Equivariant autoequivalences for finite group actions

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Abstract The familiar Fourier-Mukai technique can be extended to an equivariant setting where a finite group G acts on a smooth projective variety X. In this paper we compare the group of invariant autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X))^G$ with the group of autoequivalences of $\mathrm{D}^G(X)$. We apply this method in three cases: Hilbert schemes on K3 surfaces, Kummer surfaces and canonical quotients.

1 Introduction and Setup

It often proves useful to consider analogues of classical settings, adding the presence of a group action. Instances of this in algebraic geometry are e.g. equivariant intersection theory or the McKay type theorems. There already is a theory for derived categories of varieties with actions by algebraic groups [2]. In this article, we study the behaviour of automorphism groups of such derived categories in the case when the group is finite. The main application concerns Hilbert schemes of points on surfaces: Proposition 8 states that D-equivalent surfaces have D-equivalent Hilbert schemes, and Proposition 10 states that birational Hilbert schemes of K3 surfaces are D-equivalent. Here, two varieties are called D-equivalent if their associated bounded derived categories are equivalent as triangulated categories.

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We always work with varieties over $\mathbb{C}$. A kernel $P \in \mathrm{D^b}(X \times Y)$ gives rise to a Fourier-Mukai transform which we denote by $\mathsf{FM}_P : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$. We also introduce special notation for composition of such transforms: Let us write $\mathsf{FM}_Q \circ \mathsf{FM}_P = \mathsf{FM}_{Q \star P} : \mathrm{D^b}(X) \to \mathrm{D^b}(Z)$ for $P \in \mathrm{D^b}(X \times Y)$ and $Q \in \mathrm{D^b}(Y \times Z)$, i.e. $Q \star P = \mathbb{R}p_{XZ*}(p_{XY}^* P \otimes^{\mathbb{L}} p_{YZ}^* Q)$. This works for morphisms as well: $f : P \to P'$ and $g : Q \to Q'$ give rise to $g \star f : Q \star P \to Q' \star P'$. An object of a $\mathbb{C}$-linear category (like $\mathrm{D^b}(X)$) is called simple its endomorphism ring is $\mathbb{C}$.

1.1 Linearisations and $\mathrm{D}^G(X)$

Let X be a smooth projective variety on which a finite group G acts. A *G-linearisation* of a sheaf E on X is given by isomorphisms $\lambda_g : E \xrightarrow{\sim} g^*E$ for all $g \in G$ satisfying $\lambda_1 = \mathrm{id}_E$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. A morphism $f : (E_1, \lambda_1) \to (E_2, \lambda_2)$ is *G-invariant*, if $f = g \cdot f := \lambda_{2,g}^{-1} \circ g^* f \circ \lambda_{1,g}$ for all $g \in G$. The category of G-linearised coherent sheaves on X with G-invariant morphisms is denoted by $\mathrm{Coh}^G(X)$; note that it is abelian and that there are enough injective quasi-coherent equivariant sheaves, see [3]. Put $\mathrm{D}^G(X) := \mathrm{D^b}(\mathrm{Coh}^G(X))$ for its bounded derived category.

There is an equivalent point of view on $\mathrm{D}^G(X)$: let $\mathcal{T}$ be the category consisting of G-linearised objects of $\mathrm{D^b}(X)$, i.e. complexes $E^\bullet \in \mathrm{D^b}(X)$ together with isomorphisms $\lambda_g : E^\bullet \xrightarrow{\sim} g^*E^\bullet$ in $\mathrm{D^b}(X)$ satisfying $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$, as above.

The canonical functor $\mathrm{D}^G(X) \to \mathcal{T}$ is fully faithful: for $(E_1,\lambda_1)^\bullet, (E_2,\lambda_2)^\bullet \in \mathrm{D}^G(X)$, we have $\mathrm{Hom}_{\mathrm{D}^G(X)}((E_1,\lambda_1)^\bullet,(E_2,\lambda_2)^\bullet) = \mathrm{Hom}_{\mathrm{D}^\mathrm{b}(X)}(E_1^\bullet,E_2^\bullet)^G$ as taking G-invariants is exact (G being finite). To show that the functor is essentially surjective, let $(E^\bullet,\lambda) \in \mathcal{T}$. Choosing a bounded resolution $\varphi : E^\bullet \xrightarrow{\sim} I^\bullet$ of injective quasi-coherent sheaves, we obtain linearisations $\lambda'_g := \varphi\lambda_g\varphi^{-1} : I^\bullet \xrightarrow{\sim} g^*I^\bullet$. Since all $g^*I^\bullet$ consist of injective sheaves, the morphisms λ'_g are actually genuine complex maps. Hence $(I,\lambda')^\bullet$ is a complex with a linearisation of each sheaf, and using $\mathrm{D}^\mathrm{b}(X) \cong \mathrm{D}^\mathrm{b}_{\mathrm{Coh}(X)}(\mathrm{Qcoh}(X)) \ni (I,\tilde{\lambda})^\bullet$ (which is compatible with the G-actions), we get $\mathcal{T} \cong \mathrm{D}^G(X)$.

1.2 Motivation for equivariant Fourier-Mukai transforms

Suppose a finite group G acts on two smooth projective varieties X and X'. Consider an object $P \in \mathrm{D}^\mathrm{b}(X \times X')$ and its associated Fourier-Mukai transform $\mathsf{FM}_P : \mathrm{D}^\mathrm{b}(X) \to \mathrm{D}^\mathrm{b}(X')$. Now assume that P has a linearisation $\rho_g : P \xrightarrow{\sim} (g,g)^*P$ for the diagonal G-action on $X \times X'$. This will then allow to construct a functor on the equivariant categories by

$$\mathsf{FM}^G_{(P,\rho)} : \mathrm{D}^G(X) \to \mathrm{D}^G(X'), \quad (E,\lambda) \mapsto \mathbb{R}(p_{X'})_*(P \otimes^{\mathbb{L}} p_X^*E).$$

The complex obtained here gets a G-linearisation in the following way: the pullback p_X^*E is canonically linearised for the diagonal G-action on $X \times X'$, using λ. This is then also true for the tensor product $P \otimes^{\mathbb{L}} p_X^*E$. Finally, the pushdown onto X' equips $\mathsf{FM}^G_{(P,\rho)}(E,\lambda)$ with a G-linearisation.

Knowing this construction, it is possible to directly proceed to the applications in Section 3, granting the statement of Theorem 6 and its notation (which is introduced on page 6 immediately before the theorem).

The following Sections 1 and 2 describe how to set up an equivariant theory and relate several groups of autoequivalences. Perhaps the most notable feature is the occurence of the Schur multipliers $\mathrm{H}^2(G,\mathbb{C}^*)$ in Lemma 1.

1.3 Equivariant pushdown and Fourier-Mukai transforms

Let (X,G) and (X',G') be smooth projective varieties with finite group actions.A map between them is given by a pair of morphisms $\Phi : X \to X'$ and $\varphi : G \to G'$ such that $\Phi \circ g = \varphi(g) \circ \Phi$ for all $g \in G$. Then, we have the pull-back $\Phi^* : \mathrm{Coh}^{G'}(X') \to \mathrm{Coh}^G(X)$ (and its derived functor $\mathbb{L}\Phi^* : \mathrm{D}^{G'}(X') \to \mathrm{D}^G(X)$) which just means equipping the usual pull-back Φ^*E' with the G-linearisation $\Phi^*\lambda'_{\varphi(g)} : \phi^*E' \xrightarrow{\sim} \Phi^*\varphi(g)^*E' = g^*\Phi^*E'$.

