the transcendental lattice is the smallest lattice containing the canonical form in its complexification — there is a Hodge isometry $T_X \cong T_Y$, hence $D^b(X) \cong D^b(Y)$ and then $\mathsf{FM}(X) = \mathsf{FM}(Y)$. We therefore get maps

$$\begin{aligned}
&\tilde{c}\colon \mathcal{P}(T,L_{2d}) \to \tau^{-1}(X,h_X),\\
&c\colon \mathcal{P}(T,L_{2d}) \xrightarrow{\tilde{c}} \tau^{-1}(X,h_X) \xrightarrow{\pi} \sigma^{-1}(X,h_X).
\end{aligned}$$

with the fiber $\tau^{-1}(X,h_X)$ consisting of FM partners of (X,h_X) up to isomorphism.

The map $\tilde{c}$ is surjective (and hence c is, as well): if $(Y,h_Y) \in \mathcal{F}_{2d}$ is an FM partner of X, then we first fix a marking $\lambda_Y\colon H_Y \xrightarrow{\sim} L_{2d}$. By the derived Torelli theorem, there is a Hodge isometry $g\colon T_X \xrightarrow{\sim} T_Y$. Using g and the markings for X and Y, we produce an embedding

$$\iota\colon\ T_X \xrightarrow[g]{\sim} T_Y \hookrightarrow H_Y \xrightarrow[\lambda_Y]{\sim} L_{2d} \xrightarrow[\lambda_X^{-1}]{\sim} H_X\ .$$

This gives a point $\iota \in \mathcal{P}(T_X,H_X)$ and by construction, $\tilde{c}(\iota) = (Y,h_Y) \in \tau^{-1}(X,h_X)$.

For brevity, we temporarily introduce shorthand notation

$$\begin{aligned}
\mathcal{P}^{\mathrm{eq}}(T_X,H_X) &:= \mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H_X)\backslash\mathcal{P}(T_X,H_X),\\
\tilde{\mathcal{P}}^{\mathrm{eq}}(T_X,H_X) &:= \mathrm{O}_{\mathsf{H}}(T_X) \times \tilde{\mathrm{O}}(H_X)\backslash\mathcal{P}(T_X,H_X).
\end{aligned}$$

and the goal is to show $\tilde{\mathcal{P}}^{\mathrm{eq}}(T_X,H_X) = \tau^{-1}([X,h_X])$ and $\mathcal{P}^{\mathrm{eq}}(T_X,H_X) = \sigma^{-1}([X,h_X])$.

Now suppose that two embeddings $\iota,\iota'\colon T \hookrightarrow L_{2d}$ give the same equivalence class in $\tilde{\mathcal{P}}^{\mathrm{eq}}(T,L_{2d})$. This means that there exist isometries $g \in \mathrm{O}(T)$ and $\tilde{g} \in \tilde{\mathrm{O}}(L_{2d})$ with $\iota' \circ g = \tilde{g} \circ \iota$. Denote the associated polarised K3 surfaces by (Y,h_Y) and $(Y',h_{Y'})$; choose markings λ_Y and $\lambda_{Y'}$ as above. Then we obtain a Hodge isometry

$$\begin{tikzcd}
H_Y \arrow[r, "\sim", "\lambda_Y"'] & L_{2d} \arrow[r, "\sim", "\tilde{g}"'] & L_{2d} \arrow[r, "\sim", "\lambda_{Y'}^{-1}"'] & H_{Y'} \\
T_Y \arrow[u, hook] \arrow[r, "\sim"] & T \arrow[u, hook, "\iota"'] \arrow[r, "\sim", "g"'] & T \arrow[u, hook, "\iota'"'] \arrow[r, "\sim"] & T_{Y'} \arrow[u, hook]
\end{tikzcd}$$

and hence (Y,h_Y) and $(Y',h_{Y'})$ define the same point in $\mathcal{F}_{2d}$. Thus the map $\tilde{c}$ factorises over equivalences classes and descends to a surjective map $\tilde{c}\colon \tilde{\mathcal{P}}^{\mathrm{eq}}(T,L_{2d}) \to \tau^{-1}(X,h_X)$.

Analogous reasoning applies if ι and ι' are equivalent in $\mathcal{P}^{\mathrm{eq}}(T,L_{2d})$: we get isometries $g\in \mathrm{O}(T)$ and $\hat{g}\in \mathrm{O}(L_{2d})$ with $\iota'\circ g=\hat{g}\circ\iota$ and use a diagram similar to the one above. In this case, with the isometry $\hat{g}$ not necessarily stable, we can only derive that the period points coincide in $\hat{\mathcal{F}}_{2d}$; hence $c\colon \mathcal{P}^{\mathrm{eq}}(T,L_{2d})\to\sigma^{-1}(X,h_X)$.

