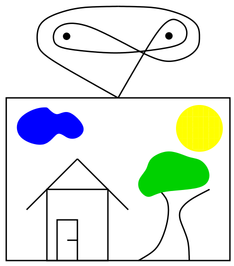
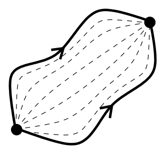
1 The fundamental group **Topology I**



Exercise: Put the picture on the wall using two nails in such a way that removing either of the nails will make the picture fall down to the floor.
The goal of the course will be to see how one can come up with a solution without experimentation. Also, the same statement holds for any number of nails, and for that, it is useful to have a way to compute the solutions.

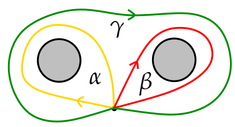
Notation.
Possibly introduce the notations $\mathbb{R}$, $\mathbb{R}^2$, $I = [0, 1]$, $\mathbb{Z}$ and how to write down mappings.

Abstraction.
The string of the picture solution is a loop, a closed path. General definition: a *path* in the plane is a continuous map $\gamma : I \to \mathbb{R}^2$. (Can be sloppy about continuity, give concrete examples by both formula and picture, mention starting and end points, define loop.)
The string of the exercise is flexible; we cater for this by a sloppy definition of homotopy: two plane paths are called homotopic if one can be moved into the other in a continuous manner. This is a purely graphical definition!



Operations on paths.
Concatenation and inversion of paths. (One can give the formulae but the pictures are more important.)

Exercise: Show, by drawing the homotopies:
$\alpha * \beta = \gamma$ and $\alpha * \alpha^{-1} = c$ (constant loop).



Fundamental groups.
The fundamental group of a subset of the plane, $X \subset \mathbb{R}^2$, can be sloppily introduced as the set of loops starting at the based point and considered up to deformations (homotopy). It is not necessary to elaborate on the group structure (unless the students already know what a group is).
Here, it is useful to introduce the notation X_n for the plane with n holes. (It might be a good idea to point out that size and location of the holes won't matter. The reason is homotopy invariance of the fundamental group; if there is time, this could also be explored by way of examples.)
Example: $\pi_1(X_0, 0) = \pi_1(\mathbb{R}^2, 0) = 0$ (only one element, the constant loop).

Exercise: How many different loops are there in X_1, up to homotopy? (Phrased in this way, many students can see the need of iterated loops and orientation reverse by themselves. In the end, everyone should understand $\pi_1(X_1) = \mathbb{Z}$.)

Exercise: In the above example of loops in the doubly pointed plane X_2, see that $\alpha * \beta$ is not homotopic to $\beta * \alpha$. (There is no graphical proof of such a statement, but it is still easy to see in the pictures.)
If there is time (or to keep eager students busy): show that the two base loops on a torus do commute.

Guide students towards $\pi_1(X_2) = F_2$, the free group with 2 generators. I usually mention the theorem of Seifert and van Kampen so that the students see something nontrivial is going on. One way of phrasing it: Any loop in X_2 can be deformed into one given only by a word consisting of letters α, α^{-1}, β and β^{-1}; two different words describe different loops (i.e. there are no relations — the above exercise is supposed to hint at that).

The solution to the initial exercise and generalisations.

Exercise: Find a word that solves the initial exercise.

Exercise: Find a word that works for three nails. (It may be necessary to draw a picture of X_3 and the base loops beforehand. It will not be necessary to repeat the whole process done for $n = 2$ already.)

Exercise: Find a solution for n nails. (Introducing $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ will help. If doing this, the typical solution will be $w_n = [\cdots[[\alpha_1, \alpha_2], \alpha_3]\cdots, \alpha_n]$.)

Exercise: If someone needs a puzzle: what about an infinite number of nails?

Some combinatorics.

If there is time left, one can have a look at the length of the words.