Suppose that φ is surjective and put $K := \ker(\varphi)$. Then, there is also an equivariant push-forward defined for $(E,\lambda) \in \mathrm{Coh}^G(X)$ in the following way: the usual push-forward Φ_*E is canonically G-linearised since φ is surjective: $\lambda_g : E \xrightarrow{\sim} g^*E$ induces $\Phi_*\lambda_g : \Phi_*E \xrightarrow{\sim} \Phi_*g^*E = \varphi(g')^*\Phi_*E$ for some g'. The choice of $g' \in \varphi^{-1}(g)$ does not matter, since the kernel K acts trivially on X'. This trivial action also allows to take K-invariants of Φ_*E. Then the subsheaf $\Phi_*^K(E,\lambda) := [\Phi_*E]^K \subset \Phi_*E$ is G'-linearised and $\mathbb{R}\Phi_*^K : \mathrm{D}^G(X) \to \mathrm{D}^{G'}(X')$ is the correct push-forward.

For objects $(E,\lambda) \in \mathrm{Coh}^G(X)$ and $(E',\lambda') \in \mathrm{Coh}^{G'}(X')$ and a G-invariant morphism $\Phi^*E' \to E$, the adjoint morphism $E' \to \Phi_*E$ has image in $\Phi_*^K E$.

Hence, the functors $\Phi^* : \mathrm{Coh}^{G'}(X') \to \mathrm{Coh}^G(X)$ and $\Phi_*^K : \mathrm{Coh}^G(X) \to \mathrm{Coh}^{G'}(X')$ are adjoint; analogously for $\mathbb{L}\Phi^*$ and $\mathbb{R}\Phi_*^K$.

As a consequence, the usual Fourier-Mukai calculus extends to the equivariant setting if we use these functors (the tensor product of two linearised objects is obviously again linearised). Explicitly, for an object $(P,\rho) \in \mathrm{D}^{G\times G'}(X \times X')$ we get a functor

$$\mathsf{FM}_{(P,\rho)} : \mathrm{D}^G(X) \to \mathrm{D}^{G'}(X'), \quad (E,\lambda) \mapsto \mathbb{R}(p_{X'})_*^G(P \otimes^{\mathbb{L}} p_X^*(E))$$

with the projections $p_{X'} : X \times X' \to X'$ and $p_X : X \times X' \to X$ (and similar projections on the group level).

1.4 Inflation and restriction

There is an obvious forgetful functor *for* : $\mathrm{D}^G(X) \to \mathrm{D}^{\mathrm{b}}(X)$. In the other direction, we have the inflation functor *inf* : $\mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^G(X)$ with $\mathit{inf}(E) := \bigoplus_{g\in G} g^*E$ and the G-linearisation comes from permuting the summands. A generalisation of *inf* to the case of a subgroup $H \subset G$ is given by

$$\mathit{inf}_H^G : \mathrm{D}^H(X) \to \mathrm{D}^G(X), \quad (E,\lambda) \mapsto \bigoplus_{[g]\in H\backslash G} g^*E$$

and the G-linearisation of the sum is a natural combination of λ and permutations.

See Bernstein/Lunts [2] for generalisations of $\mathrm{D}^G(X)$ and inf_H^G to the case of algebraic groups (neither of which is straightforward).

1.5 Invariant vs linearised objects

Obviously, a G-linearised (E,λ) object has to be G-invariant, i.e. $E \cong g^*E$ for all $g \in G$. It is a difficult question under which conditions the other direction is true. For us the following fact ([16, Lemma 3.4]) will suffice.

Lemma 1. *Let $E \in \mathrm{D}^{\mathrm{b}}(X)$ be simple and G-invariant. Then there is a group cohomology class $[E] \in \mathrm{H}^2(G,\mathbb{C}^*)$ such that E is G-linearisable if and only if $[E] = 0$. Furthermore, if $[E] = 0$, then the set of G-linearisations of E is canonically a $\hat{G}$-torsor, where $\hat{G} := \mathrm{Hom}(G,\mathbb{C}^*)$ is the group of characters.*

Proof. Note that the G-action on $\mathrm{Aut}(E) = \mathbb{C}^*$ is trivial. There are isomorphisms $\mu_g : E \xrightarrow{\sim} g^*E$ for all $g \in G$. As E is simple, we can define units $c_{g,h} \in \mathbb{C}^*$ by $\mu_{gh} = h^*\mu_g \circ \mu_h \cdot c_{g,h}$. It is a straightforward check that the map $c : G^2 \to \mathbb{C}^*$ is a 2-cocycle of G with values in $\mathbb{C}^*$, i.e. $c \in Z^2(G,\mathbb{C}^*)$. Replacing the isomorphisms μ_g with some other μ'_g yields the map $e : G \to \mathbb{C}^*$ such that $\mu'_g = \mu_g \cdot e_g$. The two cocycles $c, c' : G^2 \to \mathbb{C}^*$ derived from μ and μ' differ by the boundary coming from e by another easy computation. Hence, $c/c' = d(e)$ and thus $c = c' \in \mathrm{H}^2(G,\mathbb{C}^*)$. Thus the G-invariant object E gives rise to a unique class $[E] := c \in \mathrm{H}^2(G,\mathbb{C}^*)$. In these terms, E is G-linearisable if and only if $c \equiv 1$, i.e. $[E]$ vanishes.

For the second statement, we write $\mathrm{Lin}_G(E)$ for the set of non-isomorphic G-linearisations of E. Consider the $\hat{G}$-action $\hat{G} \times \mathrm{Lin}_G(E) \to \mathrm{Lin}_G(E)$, $(\chi,\lambda) \mapsto \chi\cdot\lambda$ on $\mathrm{Lin}_G(E)$. First take $\chi \in \hat{G}$ and $\lambda \in \mathrm{Lin}_G(E)$ such that $\chi\cdot\lambda = \lambda$. Then, there is an isomorphism $f : (E, \lambda) \xrightarrow{\sim} (E, \chi{\cdot}\lambda)$ which in turn immediately implies $\chi = 1$ using $f \in \mathrm{Aut}(E) = \mathbb{C}^*$. Thus, the action is effective. Now take two elements $\lambda, \lambda' \in \mathrm{Lin}_G(E)$ and consider $\lambda_g^{-1} \circ \lambda'_g : E \xrightarrow{\sim} g^*E \xrightarrow{\sim} E$. As E is simple, we have $\lambda_g^{-1} \circ \lambda'_g = \chi(g) \cdot \mathrm{id}_E$. It follows from the cocycle condition for linearisations that χ is multiplicative, i.e. $\chi \in \hat{G}$. In other words, $\lambda' = \chi \cdot \lambda$ and the action is also transitive. Altogether $\hat{G}$ acts simply transitive on $\mathrm{Lin}_G(E)$. $\square$

For our use of group cohomology, see [17]. The second cohomology $\mathrm{H}^2(G, \mathbb{C}^*)$ of the finite group G acting trivially (on an algebraically closed field of characteristic 0) is also known as the *Schur multiplier* of G (see [8, §25]). Two relevant facts about it are: $\mathrm{H}^2(G, \mathbb{C}^*)$ is a finite abelian group; its exponent is a divisor of $\#G$. Examples are given by $\mathrm{H}^2((\mathbb{Z}/n\mathbb{Z})^k, \mathbb{C}^*) = (\mathbb{Z}/n\mathbb{Z})^{k(k-1)/2}$ for copies of a cyclic group; $\mathrm{H}^2(D_{2n}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{H}^2(D_{2n+1}, \mathbb{C}^*) = 0$ for the dihedral groups with $n > 1$; and $\mathrm{H}^2(S_n, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$ for the symmetric groups with $n > 3$.