Finally, we show that these maps are injective, as well. Let $[\iota],[\iota']\in\tilde{\mathcal{P}}^{\mathrm{eq}}(T,L_{2d})$ be two equivalence classes of embeddings with $\tilde{c}([\iota])=\tilde{c}([\iota'])$. This implies the existence of a stable isometry in $\mathrm{O}(L_{2d})$ mapping $\omega\mapsto\omega'$ where ω and ω' are given by the construction of the map $\tilde{c}$ (they correspond to semi-polarised K3 surfaces (Y,h_Y) and $(Y',h_{Y'})$). Using markings $H_Y\xrightarrow{\sim}L_{2d}$ and $H_{Y'}\xrightarrow{\sim}L_{2d}$, we get an induced Hodge isometry $\varphi\colon H_Y\xrightarrow{\sim}H_{Y'}$ with $\varphi(\omega_Y)=\omega_{Y'}$ and $\varphi(h_Y)=h_{Y'}$. Once more invoking the minimality of transcendental lattices, we also get a Hodge isometry $\varphi_T\colon T_Y\xrightarrow{\sim}T_{Y'}$. These isometries combine to

$$\begin{tikzcd}
L_{2d} \arrow[r, "\sim", "\lambda_Y^{-1}"'] & H_Y \arrow[r, "\sim", "\varphi"'] & H_{Y'} \arrow[r, "\sim", "\lambda_{Y'}"'] & L_{2d} \\
T \arrow[u, hook, "\iota"] \arrow[r, "\sim"] & T_Y \arrow[u, hook] \arrow[r, "\sim", "\varphi_T"'] & T_{Y'} \arrow[u, hook] \arrow[r, "\sim"] & T \arrow[u, hook, "\iota'"']
\end{tikzcd}$$

the outer square of which demonstrates $[\iota]=[\iota']$.

For $[\iota],[\iota']\in\mathcal{P}^{\mathrm{eq}}(T,L_{2d})$ with $c([\iota])=c([\iota'])$, we argue analogously, only now starting with an isometry of L_{2d} mapping $\omega\to\omega'$ which is not necessarily stable. Since periods of the K3 surfaces get identified up to $\mathrm{O}(L_{2d})$ in this case, the outer square gives $[\iota]=[\iota']$ up to $\mathrm{O}_{\mathsf{H}}(T)$ and $\mathrm{O}(L_{2d})$. □

From Lemma 3.15 and Proposition 7.1, we derive the following statement.

Proposition 7.3. *Given an $2d$-polarised* K3 *surface (X,h_X), there are bijections*

$$\sigma^{-1}([X,h_X]) \overset{1:1}{=} \coprod_S \mathrm{O}_{\mathsf{H}}(T_X)\times \mathrm{O}(H_X)\backslash\mathcal{Q}(T_X,S,q_{2d})$$
$$\tau^{-1}([X,h_X]) \overset{1:1}{=} \coprod_S \mathrm{O}_{\mathsf{H}}(T_X)\times \tilde{\mathrm{O}}(H_X)\backslash\mathcal{Q}(T_X,S,q_{2d}).$$

where the unions run over isomorphism classes of even lattices S which admit an overlattice $S\oplus T_X\hookrightarrow H_X$ such that the induced embedding $S\subseteq H_X$ is primitive. The discriminant q_{2d} on the right-hand sides has $D_{q_{2d}}=\mathbb{Z}/2d$ as abelian group with $q_{2d}(1)=\frac{-1}{2d}$.

Proof. The fibres are obviously partitioned by the orthogonal complements that can occur (this is in general a bigger choice than just of an element in the genus).

Once a complement S is chosen, then the set $\mathcal{P}^{\mathrm{eq}}(T_X,S,H_X)$, i.e. the set of embeddings of T_X into H_X with complement isomorphic to S, up to Hodge isometries of T_X and isometries of H_X, coincides with $(\mathrm{O}(H_X), \mathrm{O}_{\mathsf{H}}(T_X))$-equivalence classes; this follows from Lemma 3.15. Analogous reasoning addresses the σ-fibres. □

Remark 7.4. Following Remark 7.2 and Lemma 3.12, we note that the formula for FM partners of an unpolarised K3 surface X from [24] can be written as

$$\mathsf{FM}(X) = \coprod_S \mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H^2_X) \backslash \mathcal{Q}(T_X, S, 0)$$

where S now runs through isomorphism classes of lattices admitting an overlattice $S \oplus T_X \hookrightarrow H^2_X$ such that S is primitive in H^2_X. As H^2_X is unimodular, the genus of S is uniquely determined by that of T_X (see Remark 3.18). One candidate for S is $NS(X)$ and we can describe $\mathsf{FM}(X)$ as the same union, with S running through the genus $\mathcal{G}(NS_X)$. By Lemma 3.16, the sets $\mathcal{Q}(T_X, S, 0)$ are all mutually bijective.

