THE BEST WAY HOW NOT TO HANG PICTURES ON WALLS — TOPOLOGY IN SCHOOL

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Abstract. I report about my experience on a topology course given to pupils. It works for a broad range of audiences, and I indicate reasons why. mathematical interest; fundamental group

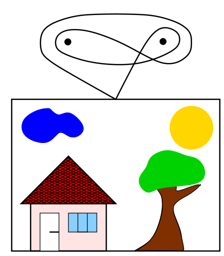
Here, I report on a topic that has served very well to get young people interested in mathematics — not the kind of mathematics they usually encounter at school. I have given the course many times to audiences of the age 15 and up. It fits reasonably into a session of about three hours, although it is possible to give a more lecture-like presentation in under an hour, or to use it for a weekly study group. No mathematical knowledge is needed. For reasons explained later, this topic can be used both for reaching weaker students as well as for engaging strong students.

At face value, the students encounter a paradoxical problem. Mathematically, we are dealing with a well-known computation from Algebraic Topology. However, the main point is that avoiding all technicalities, we can solve the curious problem in a way that everyone will understand the solution.

After stating the problem, I explain in detail the course. Afterwards, I give some educational remarks indicating why this particular topic works so well. This is followed by a section with remarks that were made by students. I then describe the content from a mathematical point of view and the assessment of lesson content. The article ends with a note on the history of the subject.

The nails-and-picture problem

Exercise 1: Hang the picture on the wall using two nails in such a way that removing either of the nails will make the picture fall down to the floor.



The goal of the course will be to see how one can come up with a solution without experimentation. The same statement holds for any number of nails, and for that, it is necessary to have a way to compute the solutions.

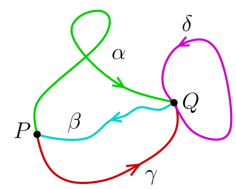
Conducting the course

As I see it, the course should not take the form of a lecture. Much better to let the students figure out as much as possible for themselves. For this purpose, I alternate between introducing notions and posing exercises.

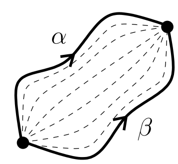
All notions are employed in a purely naive way. For example, there is no need to formalise the notion of path; it suffices to treat the curve in the plane (or on the blackboard, or in the notebook) as a path. In the following, we indicate some pictures that help students get the idea. Many more pictures can be used. Incidentally, it is more instructive to perform the drawing of a path on some kind of board than to have sheets or slides.

Abstraction. The string of the nail-and-picture problem is a loop. We will define a number of standard notions in very informal way: only via pictures! It is customary to label paths and loops with small Greek letters like α, β, γ.

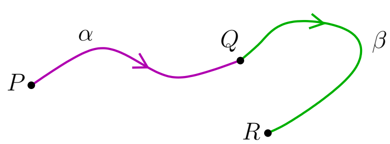
A good way to start is by drawing some *paths*. It is important to point out that a path has a start point and an end point and hence a direction. A path can cross itself, best shown in a drawing. After this, introduce a *loop* as a path which ends at the starting point. The direction still matters.

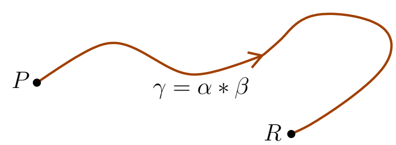


The string of the picture-and-nails problem is flexible; we cater for this by saying that two paths (with the same endpoints and direction) are *homotopic* if one can be moved into the other in a continuous manner, while fixing the end points. In other words, twisting, bending and stretching are allowed but cutting is not!
We write $\alpha \sim \beta$ if paths α and β are homotopic.

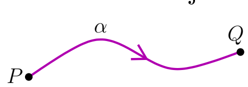


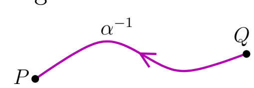
Operations on paths. The *concatenation* of a path α from P to Q with a path β from Q to R leads to a path $\gamma = \alpha * \beta$ from P to R:



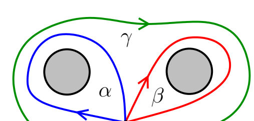


And the *inverse* of α is just α^{-1} from Q to P, travelling backwards:



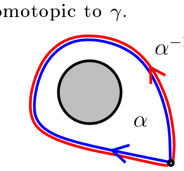


Exercise 2: Show, by drawing the homotopy: $\alpha * \beta \sim \gamma$. This takes place in the plane with two holes. The exercise is fundamental for understanding the solution to the original problem.