Note that a group with vanishing Schur multiplier has the following property: every simple G-invariant object of $\mathrm{D}^G(X)$ is G-linearisable, no matter how G acts on X.

Remark 2. The condition that E be simple in the Lemma is important. Consider an abelian surface A with the action of $G = \mathbb{Z}/2\mathbb{Z} = \{\pm\,\mathrm{id}_A\}$. Then the sheaf $E := k(a) \oplus k(-a) = \mathit{inf}(k(a))$ is G-invariant but not simple. Yet it is uniquely $\mathbb{Z}/2\mathbb{Z}$-linearisable as an easy computation shows [16, Example 3.9] (in contrast to G-invariant simple sheaves, which have precisely two non-isomorphic G-linearisations according to $\mathrm{H}^2(G, \mathbb{C}^*) = 0$ and $\hat{G} = G$ for $G = \mathbb{Z}/2\mathbb{Z}$). This behaviour is expected from geometry: by the derived McKay correspondence ([3]) one has $\mathrm{D}^G(A) \cong \mathrm{D}^{\mathrm{b}}(X)$, where X is the Kummer surface of A, a crepant resolution $\psi : X \to A/G$. Under this equivalence, skyscraper sheaves of points $x \in X$ outside of exceptional fibres of ψ are mapped to $k(\psi(x)) \oplus k(-\psi(x))$.

Example 3. If as before G acts on X, then the canonical sheaf ω_X is simple (as it is a line bundle) and G-invariant (because it is functorial). Due to this functoriality, it is actually G-linearisable: the morphism $g : X \to X$ induces a morphism of cotangent bundles $g_* : g^*\Omega_X \to \Omega_X$. Going to determinants and using adjunction yields the desired isomorphisms $\lambda_g := \det(g_*^{-1}) : \omega_X \xrightarrow{\sim} g^*\omega_X$.

2 Groups of autoequivalences

We are interested in comparing the automorphism group $\mathrm{Aut}(\mathrm{D}^G(X))$ with the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))^G := \{F \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) : g^* \circ F = F \circ g^* \ \forall g \in G\}$. It turns out that a useful intermediate step is to look at Fourier-Mukai equivalences on $\mathrm{D}^{\mathrm{b}}(X)$ which are diagonally G-linearised.

To make this precise, consider a Fourier-Mukai transform $\mathsf{FM}_P : \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(X')$. Suppose that G acts on both X and X'. Then we have the diagonal action $G \times X \times X' \to X \times X'$, $g \cdot (x, x') := (gx, gx')$ which we sometimes (especially in the case $X = X'$) for emphasis call the G_Δ-action of G on $X \times X'$. Now we are in a position to study objects $(P, \rho) \in \mathrm{D}^{G_\Delta}(X \times X')$ which give Fourier-Mukai equivalences $\mathsf{FM}_P : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$. In other words, these are ordinary kernels for equivalences $\mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X')$ which additionally have been equipped with a G_Δ-linearisation.

Not every kernel in $P \in \mathrm{D^b}(X \times X')$ has the latter property. A necessary condition is that P must be G_Δ-invariant, i.e. $(g,g)^*P \cong P$ for all $g \in G$, or, equivalently, $g^* \circ \mathsf{FM}_P = \mathsf{FM}_P \circ g^*$. Now we apply the following general fact:

Lemma 4. *If $P \in \mathrm{D^b}(X \times Y)$ is the Fourier-Mukai kernel of an equivalence $\mathsf{FM}_P : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ then P is simple, i.e. $\mathrm{Hom}_{\mathrm{D^b}(X\times Y)}(P,P) = \mathbb{C}$.*

Proof. Fix $f \in \mathrm{Hom}_{\mathrm{D^b}(X\times Y)}(P,P)$ and let Q be a quasi-inverse kernel for FM_P, i.e. $P \star Q \cong \mathcal{O}_{\Delta_Y}$. By $\mathrm{Hom}_{\mathrm{D^b}(Y\times Y)}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}) = \mathrm{Hom}_{Y\times Y}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}) = \mathbb{C}$ we have $f \star \mathrm{id}_Q = c \cdot \mathrm{id}_{\mathcal{O}_{\Delta_Y}}$ for a $c \in \mathbb{C}$. Composing again with $\mathrm{id}_P : P \to P$ gives $f = f \star \mathrm{id}_{\mathcal{O}_{\Delta_X}} = f \star (\mathrm{id}_Q \star \mathrm{id}_P) = (f \star \mathrm{id}_Q) \star \mathrm{id}_P = (c \cdot \mathrm{id}_{\mathcal{O}_{\Delta_Y}}) \star \mathrm{id}_P = c \cdot \mathrm{id}_P$. ☐

Combining Lemmas 1 and 4, we see that G_Δ-invariant kernels for equivalences are G_Δ-linearisable, provided that the obstruction class in $\mathrm{H}^2(G, \mathbb{C}^*)$ vanishes (for example, if $\mathrm{H}^2(G, \mathbb{C}^*) = 0$).

Now suppose we have an arbitrary object $(P, \rho) \in \mathrm{D}^{G_\Delta}(X \times X')$ and the accompanying functor $\mathsf{FM}_P : \mathrm{D^b}(X) \to \mathrm{D^b}(X')$. The general device of inflation allows us to produce the following equivariant Fourier-Mukai transform from (P, ρ):

$$\mathsf{FM}^G_{(P,\rho)} := \mathsf{FM}_{\mathit{inf}^{G^2}_{G_\Delta}(P,\rho)} : \mathrm{D}^G(X) \to \mathrm{D}^G(X').$$

For brevity, we set $G \cdot P := \mathit{inf}^{G^2}_{G_\Delta}(P, \rho) \in \mathrm{D}^{G^2}(X \times X')$ and call it *the inflation of* (P, ρ). The following lemma states the main properties of this assignment.

Lemma 5. *Let X, X', X'' be smooth projective varieties with G-actions and let $(P, \rho) \in \mathrm{D}^{G_\Delta}(X \times X')$ and $(P', \rho') \in \mathrm{D}^{G_\Delta}(X' \times X'')$.*
(1) $\mathsf{FM}^G_{(\mathcal{O}_{\Delta_X}, \mathsf{can})} \cong \mathrm{id} : \mathrm{D}^G(X) \to \mathrm{D}^G(X)$.
(2) For $(P, \rho) \in \mathrm{D}^{G_\Delta}(X \times X')$ there is a commutative diagram

$$\begin{tikzcd}
\mathrm{D}^G(X) \arrow[r, "\mathsf{FM}^G_{(P,\rho)}"] \arrow[d, "for"] & \mathrm{D}^G(X') \arrow[d, "for"] \\
\mathrm{D^b}(X) \arrow[r, "\mathsf{FM}_P"] & \mathrm{D^b}(X')
\end{tikzcd}$$

(3) $\mathsf{FM}^G_{(P',\rho')} \circ \mathsf{FM}^G_{(P,\rho)} \cong \mathsf{FM}^G_{(P'\star P, \rho'\star\rho)}$ where $(\rho' \star \rho)_g := \rho'_g \star \rho_g$.
(4) FM_P fully faithful $\implies$ $\mathsf{FM}^G_{(P,\rho)}$ fully faithful.
(5) FM_P equivalence $\implies$ $\mathsf{FM}^G_{(P,\rho)}$ equivalence.