Exercise: What is the length of the word w_n? (There is the obvious recursion $w_{n+1} = 2w_n + 2$ which can be turned into the closed formula $w_n = 3 \cdot 2^{n-1} - 2$.)

Exercise: Does w_n give the shortest solution? (No, take for example this word for $n = 4$ and 16 letters: $[\alpha, \beta][\gamma, \delta][\beta, \alpha][\delta, \gamma]$. This solution grows quadratically in n.)

Didactical notes

This course is pedagogically robust: Once students are interested in the initial exercise (which while absurd, is still funny and counter to intuition), they don't need to understand all the details about $\pi_1(S^1 \vee S^1)$ in order to grasp the (mathematical) solution. Instead, they can simply check the solutions produced by the machinery. So everyone should get the start and the end, and hopefully something in between.
From a mathematical point of view, note that almost no prerequisites are needed. By restricting to subsets of the plane, the concepts of topology or metric can be left implicit.

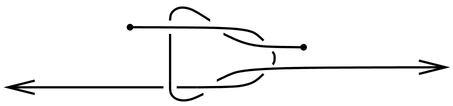
2 Knots **Topology II**

We used the first two books for the course, and consulted the third one afterwards.

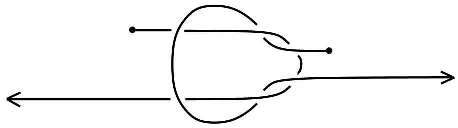
References

[BZ] Burde, Zieschang, *Knots*, (1985).
[R] D. Rolfson, *Knots and Links*, Publish or Perish (1976).
[K] Kauffmann, *Knots and Physics*, (1991).

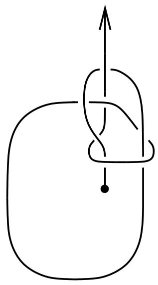
Introductory examples.
We start with some knots from real life which are used by sailors and mountain climbers. (To be honest, they are using more elaborate knots actually. Surf the web for examples.) It works well to have the students make these knots from line (working in pairs, for example).



granny knot



square knot

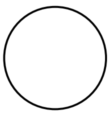


bowline

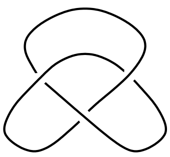
The first and second can be used to splice two ropes and the bowline is used to turn one end into a fixed loop.

Exercise: Does the square knot or the granny knot work better at splicing ropes? (The square knot is better; see below for an explanation.)

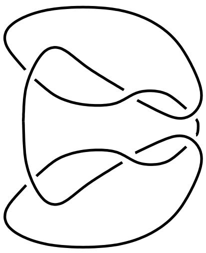
Formalising.
For mathematical purposes, it is best to connect the lose ends and treat knots as circles in space. Thus, a *knot* is a continuous embedding $K : S^1 \hookrightarrow \mathbb{R}^3$. (Need to say something about the symbols S^1 and $\mathbb{R}^3$, and also about continuity and embeddings.) Examples:



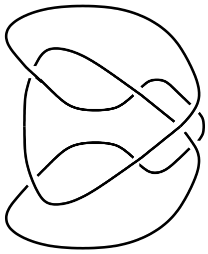
unknot



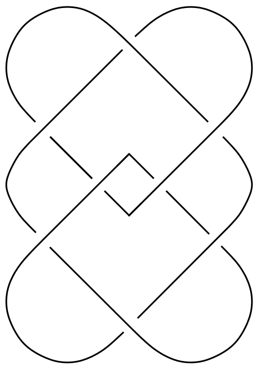
trefoil (ordinary knot)