The concatenation $\alpha * \beta$ is homotopic to γ.

Exercise 3: Show that $\alpha * \alpha^{-1}$ is homotopic to the constant loop, i.e. the idle path which is not moving at all.
For this exercise and the one above, the crucial insight is that while starting and end points are fixed, no other point of a loop is. The concatenations of the exercises can therefore be modified so as to not touch the starting point midway.



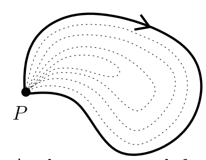
Fundamental groups. In the following, we will need some notation for the plane with holes: X_0 is the whole plane, X_1 the plane with one hole punched out, X_2 the plane with two holes etc. It is a fact that the position of the holes does not matter and neither does the size (they could be shrunk to points); this circumstance can be tacitly ignored unless questions come up.

For any subset X of the plane together with some point P in X, we want to look at all loops starting at P. Having the string of the original picture-and-nails problem in mind, we don't want to distinguish two such loops if they're homotopic. The collection of all loops up to deformation gets a curious symbol:

$\pi_1(X, P)$ *is the set of all loops in* X *starting at* P*, where homotopic loops are identified.*

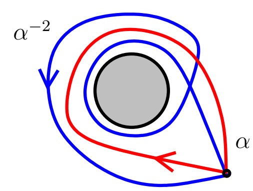
To reiterate, we treat homotopic loops as the same element of $\pi_1(X, P)$. I always mention in passing that it is called the 'fundamental group' of X (and P) but there is no need to dwell on the group structure — unless students are already familiar with groups, then it provides an interesting example. The symbol π_1 will look strange to many students, so one can just say that the Greek letter π is used in honor of the great French mathematician Henri Poincaré who invented this tool in 1895. The subscript 1 is a more recent invention by Hurewicz in 1935, who defined a whole list of 'homotopy groups' of which $\pi_1(X, P)$ is the first.

Example: For the entire plane X_0 (no holes), any loop, no matter how complicated, can be deformed into the loop that does not move at all, the 'constant loop'. Another way of saying this: in the absence of holes, any two loops can be deformed into each other. So the fundamental group $\pi_1(X_0, P)$ has only one element.



A homotopy deforming the outer loop into the constant loop.

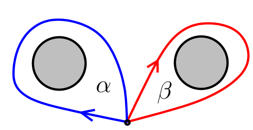
Exercise 4: How many loops are there in X_1, up to homotopy?



Phrased in this way, many students can see the need of iterated loops and orientation reverse by themselves. In the end, everyone should see $\pi_1(X_1) = \{\ldots, \alpha^{-2}, \alpha^{-1}, \alpha^0, \alpha^1, \alpha^2, \ldots\}$ where $\alpha^2 = \alpha * \alpha$ etc. In other words, for every integer there is exactly one loop in X_1 up to homotopy, the number saying how often to run around the hole and in which direction.

Exercise 5: Considering the loops α and β of Exercise 2 in the doubly pointed plane X_2, convince yourself that $\alpha * \beta$ is not homotopic to $\beta * \alpha$. (There is no graphical proof of such a statement, but it is still easy to see in the pictures.)

We finally turn to loops in X_2 — getting close to the solution. There are the two base loops α and β. We also have the various powers α^2, β^{-3} etc. Furthermore, these can be combined: $\alpha\beta$, $\alpha\beta^{-2}\alpha$ etc. (It is sloppy, but customary, to drop the $*$ symbols.) As it happens, we get all loops in X_2 up to homotopy in this way:



Loops in X_2 *up to homotopy are given by words in the two letters* α *and* β*.*

Phrased in a different manner, $\pi_1(X_2, P)$ consists of all finite expressions (the 'words') $\alpha^{i_1}\beta^{j_1}\alpha^{i_2}\beta^{j_2}\cdots\alpha^{i_r}\beta^{j_r}$ of any length $r \geq 0$ with integers $i_1, j_1, \ldots, i_r, j_r$.

It is clear that all those words give rise to loops in X_2. The above assertion is twofold: there are no loops we're missing, and there are no relations among them. The former is easy to believe and the latter is hinted at by Exercise 5.

The statement about $\pi_1(X_2, P)$ follows from the famous theorem of Seifert and van Kampen [Ha, §1.2]. In any case, for our phenomological investigation this is not relevant but I like to indicate that some effort is needed to actually prove something.