We will temporarily work with the sets $\mathcal{P}(T_X, S, H^2_X)$ instead. On each such set, $\mathrm{O}_{\mathsf{H}}(T_X)$ and $\mathrm{O}(H^2_X)$ act in the natural way. Hence, there are not just bijections $\mathcal{P}(T_X, S, H^2_X) \overset{1:1}{=} \mathcal{P}(T_X, S', H^2_X)$ but also bijections of the quotients by $\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H^2_X)$. We thus get

$$\begin{aligned}\mathsf{FM}(X) &\overset{1:1}{=} \mathcal{G}(NS_X) \times \big(\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H^2_X) \backslash \mathcal{P}(T_X, NS_X, H^2_X) \big) \\ &\overset{1:1}{=} \mathcal{G}(NS_X) \times \big(\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H^2_X) \backslash \mathcal{Q}(T_X, NS_X, 0) \big).\end{aligned}$$

However, by Corollary 3.17, this implies

$$\mathsf{FM}(X) \overset{1:1}{=} \mathcal{G}(NS_X) \times \big(\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(NS_X) \backslash \mathrm{O}(D_{NS_X}) \big).$$

Similar formulae in the polarised (hence non-unimodular) case are generally wrong.

8 Examples

Proposition 7.1 phrases the problem of classifying polarised K3 surfaces up to derived equivalence in lattice terms. Using the results of Section 3 this can be rephrased as Proposition 7.3 which clearly makes this a finite problem. Given $h_X \in L_{\mathrm{K3}}$ primitive and $T_X \subseteq H_X = h_X^{\perp}$, or equivalently, $T_X \subset L_{2d}$, one can (in principle) list all potential subgroups H of the discriminant group. This, together with the fact that $\mathrm{Hom}(L_{2d}, H)$ is finite, makes it possible to test all potential overlattice groups.

Picard rank one

We consider the special case of Picard rank 1. Here, $h_X^\perp = T_X$. Also, any FM partner of a $2d$-polarised K3 surface is again canonically $2d$-polarised (since the orthogonal complement of the transcendental lattice is necessarily of the form $\langle -2d\rangle$). Oguiso showed that the number of non-isomorphic FM partners is $2^{p(d)-1}$ (where $p(d)$ is the number of prime divisors) [36]. This is also half of the order of $\mathrm{O}(D_{L_{2d}})$.

Stellari [46, Theorem 2.2] shows: the group $\mathrm{O}(D_{L_{2d}})/\{\pm\mathrm{id}\}$ acts simply transitively on the fibre $\tau^{-1}(X, h_X)$. In particular, σ is one-to-one on these points.

We look at the situation from the point of view of Proposition 7.3. In this case $T_X = H_X$ and $\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H_X) \backslash \mathcal{P}(T_X, H_X)$ clearly contains only one element, which says that the fibre of σ contains only one element. The situation is different for τ. For this we have to analyse the action of the quotient $\mathrm{O}_{\mathsf{H}}(T_X) \times \mathrm{O}(H_X) / \mathrm{O}_{\mathsf{H}}(T_X) \times \tilde{\mathrm{O}}(H_X) \cong O(D_{L_{2d}})$. We note that $-\mathrm{id}$ is contained in both $\mathrm{O}_{\mathsf{H}}(T_X)$ and $\mathrm{O}(H_X)$, and the element $(-\mathrm{id}, -\mathrm{id})$ acts trivially on $\mathcal{P}(T_X, H_X)$. On the other hand, since the Picard number of X is 1 it follows that every element in $\mathrm{O}_{\mathsf{H}}(T_X)$ extends to an isometry of $H^2(X)$ which maps h to $\pm h$. Hence the group $O(D_{L_{2d}})/\langle \pm 1\rangle$ acts transitively and freely on the fibre of τ showing again that the number of FM-partners equals $2^{p(d)-1}$.

Large Picard rank

For Picard ranks of 12 or more, derived equivalence implies isomorphism since any Hodge isometry of T_X lifts to an isometry of H^2_X, using [35, 1.14.2]: we have $\ell(NS_X) = \ell(T_X) \leq \mathrm{rk}(T_X) = 22 - \varrho(X)$ for the minimal number of generators of D_{NS_X}; the lifting is possible if $2 + \ell(NS_X) \leq \mathrm{rk}(NS_X) = \varrho(X)$ — hence $\ell \leq 10$ and $\varrho \geq 12$. Still, there can be many non-isomorphic polarisations on the same surface. See below for an example where the fibres of τ and σ can become arbitrarily large in the case of $\varrho = 20$ maximal.