square knot



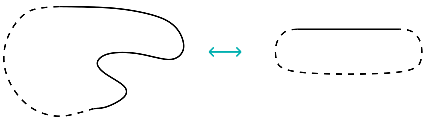
granny knot



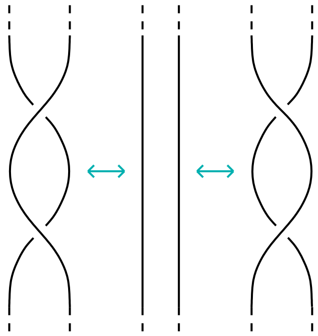
true lover's knot

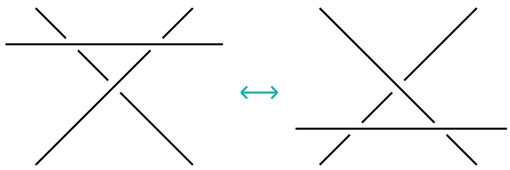
Often, embeddings $S^1 \hookrightarrow S^3$ are considered instead. Without further notice, we'll be only talking about *tame*, i.e. polygonal knots, in order to avoid pathologies. Finally, it should be pointed out that there are differences between $\mathcal{C}^0$, PL and $\mathcal{C}^\infty$ categories.

Guided by real life ropes, we will consider two knots to be equivalent, if there is a continuous deformation of one into the other in 3-space. More formally, this notion is *ambient isotopy*; see (BZ) for a proof. For the purposes of this course, it is easiest to define this equivalence by the *Reidemeister moves*. These gives rules for manipulating a part of the knot diagram — all of the rest is meant to be kept. Note that rules I, II, III deal with crossing of one, two, and three strands, respectively.

0. 

I. 

II. 

III. 

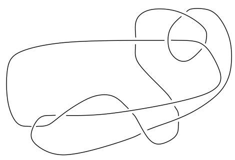
There are several notions of equivalence for knots, which are different in general:

pair equivalence:	there is a homeomorphism $h : \mathbb{R}^3 \xrightarrow{\sim} \mathbb{R}^3$ with $h(K(S^1)) = K'(S^1)$.
map equivalence:	there is a homeomorphism $h : \mathbb{R}^3 \xrightarrow{\sim} \mathbb{R}^3$ with $hK = K'$.
oriented versions:	for the above, h has to respect orientations of $\mathbb{R}^3$ and S^1.
ambient isotopy:	there is a homotopy $h_t : \mathbb{R}^3 \xrightarrow{\sim} \mathbb{R}^3$ of homeomorphisms with $h_0 = \mathrm{id}$, $h_1 = h$.
regular isotopy:	defined by Reidemeister moves 0, II, III.

There is no difference between pair and map equivalence (in our setting)?

Note that ambient isotopies are orientation preserving.

Exercise: Show that the following knot is (equivalent to) the unknot, both by experiment and Reidemeister moves.



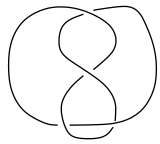
Invariants.

It can be complicated to check if two knots are (not) equivalent. For this purpose, we try to assign properties or numbers (or more complicated objects) to knots, in such a way that equivalent knots get assigned the same property or number. One way to prove this is by checking that the property or number is invariant under the Reidemeister moves.

Crossing number. The *crossing number* of a knot K is the smallest number of crossings of all knots equivalent to K. Thus the crossing number of the unknot is zero, and the crossing number of the trefoil is three (the latter is not trivial, but intuitively obvious).

Exercise: Find a knot with crossing number four.

eight knot



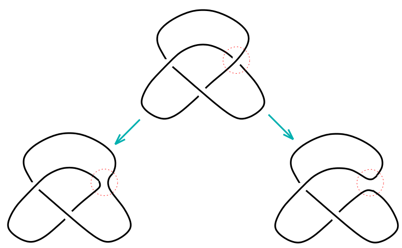
The crossing number is an invariant by definition, but tricky to determine. For example, there are trivial knots where no Reidemeister move leads to a simplification. It it used as a coarse approximation of difficulty: the classification proceeds by crossing numbers.