The solution to the initial exercise and generalisations. Now we think about the string of the original problem as a loop in the doubly pointed plane (the nails stick out), so as a word in the letters α and β. What does it mean to pull out a nail? Obviously, this allows for more freedom of the string. In particular, a loop only running around that nail simply disappears — it can now be contracted to the constant loop.

Thus, ignoring the left-hand hole (nail) causes all α letters in the word to vanish, leaving a word in only β, i.e. a single power of β. Vice versa getting rid of the other nail kills all letters β.

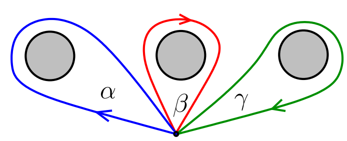
What we are then looking for is a word made up from powers of α and β which is not empty (e.g. $\alpha\alpha^{-1}\beta\beta^{-1}$ will *not* do; each letter and its inverse cancel each other right away) and has the following property: the word collapses into nothing when α is canceled, and likewise when β is removed.

Exercise 6: Find such a word. Draw the corresponding loop and see that, after a deformation as in Exercise 2 we get the desired solution.

The first part of this exercise, i.e. finding a word, can be solved without recourse to paths and homotopy. That is why it should be posed as explicitly as possible — this exercise is an opportunity for lost students to jump back into action, even if they missed something before.

Having invested all this work, we immediately pluck some low-hanging fruit:

Exercise 7: Find a word that works for three nails. The picture of the three base loops will help. It is not necessary to repeat the whole process done for two nails.



Exercise 8: Find a solution for $k > 2$ nails.
(Introducing $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ will help. Doing this, the typical solution will be $[\cdots[[\alpha_1, \alpha_2], \alpha_3]\cdots, \alpha_k]$.)

Some combinatorics. If there is time left, one can have a look at word lengths.

Exercise 9: What is the length of the word $w_k := [\cdots[[\alpha_1, \alpha_2], \alpha_3]\cdots, \alpha_k]$?
(There is the obvious recursion $w_{k+1} = 2w_k + 2$ which can be turned into the closed formula $w_k = 3 \cdot 2^{k-1} - 2$.)

Exercise 10: Does w_k give the shortest solution? (Hint: 4 nails.)
(No, take for example this word for $k = 4$ of length 16: $[\alpha, \beta][\gamma, \delta][\beta, \alpha][\delta, \gamma]$. It is shorter than w_4 which has length 22.)

Finding the shortest solutions for all k is tricky. It is a simple observation that one can restrict to exponents $+1$ and -1. A deeper insight is that we can obtain solutions for k nails by *splitting* $k = k_1 + k_2$: If W_1 is a solution word of length l_1 for k_1 nails and W_2 is a solution word of length l_2 for k_2 nails, then $[W_1, W_2] = W_1 W_2 {W_1}^{-1} {W_2}^{-1}$ will solve for $k_1 + k_2$ nails and is of length $2(l_1 + l_2)$. (The above word of length 16 is an example where $k_1 = k_2 = 2$ and $l_1 = l_2 = 4$.)

Examples:
$k = 5$: The word w_5 from Exercise 9 has length 46. Splitting $4 = 2 + 3$, we get a solution of shorter length $2(4 + 10) = 28$.
$k = 6$: Here we a choice of splitting, since $6 = 3 + 3 = 2 + 4$. However, the two solutions have the same length $2(4 + 16) = 2(10 + 10) = 40$.
$k = 7$: In this case, the splitting $7 = 2 + 5$ gives a solution of length $2(4 + 28) = 64$ whereas the splitting $7 = 3 + 4$ yields length $2(10 + 16) = 52$.

It is easy and fun to tabulate the numbers obtained in these ways:

k nails	2	3	4	5	6	7	8	9	10	11	12
$3 \cdot 2^{k-1} - 2$	4	10	22	46	94	190	382	766	1534	3070	6142
splitting	4	10	16	28	40	52	64	88	112	136	160

Educational remarks

In this section, I collect some observations regarding the instructional component. The course introduced here has certain desirable properties. Thinking about those in terms of the psychological and educational sciences allows to understand better why this concept works so well, and makes it easier to improve the presentation for this, and other, topics.

On a personal note, I have conducted a number of extra-curricular presentations, with various mathematical topics. Most of them suffer from the fact that the main problem appeals to strong students but fails to motivate weaker ones. In other words, these topics are well-suited only for an adequate audience, e.g. study groups where interested schoolchildren voluntarily take part. By contrast, the picture-and-nails course is one of few topics that manage to captivate all kinds of students equally well. The following sections elucidate that effect from different perspectives.