Positive definite transcendental lattice

We consider the following candidates for transcendental lattices: $T = \left(\begin{smallmatrix} 2a & 0 \\ 0 & 2b \end{smallmatrix}\right)$ with $a > b > 0$. We denote the standard basis vectors for T by u and v, so that $u^2 = 2a$ and $v^2 = 2b$. Note that the only isometries of this lattice are given by sending the basis vectors u, v to $\pm u, \pm v$. In the lattice $L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d\rangle$, denote by l a generator of the non-unimodular summand $\langle -2d\rangle$.

In this setting, we are looking for embeddings $\iota_1, \iota_2 \colon T \hookrightarrow L_{2d}$ such that $\iota_1(v)$ and $\iota_2(v)$ belong to different $\mathrm{O}(L_{2d})$-orbits. This would then immediately imply that the two embeddings cannot be equivalent. In order to show this, we appeal to Eichler's criterion.

Let us restrict to the special case $d = b = p^3$ for a prime p. Recall that the *divisor* of a vector w is the positive generator of the ideal (w, L_{2d}). We want the divisor of the vector v to be p^2. Setting, for $c \in \mathbb{Z}$,

$$v_c := p^2 e_2 + p(1+c^2)f_2 + cl,$$

we have $v_c^2 = 2p^3$ and $\mathrm{div}(v_c) = (p^2, p(1+c^2), 2cp^3)$. Choosing c with $1+c^2 \equiv 0\ (p)$, which enforces $p \equiv 1\ (4)$, we get $\mathrm{div}(v_c) = p^2$. Now, by Eichler's criterion, the $\tilde{\mathrm{O}}(L_{2p^3})$-orbit of v_c is determined by the length $v_c^2 = 2p^3$ and the class $[v_c/\mathrm{div}(v_c)] = [c/p^2] \in D_{L_{2p^3}}$. The latter discriminant group is cyclic of order $2p^3$. Hence, the number of orbits of vectors with length $2p^3$ and divisor p^2 equals the number of solutions of $1 + c^2 \equiv 0\ (p)$ for $c = 0, \dots p^2 - 1$. The equation $1+c^2 \equiv 0$ has two solutions in $\mathbb{Z}/p$, as $p \equiv 1\ (4)$, hence $2p$ solutions in $\mathbb{Z}/p^2$. (The above computation is a very special case of the obvious adaption of [20, Prop. 2.4] to the lattice L_{2d}.)

Together with $u := e_1 + af_1$, we get $2p$ lattices $T_c = \langle u, v_c \rangle$ embedded into L_{2p^3}, and such that these embeddings are pairwise non-equivalent under the action of $\tilde{\mathrm{O}}(L_{2p^3})$. The discriminant group of L_{2p^3} is $D_{L_{2p^3}} \cong \mathbb{Z}/2p^3$, hence $\mathrm{O}(D_{L_{2p^3}}) = \{\pm \mathrm{id}\} \cong \mathbb{Z}/2$. We have to take the action of $\mathrm{O}(L_{2p^3})/\tilde{\mathrm{O}}(L_{2p^3}) \cong \mathrm{O}(D_{L_{2p^3}})$ on the set of $\tilde{\mathrm{O}}(L_{2p^3})$-orbits into account. As this is a 2-group, there must at least p orbits under the action of $\mathrm{O}(L_{2p^3})$. We remark that $\mathrm{O}(D_{L_{2p^3}})$ is a 2-group in greater generality, see [20, Prop. 2.5].

In particular, the number of pairwise non-equivalent embeddings is finite, but unbounded.

Unimodular $(T)^{\perp}_{L_{2d}}$

We use that there are precisely two inequivalent negative definite unimodular even lattices of rank 16, namely $2E_8$ and D_{16}^+ (the latter is an extension of the non-unimodular root lattice D_{16}); see [12, §16.4]. They become equivalent after adding a hyperbolic plane: $2E_8 \oplus U \cong D_{16}^+ \oplus U$, since unimodular even indefinite lattices are determined by rank and signature. Setting $T := 2U \oplus \langle -2d \rangle$, we get

$$2E_8(-1) \oplus T \cong L_{2d} \cong D_{16}^+(-1) \oplus T.$$

Hence, since the orthogonal complements are different, there must at least be two different embeddings of T into L_{2d}. This example allows for arbitrary polarisations (in contrast to the previous one).

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