Proof. (1) The structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$ has a canonical G_Δ-linearisation, as $(g,g)^*\mathcal{O}_\Delta = \mathcal{O}_\Delta$. The inflation of $\mathcal{O}_\Delta$ is $G \cdot \mathcal{O}_\Delta = \bigoplus_{g\in G} \mathcal{O}_{(g,1)\Delta}$, and its G^2-linearisation is given by the permutation of summands via $G \to G$, $g \mapsto kgh^{-1}$.

Using this one can check by hand that $\mathsf{FM}^G_{(\mathcal{O}_\Delta, \mathsf{can})}$ maps any $(E, \lambda) \in \mathrm{D}^G(X)$ to itself (see [16, Example 3.14]).

(2) Take any object $(E, \lambda) \in \mathrm{D}^G(Y)$. Then, we have by definition of equivariant Fourier-Mukai transforms $\mathsf{FM}^G_P : \mathrm{D}^G(Y) \to \mathrm{D}^G(Y)$, $(E, \lambda) \mapsto [\mathbb{R}p_{2*}(G \cdot P \otimes^{\mathbb{L}} p_1^* E)]^{G\times 1}$. The $G \times 1$-linearisation of $G \cdot P$ is given by permutations (the G_Δ-linearisation of P does not enter). Since $\mathbb{R}p_{2*}((g,1)^*P \otimes^{\mathbb{L}} p_1^*E) = \mathbb{R}p_{2*}(g,1)^*(P \otimes^{\mathbb{L}} p_1^* g^{-1*} E) = \mathbb{R}p_{2*}(P \otimes^{\mathbb{L}} p_1^* E)$, we see that $\mathbb{R}p_{2*}(G \cdot P \otimes^{\mathbb{L}} p_1^* E) \cong \bigoplus_G \mathbb{R}p_{2*}(P \otimes^{\mathbb{L}} p_1^* E)$ and $G \times 1$ acts with permutation matrices where the 1's

are replaced by $p_1^*\lambda_g$'s. Taking $G \times 1$-invariants singles out a subobject of this sum isomorphic to one summand.

A morphism $f : E_1 \to E_2$ in $\mathrm{D}^G(X)$ is likewise first taken to a G-fold direct sum. The final taking of $G \times 1$-invariants then leaves one copy of $\mathsf{FM}_P(f)$.

(3) The composite $\mathsf{FM}_{G\cdot P'} \circ \mathsf{FM}_{G\cdot P}$ has the kernel

$$\begin{aligned}
(G\cdot P') \star (G \cdot P) &= [\mathbb{R}p_{13*}(p_{12}^*(G\cdot P) \otimes^{\mathbb{L}} p_{23}^*(G\cdot P'))]^{1\times G\times 1} \\
&= [\mathbb{R}p_{13*}(p_{12}^* \bigoplus_{g\in G} (g,1)^* P \otimes^{\mathbb{L}} p_{23}^* \bigoplus_{h\in G} (h,1)^* P')]^{1\times G\times 1} \\
&\cong_{\rho'} [\mathbb{R}p_{13*}(p_{12}^* \bigoplus_{g\in G} (g,1)^* P \otimes^{\mathbb{L}} p_{23}^* \bigoplus_{h\in G} (1,h^{-1})^* P')]^{1\times G\times 1} \\
&= [\bigoplus_{g,h\in G} \mathbb{R}p_{13*}(g,1,h^{-1})^*(p_{12}^*P \otimes^{\mathbb{L}} p_{23}^*P')]^{1\times G\times 1} \\
&= [\bigoplus_{g,h\in G} (g,h^{-1})^*\mathbb{R}p_{13*}(p_{12}^*P \otimes^{\mathbb{L}} p_{23}^*P')]^{1\times G\times 1}.
\end{aligned}$$

Now note that $(1,c,1) \in 1 \times G \times 1$ acts on $(G\cdot P') \star (G\cdot P)$ via permutations (inverse multiplications from left) and ρ on P, and $(1,c,1)$ acts purely by permutations (which are multiplications from right) on P'. Plugging this into the last equation, we find that after taking invariants we end up with $\bigoplus_{d\in G}(d^{-1},d)^*\mathbb{R}p_{13*}(p_{12}^*P \otimes^{\mathbb{L}} p_{23}^*P')$. Since the (d^{-1},d)'s give all classes in $G_\Delta \setminus G^2$, we find that $(G\cdot P') \star (G\cdot P) \cong G\cdot (P' \star P)$.

(4) Fix two objects $(E_1,\lambda_1),(E_2,\lambda_2) \in \mathrm{D}^G(X)$. The injectivity of the natural map $\mathrm{Hom}_{\mathrm{D}^G(X)}((E_1,\lambda),(E_2,\lambda_2)) \to \mathrm{Hom}_{\mathrm{D}^G(X')}(\mathsf{FM}_P^G(E_1,\lambda_1),\mathsf{FM}_P^G(E_2,\lambda_2))$ is a consequence of two facts: $\mathrm{Hom}_{\mathrm{D}^G(X)}(\cdot,\cdot) = \mathrm{Hom}_{\mathrm{D^b}(X)}(\cdot,\cdot)^G \subset \mathrm{Hom}_{\mathrm{D^b}(X)}(\cdot,\cdot)$ by the definition of morphisms in $\mathrm{D}^G(X)$ on one hand and $\mathrm{Hom}_{\mathrm{D^b}(X)}(E_1,E_2) \cong \mathrm{Hom}_{\mathrm{D^b}(X')}(\mathsf{FM}_P(E_1),\mathsf{FM}_P(E_2))$ by hypothesis.

Similarly, the surjectivity uses the same facts. One just replaces the embedding $\mathrm{Hom}_{\mathrm{D^b}(X)}(\cdot,\cdot)^G \subset \mathrm{Hom}_{\mathrm{D^b}(X)}(\cdot,\cdot)$ with the averaging projection (Reynolds operator) $\theta : \mathrm{Hom}_{\mathrm{D^b}(X)}(E_1,E_2) \twoheadrightarrow \mathrm{Hom}_{\mathrm{D^b}(X)}(E_1,E_2)^G$ given by $\theta(f) := \frac{1}{\#G}\sum_{g\in G}\lambda_{2,g}^{-1}\circ g^*f\circ\lambda_{1,g}$.