Interest. While the original problem is absurd, it is funny and defies intuition. This immediately catches the students' attention, which is particularly useful for schoolchildren unmotivated by their mathematics class.

From practice I can confirm that most of the students were able to solve Exercises 2 to 8 of the course. Hence they experience competence in a new area. This is one of the basic needs of the self-determination theory of motivation by Deci and Ryan [DR].

It may be premature to talk about development of *stable actual interest*, in the sense of Krapp [K], but a *first development of actual interest* is certainly involved — this is Krapp's 'catch component'.

Scaffolding. In this section I assume that circumstances allow the lecturer to talk to students individually or in pairs. This means enough time and not too many students.

Given this, the approach for this course is closely related to *Scaffolding* and Vygotsky's *zones of proximal development* [V]. *Constructivism* assumes that learners construct knowledge out of previous knowledge and experience. In practice, this is achieved by either self-exploration or providing stimulus appropriate to each learner's current state of knowledge.

For the topic treated here, students are not expected to solve the initial problem (Exercise 1). While in fact a few actually might do, even those will not be able to master the more general problem of an arbitrary number of nails. More importantly, one can assume that students have no prior knowledge at all, so they all start on the same level.

The approach of alternating between presenting notions and posing exercises allows the lecturer to give just those impulses which are needed to enable students solving the exercises on their own. Ideally, this can be done talking to students one on one, addressing their specific cognitive situation.

Why the course is suited for mixed audiences. Generally speaking, mathematical education as known to pupils rarely focuses on problem solving competence. Even if it does, weaker students are often disadvantaged beyond their lack of competence: once they fail to understand one part of the explanation, they tend to miss the subsequent reasoning as well.

For the course examined here, a significant difference emerges: Naturally, it can happen that students get lost. For example, the introduction of the fundamental group is a potential stumbling block. Nevertheless, those students can jump in again for Exercise 6. Although they won't be able to explain why this particular word is sought after (hence missed a Vygotsky zone they might have achieved), this part of Exercise 6 can be solved ad hoc. If they also understood (not necessarily solved themselves) Exercise 2 — as almost all students do —, then they can put together the two pieces in order to come up with their own solution of the original picture-and-nails problem. Despite missing part of the course, this will leave them with a feeling of success, rather than failure!

Comments and student feedback

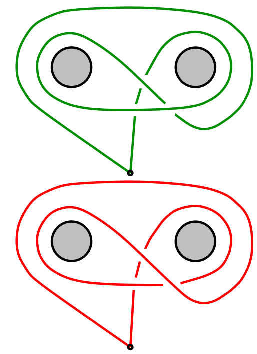
The first hurdle is to convince the students that this is a genuine problem, not a hoax. This is best done by simply drawing some loops on the board and telling them that a solution looks like this, only a little more complicated.

Is it alright if students come up with the solution to the picture-and-nails problem right away. There is always disbelief about having solutions for thousand nails.

Students may get lost when the fundamental group is introduced. However, they can take part in solving the word problem. For this, it is necessary to make that problem as explicit as possible.

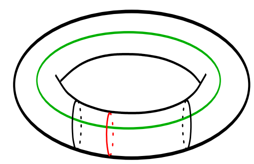
Sometimes, students pose interesting questions:

Knots. Our modeling of the original problem projects three-dimensional twine onto a two-dimensional image. In other words, the crossing information (which part of the twine goes above or below the other) is lost. Once, a student pointed this out to me. It does not affect our solution, however: since the loop is supposed to come from a physical string, the crossings will all match up.
As an example, the red string shows an ordinary way of winding the string around the nails. By contrast, the green string does not arise without artificially introducing a knot. In particular, removing the right-hand nail will *not* cause the picture to fall down!



Combinatorics. It was (another) student who asked about the length of solutions for many nails. He also came up with the shorter solution of Exercise 10.

Loop relations. At one time, a curious student asked whether the fundamental group always has the shape we encounter, i.e. as a free group. An interesting counterexample is the 'torus' (donut surface): Find two loops α and β on the torus which are not homotopic. Then show that $\alpha * \beta \sim \beta * \alpha$. This also makes a good additional exercise under certain circumstances. If you intend to cover this, bring a torus-shaped item.



Mathematical description and assessment of lesson content

A good source for the content of this course is Hatcher's book [Ha]. It is available in a free electronic version on the author's web page. The fundamental group is treated in the first chapter. Reading this chapter requires some familiarity with notions from undergraduate mathematics, like sets, maps, groups.