Writhe. The *writhe* or *twisting number* of an oriented knot is defined as the sum $w(K) = \sum_p \epsilon(p)$ over all crossings p with $\epsilon(\times) = 1$ and $\epsilon(\times) = -1$. Writhe is an invariant for regular isotopy, but not for ambient isotopy. For example, the writhe of the trefoil T is $w(T) = 3$ (regardless of the orientation) and also $w(T^*) = -3$, yet this does not prove $T \ncong T^*$.

Knot group. A very important invariant is the *knot group* $\pi_1(\mathbb{R}^3 \setminus K)$. A simple exercise is $\pi_1(\mathbb{R}^3 \setminus U) = \mathbb{Z}$ for the unknot U. Some more facts:
$\pi_1(K) = \pi_1(S^1) = \mathbb{Z}$ is an invariant, but a useless one.
$H_1(\mathbb{R}^3 \setminus K) = \mathbb{Z}$ is also useless (the equality follows from Alexander duality, for example).
$\pi_1(\mathbb{R}^3 \setminus K) = \pi_1(S^3 \setminus K)$ by Seifert-van Kampen.
$\pi_1(\mathbb{R}^3 \setminus K) = \mathbb{Z} \iff K$ is an unknot.
$\pi_1(\mathbb{R}^3 \setminus K)$ has non-trivial center $\iff K$ is a torus knot.
The *Wirtinger presentation* gives an explicit description of the knot group by generators and relations.

Alexander invariant. Given a knot K, there is a canonical infinite cyclic covering $\tilde{X} \to S^3 \setminus K$ (the universal abelian covering space) and its total homology $H_*(\tilde{X}, \mathbb{Z})$ is obviously a knot invariant. After choosing a generator t of the group of deck transformations, $H_*(\tilde{X}, \mathbb{Z})$ becomes a $\mathbb{Z}[t, t^{-1}]$-module, the *Alexander invariant*.
For classical knots, only $H_1(\tilde{X})$ is important. Choosing a presentation matrix for $H_1(\tilde{X})$ (which is square in the case of classical knots), the determinant is the Alexander polynomial. Traditionally, it is conconcted from the diagram in a rather roundabout way. (For more general knots, one looks at the Fitting ideal of the presentation matrix which is generated by all maximal minors.) See (R, 8C, 7A) for details.

Bracket polynomial. Considering all ways in which a knot can be spliced leads to a tree of diagrams. Note that links appear here. After an appropriate normalisation (using writhe), one obtains a Reidemeister-invariant polynomial. It is strong enough to see $T \not\sim T^*$. This approach is related to the Jones polynomial. See (K, I.3, I.7) for details.
The bracket polynomial could be investigated in a course.

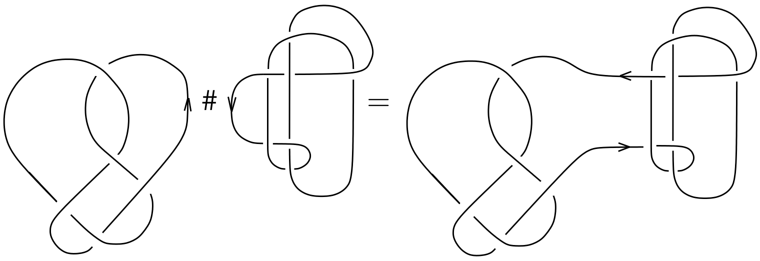


Construction of knots

Dual knots. Given a knot K, the *dual knot* K^* is obtained by changing all overcrossings into undercrossings and vice versa. Another description is given by rK for an appropriate reflection r. (Give drawings for $T^* = rT$ for the trefoil T.)

Exercise: Show $T \ncong T^*$ for the trefoil T. Show $E \cong E^*$ for the eight knot E.

Connected sums. Given two oriented knots K and K', one can form the *connected sum* $K\#K'$:



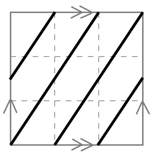
Exercise: $T\#T^* = S$ and $T\#T = G$, where T is the trefoil, G is the granny knot and S is the square knot (and also $T^*\#T^* = G^*$). This explains why the square knot is superior to the granny knot: the latter is just two ordinary knots in succession.