(5) follows from (3) and (1): let $\mathsf{FM}_P : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X')$ be an equivalence. Then, $Q = \mathbb{R}\mathcal{H}om(P,\mathcal{O}_{X\times X'}) \otimes p_X^*\omega_X[\dim(X)]$ is the Fourier-Mukai kernel of a quasi-inverse for FM_P. As the canonical bundle ω_X is G-linearisable (see Example 3), Q inherits a G_Δ-linearisation from those of P and $p_X^*\omega_X$. We have $(G\cdot P) \star (G\cdot Q) = G\cdot(P\star Q) = G\cdot\mathcal{O}_\Delta$. There are precisely $\#\hat{G}$ different G-linearisations for ω_X as well as $\#\hat{G}$ different G_Δ-linearisations for $\mathcal{O}_\Delta$. Thus, exactly one choice of G-linearisation for ω_X will equip the composition $P\star Q = \mathcal{O}_\Delta$ with the canonical G_Δ-linearisation. But then (1) shows that this is the kernel of the identity on $\mathrm{D}^G(X')$. Hence, $G\cdot P$ is an equivalence kernel as was P. $\square$

Let us consider the following automorphism groups:

$$\begin{aligned}
\mathrm{Aut}(\mathrm{D^b}(X))^G &\cong \{\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D^b}(X)) \,:\, (g,g)^*P \cong P \ \forall g \in G\} \\
\mathrm{Aut}(\mathrm{D}^G(X)) &\cong \{\mathsf{FM}_{\widetilde{P}} : \mathrm{D}^G(X) \xrightarrow{\sim} \mathrm{D}^G(X) \,:\, \widetilde{P} \in \mathrm{D}^{G^2}(X\times X)\} \\
\mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(X)) &:= \{(P,\rho) \in \mathrm{D}^{G_\Delta}(X\times X) \,:\, \mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D^b}(X))\}.
\end{aligned}$$

The first isomorphism uses the action $G \times \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\mathrm{D^b}(X))$ given by $g \cdot F := (g^{-1})^* \circ F \circ g^*$ and the formula $\mathsf{FM}_{(g,g)^*P} \cong g^* \circ \mathsf{FM}_P \circ (g^{-1})^*$ and finally Orlov's theorem on the existence and uniticity of Fourier-Mukai kernels [14, Theorem 2.2]. Orlov's result has been extended by Kawamata to smooth stacks associated to normal projective varieties with quotient singularities [11]. In view of $\mathrm{Coh}([X/G]) \cong \mathrm{Coh}^G(X)$ (where $[X/G]$ denotes the stack), this implies the second relation. Finally we turn $\mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(X))$ into a group by Lemma 5: (3) settles the composition, (1) the neutral element and (5) the inverses.

The next theorem [16, Proposition 3.17] attempts to compare these groups.

Theorem 6. *Suppose that the finite group G acta faithfully on the smooth projective variety X.*

(1) The construction of inflation gives a group homomorphism inf which fits in the following exact sequence, where $Z(G) \subset G$ is the centre of G:

$$\begin{array}{ccccccc} 0 \longrightarrow Z(G) \longrightarrow & \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(X)) & \xrightarrow{\mathit{inf}} & \mathrm{Aut}(\mathrm{D}^G(X)) \\ & (P,\rho) & \longmapsto & \mathsf{FM}^G_{(P,\rho)} \end{array}$$

(2) Forgetting the G_Δ-linearisation gives a group homomorphism which sits in the following exact sequence; here $G_{\mathrm{ab}} := G/[G,G]$ is the abelianisation, which is non-canonically isomorphic to $\mathrm{Hom}(G,\mathbb{C}^*) = \mathrm{H}^1(G,\mathbb{C}^*)$:

$$\begin{array}{ccccccc} 0 \longrightarrow G_{\mathrm{ab}} \longrightarrow & \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(X)) & \xrightarrow{\mathit{for}} & \mathrm{Aut}(\mathrm{D^b}(X))^G & \longrightarrow & \mathrm{H}^2(G,\mathbb{C}^*) \\ & (P,\rho) & \longmapsto & \mathsf{FM}_P & \longmapsto & [P] \end{array}$$

Proof. (1) It follows from Lemma 5 that *inf* is a group homomorphism. The kernel $\ker(\mathit{inf})$ consists of $(P,\rho) \in \mathrm{D}^{G_\Delta}(X^2)$ giving equivalences such that $G \cdot P \cong G \cdot \mathcal{O}_\Delta$. Obviously, this forces P to be a sheaf of type $P \cong (g,1)^*\mathcal{O}_\Delta$ for some $g \in G$. Now $(g,1)^*\mathcal{O}_\Delta$ is G_Δ-invariant if and only if $g \in Z(G)$ as $(h,h)^*(g,1)^*\mathcal{O}_\Delta = (gh,h)^*\mathcal{O}_\Delta \cong (gh,h)^*(h^{-1},h^{-1})^*\mathcal{O}_\Delta = (h^{-1}gh,1)^*\mathcal{O}_\Delta$. This in turn implies that the isomorphism $(g,1) : X^2 \xrightarrow{\sim} X^2$ is a G_Δ-map. In particular, $P \cong (g,1)^*\mathcal{O}_\Delta$ gets the pulled back G_Δ-linearisation. Giving $\mathcal{O}_\Delta$ a G_Δ-linearisation $\lambda \in \hat{G}$ different from the canonical one yields $G \cdot (\mathcal{O}_\Delta, \lambda) \ncong G \cdot (\mathcal{O}_\Delta, \mathsf{can})$; this follows for example from the uniqueness of Fourier-Mukai kernels. Both facts together imply $\ker(\mathit{inf}) \cong Z(G)$. Here is the only place where we use that the action $\Xi : G \to \mathrm{Aut}(X)$ is faithful. For a more general action, the kernel is $\ker(\mathit{inf}) = \Xi^{-1}(Z(\mathrm{im}(\Xi)))$.

(2) It is obvious from the definition of $\mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(X))$ that *for* is a group homomorphism. The kernel of *for* corresponds to the G_Δ-linearisations on $\mathcal{O}_\Delta$. From Lemma 1, we see that they form a group isomorphic to $\hat{G} = \mathrm{Hom}(G,\mathbb{C}^*) \cong G_{\mathrm{ab}}$. Note also $\mathrm{Hom}(G,\mathbb{C}^*) = \mathrm{H}^1(G,\mathbb{C}^*)$ as G acts trivial on $\mathbb{C}^*$. Given $\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D^b}(X))^G$, we know from Lemma 4 that its Fourier-Mukai kernel P is simple. Furthermore, it is G_Δ-invariant by assumption, so that the map $\mathsf{FM}_P \mapsto [P]$ is defined as in Lemma 1. To see that it is a group homomorphism, take two G_Δ-invariant kernels $P, Q \in \mathrm{D^b}(X^2)$. Choose isomorphisms $\lambda_g : P \xrightarrow{\sim} (g,g)^*P$ and $\mu_g : Q \xrightarrow{\sim} (g,g)^*Q$ for all $g \in G$. Then we have $\lambda_{gh} = (h,h)^*\lambda_g \circ \lambda_h \cdot [P]_{g,h}$ and likewise for Q, μ. Furthermore, the composition of λ_g and μ_g gives an isomorphism $\mu_g \star \lambda_g : Q \star P \xrightarrow{\sim} ((g,g)^*Q) \star ((g,g)^*P)$ and the latter term is canonically isomorphic to $(g,g)^*(Q \star P)$. Then $\mu_{g,h} \star \lambda_{gh} = (h,h)^*(\mu_g \star \lambda_g) \circ$