The problem is an application of Algebraic Topology: The nails are points N, M in the plane $\mathbb{R}^2$, i.e. the topological space under consideration is $X_2 := \mathbb{R}^2 \setminus \{N, M\}$ with the standard (Euclidean, i.e. norm) topology. A path is a continuous map $\gamma\colon [0,1] \to X_2$. The string corresponds to a loop with base point P, this is the requirement $\gamma(0) = \gamma(1) = P$. This loop gives an element $[\gamma] \in \pi_1(X_2, P)$.

The goal is to find non-trivial γ (not homotopic to the constant loop) in $X_2 := \mathbb{R}^2 \setminus \{N, M\}$ such that γ becomes trivial when considered as a loop in $\mathbb{R}^2 \setminus \{N\}$ and in $\mathbb{R}^2 \setminus \{M\}$. In other words, using the inclusions $i\colon X_2 \to \mathbb{R}^2 \setminus \{N\}$ and $j\colon X_2 \to \mathbb{R}^2 \setminus \{M\}$, we want that $i_*[\gamma] = 1$ is the trivial element of $\pi_1(\mathbb{R}^2 \setminus \{N\}, P)$ and that $j_*[\gamma] = 1$ is trivial in $\pi_1(\mathbb{R}^2 \setminus \{M\}, P)$.

The fundamental group of the plane with two points removed is known to be the free group with two generators, which can be chosen as simple loops around each removed point. This is a consequence of the theorem of Seifert and van Kampen [Ha, §1.2].

Therefore $\pi_1(X, P)$ consists of words in two letters, α and β. Applying i_* to a word means to drop all letters α and analogously for j_*. The shortest solutions are words of length 4, for example $\alpha\beta\alpha^{-1}\beta^{-1}$. A loop homotopic to this path is depicted adjacent to Exercise 1. The concatenation is deformed so as to touch the base point only at the start and end.

Similarly, the space X_k obtained removing k points from the plane $\mathbb{R}^2$ has the free group on k letters as fundamental group. This allows for solving the generalised problem.

In the text, it was mentioned that size and position of the holes do not matter: Given two subsets X_k and Y_k of the plane, each with k holes, then X_k and Y_k are *homotopy equivalent*. In particular, their fundamental groups coincide [Ha, Proposition 1.18].

Basic notions. For the purpose of the talk, notions like topological space or metric are suppressed by only talking about 'the plane' and relying on the intuitive distance.

The lecturer has to introduce the notions of path, loop as well as homotopies among them. Of these, only 'homotopy' may pose difficulties. Mentioning the concept of 'continuity' (crucial in a rigorous development of the theory) is only useful when the students are familiar with continuous real functions. Concatenation of paths is a simple concept and Exercise 2 allows students to put the various notions together. Likewise, Exercise 3 does the same with inversion of paths.

Students are not expected to know about sets theory, e.g. how to write down sets and subsets. Because of this, it is prudent to introduce the relevant space X_k in the most simple fashion, as the k-pointed plane. I found it also worthwhile to hold off variable symbols as far as possible, which is why I prefer to speak of 'X_0, X_1, etc.' rather than 'X_k for $k \geq 0$'.

Fundamental group: introduction and X_1. Introducing 'fundamental group' as a technical term with the proper mathematical notation $\pi_1(X, x_0)$ has several purposes:

(1) Most importantly, explicit mention of the space X keeps track of crucial information: what freedom do the loops have when they undergo deformations. This could be done in other ways, but generally more clumsily.
(2) The fundamental group is the most advanced mathematical notion of the course. If students want to consult books or web pages, this is the phrase they need.
(3) In my experience, hinting at the existence of a 'proper' theory is quite often appealing to students.

I treat $X_0 = \mathbb{R}^2$ as an example rather than an exercise because it can be conceptionally more difficult to state absence rather than presence. In Exercise 4, students are asked to enumerate $\pi_1(X_1, P)$. All students will be aware of one non-trivial element, the loop going around the hole once. If necessary, it is easy to guide students towards the existence of further elements: The first insight is concatenation, i.e. going from α to α^2, α^3 etc. The second insight is inversion, i.e. listing α^{-1}, α^{-2} etc. Finally, it may be necessary to point out the remaining element α^0, the constant path. It does frequently happen that students come up with full solution, given a few minutes.