The connected sum is only well-defined for oriented knots in the oriented S^3. Such knots form an abelian semigroup under #. There are no units except for the unknot by the Cancellation Theorem: if $K\#K'$ is the unknot (for tame knots K, K'), then K and K' are unknots. Introducing the stronger equivalence relation of concordance (knot cobordism), knots form an abelian group with $-[K] = [r\overline{K}]$ where $r\overline{K}$ is the dual knot with the opposite orientation. See (R, 4B, 8F).

Special types of knots

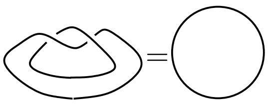
Plane knots. By the Jordan Curve Theorem, all plane knots $K : S^1 \hookrightarrow \mathbb{R}^2 \subset \mathbb{R}^3$ are unknots.

trefoil

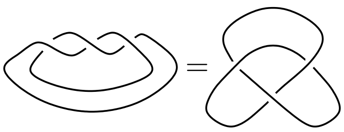


Torus knots. Let $T^2 = S^1 \times S^1$ be the 2-torus. Introduce it as both the donut and the glued square.

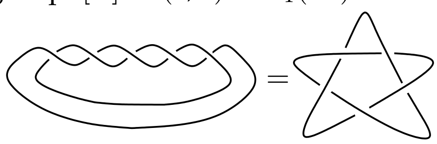
Exercise: Find a non-trivial torus knot.
Any torus knot $K : S^1 \hookrightarrow T^2$ has a class in the fundamental group: $[K] = (a, b) \in \pi_1(T^2) = \mathbb{Z}^2$.



$[K] = (2, 1)$ unknot



$[K] = (2, 3)$, the trefoil



$[K] = (2, 5)$, Salomon's knot

Up to unoriented equivalence, torus knots are classified by their invariant $[K] = (a, b) \in \mathbb{Z}^2$ as follows: a and b have to be coprime; if $a = \pm 1$ or $b = \pm 1$, then K is an unknot; else (a, b), $(\pm a, b)$ and (b, a) give equivalent knots. For oriented equivalence, (a, b) and $(-a, b)$ are not equivalent (as can be seen for the trefoil).

Fibred knots. $K \subset S^3$ is called *fibred* if there is a fibration $\pi : S^3 \setminus K \to S^1$ with a framed neighbourhood $\varphi : S^1 \times D^2 \xrightarrow{\sim} U(K) \subset S^3$ such that $\varphi(S^1 \times \{0\}) = K$ and $\pi|_{U'} : U' \to S^1$, $(x, y) \mapsto y/|y|$ (where $U' := \varphi(U \setminus K)$).
Examples are the unknot, trefoil, eight knot, true lover's knot. The connected sum of fibred knots is again fibred. See (R, 10H) for details.

Classification

There are infinitely many inequalent knots. But for fixed crossing number, there are is only a finite number of different types. The following table lists different knots in the traditional way (Alexander and Biggs 1926, Conway 1976), i.e. ignoring decomposable knots (with respect to #), and up to non-oriented equivalence:

3	4	5	6	7	8	9	10
1	1	2	3	7	21	49	166

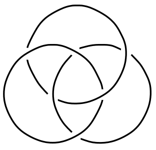
Summary and didactical notes

The theory of knots exhibits three typical stages of mathematical research:
1. Formalise the problem (includes notion of equivalence).
2. Decide when two objects are equivalent (this leads to invariants).
3. Attempt classification.

In order to reach the students, the exercises (bring string and scissors) are mandatory. Only drawing examples on the blackboard (and let the students solve exercise by drawing) will not work as well.

Further topics.

Links. Natural generalisation of knots, necessary for the bracket polynomial. Linking numbers. Example: the Borromean rings (which are non-trivially linked, but pairwise unlinked).



Differential topology.

Seifert surfaces. Surgery (e.g. Dehn twists).