$(\mu_h \star \lambda_h) \cdot [Q \star P]_{g,h}$. The formula $(B \circ A) \star (D \circ C) \cong (B \star D) \circ (A \star C)$ for morphisms $P'' \xrightarrow{B} P' \xrightarrow{A} P$ and $Q'' \xrightarrow{D} Q' \xrightarrow{C} Q$ implies $[Q \star P]_{g,h} = [Q]_{g,h} \cdot [P]_{g,h}$. Now it is obvious that $[\cdot] \circ \mathit{for} = 0$ and finally $\mathsf{FM}_P \in \mathrm{Aut}(\mathrm{D^b}(X))^G$ with $[P] = 0$ implies that P is G_Δ-linearisable by Lemma 1. □

3 Applications

The applications will make use of the derived McKay correspondence: assume that X is a smooth (quasi-)projective variety with an action of the finite group G. Then there is a G-Hilbert scheme G-Hilb(X) which parametrises G-clusters in X, i.e. points $\xi \in G$-Hilb(X) correspond to 0-dimensional subschemes $Z_\xi \subset X$ such that $\mathrm{H}^0(\mathcal{O}_{Z_\xi}) \cong \mathbb{C}[G]$, as G-representations. Typical examples of such clusters are free G-orbits. The formal definition of G-Hilb(X) uses that it represents the relevant functor. Let $\widetilde{X} \subset G$-Hilb(X) be the connected component which contains the free orbits. Then there is a birational morphism (the Hilbert-Chow map) $\widetilde{X} \to X/G$. Combined with the projection $X \to X/G$, this yields a universal subscheme $\mathcal{Z} \subset \widetilde{X} \times X$; note that canonically $\mathcal{O}_\mathcal{Z} \in \mathrm{D}^{1 \times G}(\widetilde{X} \times X)$. We refer to [3] for

Theorem 7 (Bridgeland-King-Reid)**.** *Suppose that ω_X is locally trivial in* $\mathrm{Coh}^G(X)$ *and* $\dim(\widetilde{X} \times_{X/G} \widetilde{X}) \leq \dim(X)+1$. *Then* $\mathsf{FM}_{\mathcal{O}_\mathcal{Z}} : \mathrm{D^b}(\widetilde{X}) \xrightarrow{\sim} \mathrm{D}^G(X)$ *is an equivalence.*

3.1 Hilbert schemes on K3 surfaces

At first consider two smooth projective varieties X and Y and the n-fold products X^n, Y^n with their natural S_n-actions. Let $P \in \mathrm{D^b}(X \times Y)$ be the kernel of a Fourier-Mukai transform $\mathsf{FM}_P : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$. Then, the exterior tensor product $P^{\boxtimes n} = p_1^* P \otimes \cdots \otimes p_n^* P \in \mathrm{D^b}(X^n \times Y^n)$ yields the functor $F^n = \mathsf{FM}_{P^{\boxtimes n}} : \mathrm{D^b}(X^n) \to \mathrm{D^b}(Y^n)$. Furthermore, $P^{\boxtimes n}$ has an obvious $(S_n)_\Delta$-linearisation via permutation of tensor factors. Hence, using inflation, we get the new functor $F^{[n]} := \mathsf{FM}^{S_n}_{P^{\boxtimes n}} : \mathrm{D}^{S_n}(X^n) \to \mathrm{D}^{S_n}(Y^n)$. If we restrict to the case $X = Y$ and autoequivalences $F : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$, we get a group homomorphism by Theorem 6

$$\mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}^{(S_n)_\Delta}(\mathrm{D^b}(X^n)) \to \mathrm{Aut}(\mathrm{D}^{S_n}(X)), \quad \mathsf{FM}_P \mapsto \mathsf{FM}^{S_n}_{(P^{\boxtimes n}, \mathsf{perm})}.$$

From now on we need the provision $\dim(X) = \dim(Y) = 2$. It is well-known that for surfaces $\mathrm{Hilb}_n(X)$ is a crepant resolution of X^n/S_n. Furthermore, a theorem of Haiman states $\mathrm{Hilb}_n(X) \cong S_n$-Hilb$(X^n)$, see [7]. If we additionally assume $\omega_X \cong \mathcal{O}_X$ and $\omega_Y \cong \mathcal{O}_Y$, then we can invoke Theorem 7 in order to obtain $\mathrm{D^b}(\mathrm{Hilb}_n(X)) \cong \mathrm{D}^{S_n}(X^n)$ because in this case X^n and Y^n are symplectic manifolds and the condition $\dim(\mathrm{Hilb}_n(X) \times_{X^n/S_n} \mathrm{Hilb}_n(X)) < 1 + n\dim(X)$ of Theorem 7 is automatically fulfilled; see [3, Corollary 1.3]. However, a posteriori this inequality is true for general surfaces as the dimension of the fibre product is a local quantity which may be computed with any (e.g. affine or symplectic) model. The above homomorphism of groups of autoequivalences is now

$$\mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}(\mathrm{D^b}(\mathrm{Hilb}_n(X))).$$

It is always injective: this is clear for $n > 2$ since the centre of S_n is trivial in this case. For $n = 2$ one can check that the sheaf $\mathcal{O}_{\Delta_X \times \Delta_X}$ with the non-canonical $(S_2)_\Delta$-linearisation is not in the image of $\mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut}^{(S_2)_\Delta}(\mathrm{D^b}(X^2))$.

Let us introduce the shorthand $X^{[n]} := \mathrm{Hilb}_n(X)$, so that $\mathrm{D^b}(X^{[n]}) \cong \mathrm{D}^{S_n}(X^n)$ by the above and $\mathrm{Aut}(\mathrm{D^b}(X)) \hookrightarrow \mathrm{Aut}(\mathrm{D^b}(X^{[n]}))$. The above technique shows

Proposition 8. *If X and Y are two smooth projective surfaces with $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$, then $\mathrm{D^b}(X^{[n]}) \cong \mathrm{D^b}(Y^{[n]})$.* □

Remark 9. A birational isomorphism $f : X \dashrightarrow Y$ of smooth projective surfaces induces a birational map $f^{[n]} : X^{[n]} \dashrightarrow Y^{[n]}$ between their Hilbert schemes. There is a derived analogue: a Fourier-Mukai transform (resp. equivalence) $F = \mathsf{FM}_P : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ induces a functor (resp. equivalence) $F^{[n]} = \mathsf{FM}^{S_n}_{P^{\boxtimes n}} : \mathrm{D^b}(X^{[n]}) \to \mathrm{D^b}(Y^{[n]})$. Since for the time being it is unknown whether every functor $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ is of Fourier-Mukai type, we have to restrict to those (which include equivalences by [14, Theorem 2.2]).

Finally, we specialise to K3 surfaces.