Words and the fundamental group of X_2. Exercise 5 exhibits a non-commutativity behaviour. Typcially, this is a new phenomenon for the students. The plain fact is also important for calculations later on, to convince everyone that different letters in words may not be interchanged.

The fact that $\pi_1(X_2, P)$ is the free group on two generators is difficult to prove, yet easily believed. This is why it should be presented in a simple manner. Mentioning the 'theorem of Seifert and van Kampen' again indicates the existence of a larger theory.

I avoid a proper definition of what a 'word on letters α and β' is but it will be very useful to give many examples. These should include

- $\alpha^2 = \alpha\alpha$, $\alpha^{-2} = \alpha^{-1}\alpha^{-1}$ (this also explains the exponential notation)
- $1 = \alpha\alpha^{-1} = \alpha^{-1}\alpha$ (cancellation as for fractions)
- $1 = \beta\beta^{-1} = \beta^{-1}\beta$
- $1 = \alpha^0 = \beta^0$ represents the constant path
- $\alpha\beta \neq \beta\alpha$ (no commutativity)

Induced homomorphisms $\pi_1(X_2, P) \to \pi_1(X_1, P)$ and the solution. It is easily explained what happens when a hole disappears: the corresponding base loop becomes contractible, so the letter vanishes from words describing loops in X_2. Given time, this also makes for another good exercise.

This explains the first part of Exercise 6: a word with these properties will solve the original problem. To emphasize once more: given a careful introduction of words and how to calculate with them, as well as a precise statement of the properties the sought-after word should have, students who already lost the plot can participate again.

Exercises 7 and 8 have the purpose of showing the power of the technique. It is sensible to try and come up with an experimental solution for two nails. This is also how mathematical results are guessed. After this, the systematic approach is employed to get a full view and will reap further rewards — in our case simple solutions for an arbitrary number of nails.

Combinatorics. I at least mention that the word from Exercise 8 is not optimal; if there is time, I let students explore this themselves. Elaborating on this is done for the following reasons:

- In typical mathematical fashion, one question leads to the next: after knowing existence, it is natural to ask for efficiency. For our case, as soon as we know that there is a solution (for example via Exercise 8), we would like to get the best, or at least a good, solution. Incidentally, it is also typical that the first solution is hopeless from a practical point of view: the word length from Exercise 8 grows exponentially in k, whereas the splitting solution only grows quadratically.
- Mathematical areas are connected to each other. The picture-and-nails problem starts out as geometry, then is abstracted using topology, and solving by turning it into a question on abstract groups. Asking for the best solution connects to combinatorics.

Historical note

The notion of homotopic curves was introduced in 1866 by Camille Jordan under the label 'irreducible' in an informal — by current standards — way, just as in the course above:

Any two closed contours, drawn on a given surface, are called reducible into one another, if one can pass from the one to the other by a progressive deformation.

He also states that in the the whole plane, any two loops are homotopic (cited from [Ja, §2]).

This piece of historical information can be useful to university students: In mathematics, there is a gap between intuitive ideas and the formalism for expressing them rigorously. For students, it can be very instructive to see how far the antecessors went while relying almost exclusively on intuition. It is true, and should be mentioned, that this approach is prone to errors of all kinds. However, one does get the impression that rigor and intuition require each other, and that the former shouldn't always be stressed at the expense of the latter.

The fundamental group was introduced in 1895 by Henri Poincaré in the groundbreaking article Analysis Situs [P]. An English translation has recently been published [PS]. It should be mentioned that Poincaré wrote his treatise when neither point-set topology nor group theory were developed. This is why he used manifolds as spaces (later polyhedra as well) and employed permutations in order to describe the groups they generate. Neither definition nor notation from [P, §12] are similar to the one given here. However, Poincaré was surely aware of the fact that the fundamental group of the plane with k holes has k generators without relations. Poincaré goes far beyond the mere definition; he develops a substantial theory, culminating in a famous conjecture which now bears his name, only proved more than hundred years after his work.

I couldn't trace back the first use of $\pi(X,x)$ for a fundamental group. The notation $\pi_1(X,x)$ (i.e. with the subscript) was introduced in 1935 by Witold Hurewicz [Hu] when he defined the homotopy groups of topological spaces, the first one of which is the fundamental group. (Higher homotopy groups were independently mentioned by Max Dehn and Eduard Čech before that.)

Finally, the picture-and-nails problem is folklore in topology circles. I couldn't find a proper source for the problem. It seems to be lesser known than classics like the 'hairy ball theorem'.

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