Proposition 10. *Let X_1 and X_2 be two projective K3 surfaces. If there is a birational isomorphism $X_1^{[n]} \dashrightarrow X_2^{[n]}$ of their Hilbert schemes, then the derived categories are equivalent: $\mathrm{D^b}(X_1^{[n]}) \cong \mathrm{D^b}(X_2^{[n]})$.*

Proof. A birational isomorphism $f : X_1^{[n]} \dashrightarrow X_2^{[n]}$ induces an isomorphism on second cohomology, $f^* : \mathrm{H}^2(X_1^{[n]}) \xrightarrow{\sim} \mathrm{H}^2(X_2^{[n]})$, respecting the Hodge structures, because Hilbert schemes of symplectic surfaces are symplectic manifolds. From the crepant resolution $X_1^{[n]} \to X_1^n/S_n$, we find $\mathrm{H}^2(X_1) \subset \mathrm{H}^2(X_1^{[n]})$, and only the exceptional divisor $E_1 \subset X_1^{[n]}$ is missing: $\mathrm{H}^2(X_1^{[n]}) = \mathrm{H}^2(X_1) \oplus \mathbb{Z} \cdot \delta_1$ with $2\delta_1 = E_1$. In particular, as $[E_1]$ is obviously an algebraic class, the transcendental sublattices coincide: $T(X_1) = T(X_1^{[n]})$. Hence, the birational isomorphism furnishes an isometry $T(X_1) \cong T(X_2)$. Orlov's derived Torelli theorem for K3 surfaces [14] then implies $\mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$. But now we apply the above result on lifting equivalences from K3 surfaces to Hilbert schemes and deduce $\mathrm{D^b}(X_1^{[n]}) \cong \mathrm{D^b}(X_2^{[n]})$, as claimed. □

Remarks 11. (1) follows from Remark 9 and (2) from Proposition 10.

(1) D-equivalent abelian or K3 surfaces X_1 and X_2 have D-equivalent Hilbert schemes $X_1^{[n]}$ and $X_2^{[n]}$.
(2) Considering only birational equivalence classes of Hilbert schemes on K3 surfaces, we find that each such class is finite, since K3 surfaces have only finitely many Fourier-Mukai partners [4].
(3) Markman [12] gives an example of non-birational Hilbert schemes $X_1^{[n]}$ and $X_2^{[n]}$ with $\mathrm{H}^2(X_1^{[n]}) \cong \mathrm{H}^2(X_2^{[n]})$. The above arguments still yield $\mathrm{D^b}(X_1) \cong \mathrm{D^b}(X_2)$ and $\mathrm{D^b}(X_1^{[n]}) \cong \mathrm{D^b}(X_2^{[n]})$, i.e. $X_1^{[n]}$ is D-equivalent to $X_2^{[n]}$.

3.2 Kummer surfaces

Let A be an abelian variety and consider the action of $G := \{\pm\,\mathrm{id}_A\} \cong \mathbb{Z}/2\mathbb{Z}$. Denote by $\hat{A}$ the dual abelian variety and by X the Kummer surface associated to A. In order to investigate $\mathrm{Aut}(\mathrm{D}^G(A))$, we start with the exact sequence (see Orlov's article [15]):

$$0 \longrightarrow \mathbb{Z} \times A \times \hat{A} \xrightarrow{\eta} \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A)) \xrightarrow{\gamma} \mathrm{Sp}(A \times \hat{A}) \longrightarrow 0\ ,$$

the first morphism η maps a triple (n, a, ξ) to the autoequivalence $t_a^* \circ \mathsf{M}_\xi[n]$, where $t_a : A \xrightarrow{\sim} A$ denotes the translation by a and $\mathsf{M}_\xi : \mathrm{D}^{\mathrm{b}}(A) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(A)$ the line bundle twist with ξ. Note that shifts, translations and twists by degree 0 line bundles commute. Before turning to the second morphism γ, we set

$$\mathrm{Sp}(A \times \hat{A}) := \left\{ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \in \mathrm{Aut}(A \times \hat{A}) \ : \ \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} \hat{f}_4 & -\hat{f}_2 \\ -\hat{f}_3 & \hat{f}_1 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_{\hat{A}} \end{pmatrix} \right\}.$$

Now given $F \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$, there is a functorial way to attach an equivalence $\Phi_F : \mathrm{D}^{\mathrm{b}}(A \times \hat{A}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(A \times \hat{A})$ which sends skyscraper sheaves to skyscraper sheaves. Hence Φ_F yields an automorphism $\gamma(F) : A \times \hat{A} \xrightarrow{\sim} A \times \hat{A}$ which turns out to be in $\mathrm{Sp}(A \times \hat{A})$ (see the original [15, §2] by Orlov or [16, §4] for a slightly different presentation).

Note that G acts on $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A))$ via conjugation, i.e. $(-1) \cdot F := (-1)^* \circ F \circ (-1)^*$. This induces an action on $\mathrm{Sp}(A \times \hat{A})$, which is trivial since $\gamma((-1)^*) = -\mathrm{id}_{A \times \hat{A}}$ is central. Taking G-invariants of Orlov's exact sequence, we get

$$0 \longrightarrow \mathbb{Z} \times A[2] \times \hat{A}[2] \xrightarrow{\eta^G} \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A))^G \xrightarrow{\gamma^G} \mathrm{Sp}(A \times \hat{A}) \longrightarrow 0\ .$$

Here, $A[2] \subset A$ and $\hat{A}[2] \subset \hat{A}$ denote the 2-torsion subgroups; the surjectivity of γ^G uses $\mathrm{H}^1(G, \mathbb{Z} \times A \times \hat{A}) = 0$; see [16, Proposition 4.8]. Hence, any autoequivalence $F \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A))$ differs from a G-invariant one just by translations and degree 0 line bundle twists.

Assume from now on that A is an abelian *surface*. Let X be the corresponding Kummer surface, which is a crepant resolution of A/G. A realisation is $X = G\text{-Hilb}(A)$ and we use derived McKay correspondence (Theorem 7) to infer $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^G(A)$. From Theorem 6 we get group homomorphisms

$$\begin{tikzcd} \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A))^G & \mathrm{Aut}^{G_\Delta}(\mathrm{D}^{\mathrm{b}}(A)) \arrow[l, two heads, "for"', "2:1"] \arrow[r, "inf", "2:1"'] & \mathrm{Aut}(\mathrm{D}^G(A)) \end{tikzcd}$$

In our situation both *for* and *inf* are 2:1, and *for* is surjective as $\mathrm{H}^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0$. There does not seem to be a homomorphism between the outer groups making a commutative triangle.

However, going to cohomology (here with $\mathbb{Q}$ coefficients throughout), the maps can be completed to a commutative diagram (see [16, Proposition 4.14]):

$$\begin{tikzcd} & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(A)) \arrow[r, "(\cdot)^{\mathrm{H}}"] & \mathrm{Aut}(\mathrm{H}^{2*}(A)) \\ \mathrm{Aut}^{G_\Delta}(\mathrm{D}^{\mathrm{b}}(A)) \arrow[ur, "for"] \arrow[dr, "inf"'] & & \\ & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \arrow[r, "(\cdot)^{\mathrm{H}}"'] & \mathrm{Aut}(\mathrm{H}^*(X)) \arrow[uu, "res"'] \end{tikzcd}$$

Here, $(\cdot)^{\mathrm{H}} : \mathrm{Aut}(\mathrm{D^b}(A)) \to \mathrm{Aut}(\mathrm{H}^{2*}(A))$ sends a Fourier-Mukai equivalence to the corresponding isomorphism on cohomology. Further we use that the image of $(\cdot)^{\mathrm{H}} \circ \mathit{inf}$ lies inside the subgroup of isometries preserving the exceptional classes,

$$\mathrm{Aut}(\mathrm{H}^*_{\mathrm{ex}}(X)) := \{\varphi \in \mathrm{Aut}(\mathrm{H}^*(X)) \,:\, \varphi(\Lambda) = \Lambda\}$$

where $\Lambda \cong \mathbb{Q}^{16}$ is the lattice spanned by the (-2)-classes arising from the Kummer construction; the morphism $\mathit{res} : \varphi \mapsto \varphi|_\Lambda$ is then the obvious restriction.

3.3 Canonical quotients

Let X be a smooth projective variety whose canonical bundle is of finite order. Suppose that $n > 0$ is minimal with $\omega_X^n \cong \mathcal{O}_X$. Then there is an étale covering $\widetilde{X} \xrightarrow{\pi} X$ of degree n with $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$. One concrete definition is $\widetilde{X} = \mathrm{Spec}(\mathcal{O}_X \oplus \omega_X \cdots \oplus \omega_X^{n-1})$, see [1, §I.17]. The group $G := \mathbb{Z}/n\mathbb{Z}$ then acts freely on $\widetilde{X}$ with $\widetilde{X}/G = X$. Fix a generator $g \in G$ and note that a G-linearisation for some $\widetilde{E} \in \mathrm{D^b}(\widetilde{X})$ is completely determined by an isomorphism $\lambda : \widetilde{E} \xrightarrow{\sim} g^*\widetilde{E}$ subject to $(g^{n-1})^*\lambda \circ \cdots \circ g^*\lambda \circ \lambda = \mathrm{id}_{\widetilde{E}}$ as G is cyclic. Here we have an equivalence $\mathrm{Coh}(X) \cong \mathrm{Coh}^G(\widetilde{X})$ already on the level of abelian categories (see [13, §7]). Hence, $\mathrm{D^b}(X) \cong \mathrm{D}^G(\widetilde{X})$ as well, a fact which also follows from the derived McKay correspondence using the trivially crepant resolution $X \xrightarrow[\mathrm{id}_X]{} X$. Then,

$$\mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G \xleftarrow[n:1]{\mathit{for}} \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(\widetilde{X})) \xrightarrow[n:1]{\mathit{inf}} \mathrm{Aut}(\mathrm{D^b}(X)) = \mathrm{Aut}(\mathrm{D}^G(\widetilde{X})).$$

Bridgeland and Maciocia consider in [5] canonical quotients from the point of view of derived categories. They introduce the set of all *equivariant* equivalences by

$$\mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X})) := \{(F,\mu) \in \mathrm{Aut}(\mathrm{D^b}(\widetilde{X})) \times \mathrm{Aut}(G) \,:\, g_* \circ \widetilde{F} \cong \widetilde{F} \circ \mu(g)_* \; \forall g \in G\};$$

this is actually a group. There is an exact sequence

$$0 \longrightarrow \mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G \longrightarrow \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X})) \longrightarrow \mathrm{Aut}(G) \longrightarrow 0$$

where the latter morphism maps $(\widetilde{F},\mu) \mapsto \mu$ and $\mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G$ is by definition the group of all equivariant equivalences with $\mu = \mathrm{id}_G$. Note that $G \cong \mathbb{Z}/n\mathbb{Z}$ implies $\mathrm{Aut}(G) \cong \mathbb{Z}/\varphi(n)\mathbb{Z}$. Furthermore, we have a subgroup $G \hookrightarrow \mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G$, $g \mapsto g_* = (g^{-1})^*$, or also $G \hookrightarrow \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))$, $g \mapsto (g_*, \mathrm{id}_G)$. The latter is a normal subgroup in view of $(\widetilde{F},\mu)^{-1} \circ (g_*, \mathrm{id}_G) \circ (\widetilde{F},\mu) = (\widetilde{F}^{-1},\mu^{-1}) \circ (g_* \circ \widetilde{F},\mu) = (\mu(g)_*, \mathrm{id}_G)$. By [5, Theorem 4.5] every equivalence $F \in \mathrm{Aut}(\mathrm{D^b}(X))$ has an equivariant lift $\widetilde{F} \in \mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))$, i.e. $\pi_* \circ \widetilde{F} \cong F \circ \pi_*$ and $\pi^* \circ F \cong \widetilde{F} \circ \pi^*$. If $\widetilde{F}_1$ and $\widetilde{F}_2$ are two lifts of F, then $\widetilde{F}_2^{-1} \circ \widetilde{F}_1$ is a lift of $\mathrm{id}_{\mathrm{D^b}(X)}$ and thus $\widetilde{F}_2^{-1} \circ \widetilde{F}_1 \cong g_*$ for a $g \in G$ ([5, Lemma 4.3(a)]). Thus the equivariant lift $\widetilde{F} \in \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))$ is unique up to the action of G and we get a group homomorphism $\mathit{lift} : \mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))/G$. [5, Lemma 4.3(b)] states that if $F, F' \in \mathrm{Aut}(\mathrm{D^b}(X))$ both lift to $\widetilde{F} \in \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))$, then they differ by a line bundle twist: $F \cong F' \circ \mathsf{M}_{\omega_X^i}$ with $0 \le i < n$. Thus *lift* is $n:1$, and we propose the following commutative pentagon

$$\begin{tikzcd}
 & \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(\widetilde{X})) \arrow[dl, twoheadrightarrow, "for"', "n:1"] \arrow[dr, "inf", "n:1"'] & \\
\mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G \arrow[d, hook, "1:\varphi(n)"] & & \mathrm{Aut}(\mathrm{D^b}(X)) = \mathrm{Aut}(\mathrm{D}^G(\widetilde{X})) \arrow[d, "n:1"', "lift"] \\
\mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X})) \arrow[rr, twoheadrightarrow, "n:1"] & & \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))/G
\end{tikzcd}$$

The commutativity of this diagram boils down to the following question: given a kernel $(P,\rho) \in \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(\widetilde{X}))$, is $\mathsf{FM}_P : \mathrm{D^b}(\widetilde{X}) \xrightarrow{\sim} \mathrm{D^b}(\widetilde{X})$ a lift of $\mathsf{FM}^G_{(P,\rho)} : \mathrm{D}^G(\widetilde{X}) \xrightarrow{\sim} \mathrm{D}^G(\widetilde{X})$ (where we identify $\mathrm{D}^G(\widetilde{X}) \cong \mathrm{D^b}(X)$)? However, this is clear from $\pi_* : \mathrm{D^b}(\widetilde{X}) \to \mathrm{D}^G(\widetilde{X})$, $E \mapsto inf(E)$ and $\pi^* : \mathrm{D}^G(\widetilde{X}) \to \mathrm{D^b}(\widetilde{X})$, $(F,\lambda) \mapsto [F,\lambda]^G$.

Example 12. In the case of a double covering (e.g. X an Enriques surface), we have $n = 2$ and hence $\mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G = \mathrm{Aut_{eq}}(\mathrm{D^b}(\widetilde{X}))$. Then the diagram looks like

$$\begin{tikzcd}
 & \mathrm{Aut}^{G_\Delta}(\mathrm{D^b}(\widetilde{X})) \arrow[dl] \arrow[dr, "inf"] & \\
\mathrm{Aut}(\mathrm{D^b}(\widetilde{X}))^G/G & & \mathrm{Aut}(\mathrm{D^b}(X)) \arrow[ll, "lift"]
\end{tikzcd}$$

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