Supercomplete Extenders and Type 1 Mice, I

by Q. Feng* and R. Jensen**

ABSTRACT We study type 1 premice equipped with supercomplete extenders. In this paper, we show that such premice are normally iterable and all normal iteration trees of type 1 premice has a unique cofinal branch. We give a construction of an K^C type model using supercomplete type 1 extenders.

Keywords: iteration trees, premice, mice, extenders, supercompleteness, inner model, large cardinals.

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§1 Introduction

In [7], Steel constructs an alternative inner model which he calls K^C and then derives a true core model K from it to anticipate one Woodin cardinal. It is assumed that there is a measurable cardinal in the working universe. A natural question is whether such measurable cardinal is needed in order to construct such a core model. Steel conjectures that one can do without such hypothesis. It is therefore desirable to understand where the complexity in deriving this core model occurs at which level of large cardinals. Schindler [6] isolates the core model below the level where an overlapping pair occurs by rediscovering an idea of Dodd. In his construction, Schindler requires that the extenders are ω–complete and are of type 0. By constraining to type 0 extenders, iterations are almost linear iterations, namely, iteration trees are finite branching and each extender is applied somewhere on the same branch leading to the point where the extender is produced.

In this paper, we start our investigation along this line. We shall deal with type 1 extenders. We allow one level overlapping but not two levels. In order to handle such extenders, we require that the extenders are supercomplete, a property that is stronger than being ω–complete but derivable from other known "backgrand conditions". After a short introduction to our fine structure set up in this section, we carry out our analysis of normal iteration trees of type 1 premice in Section 2. The main result of Section 2 is that a normal iteration tree of type 1 premice of limit length has a unique cofinal branch. In Section 3, we prove that every type 1 premouse with only supercomplete extenders is simply normally iterable. The proof shall indicate the main idea which shall be applied in the last section to show that each N_ξ is normally iterable. In Section 4, we carry out a K^C type construction of an inner model by putting only type 1 supercomplete extender on at each potential level and show that the levels of the construction are normally iterable and, granting an inaccessible cardinal in the universe, they are fully iterable. This type of construction is basically the one given in [5].

Our general set theory usage is [1], which we take as our basic reference book. We refer [8] as our fine structure theory standard reference book. Any undefined terms are taken from there, although we try to minimize the number of times of checking with the book while reading this paper.

We intend to construct an inner model to host larger cardinals by changing our current λ–indexing schema to a new Λ–indexing schema, as called by now T–indexing in [4], in a sequential work. Since it has been a tradition to use T to denote an iteration tree relation, we decide to change T–indexing to Λ–indexing. This in fact is our main interest and the current work is just a warm up. We also intend to work out a fully iterable K^C model by requiring a stronger backgrand condition and by removing the hypothesis of an inaccessible cardinal which we put in our last section in our later reports.

Fine Structure Theory Set Up

Let us recall that a structure M of set theory is *amenable* if its universe is transitive and if A is a $\Sigma_0(M)$, and u is in the universe of M, then $A \cap u$ is in the universe of M.

Also recall that a *J–structure* is an amenable structure M of the form

$$\langle J_\alpha^{A_1,\cdots,A_m}, B_1, \cdots, B_n \rangle.$$

Definition 1.1 A J–structure $M = \langle J^A_\alpha, B\rangle$ is *acceptable* if and only if for all $\xi < \alpha$, for all $\tau < \omega\xi$, if $P(\tau) \cap (J^A_{\xi+1} - J^A_\xi) \neq \emptyset$, then there is an $f \in J^A_{\xi+1}$ such that f is a surjective function from τ to $\omega\xi$.

The language in our fine structure theory set up is the $\mathcal{L}^*$ language which consists of predicate symbols $\in$, $=$, B, D and variables ν^i of type $i \in \omega$.

$\Sigma_0^{(n)}$ is the smallest set Σ of formulae containing all basic formulae $\nu^i \in \nu^j$, $\nu^i = \nu^j$, $B(\nu^i)$, $D(\nu^j)$; containing all formulae which are in $\Sigma_1^{(m)}$; closed under propositional logic connectives; and closed under bounded quantifiers of the forms: $(\forall\, \nu^n \in \nu^i)$, $(\exists\, \nu^n \in \nu^i)$, where $i \geq n > m$.

$\Sigma_1^{(n)}$ is the set of all formulae of the form $\exists\, \nu^n\, \phi$, where $\phi \in \Sigma_0^{(n)}$.

$\Sigma^* = \bigcup_{n\in\omega} \Sigma_0^{(n)} = \bigcup_{n\in\omega} \Sigma_1^{(n)}$.

Definition 1.2 Let M and N be acceptable J–structures.

$\pi : M \rightarrow_{\Sigma_l^{(n)}} N$ if and only if π preserves all $\Sigma_l^{(n)}$ properties.

$\pi : M \rightarrow_{\Sigma^*} N$ if and only if π preserves all $\Sigma_0^{(n)}$ (for $n < \omega$) properties (if and only if $\pi : M \rightarrow_{\Sigma_0^{(n)}} N$ for all $n < \omega$, if and only if $\pi : M \rightarrow_{\Sigma_1^{(n)}} N$ for all $n < \omega$.)

Let $M = \langle J^A_\nu, B\rangle$ be an acceptable J–structure. Fine structure theory provides us the following objects:

(1) $n(M) \in \omega$.

(2) A sequence of *projectums*:

$$\nu = ht(M) = \rho^0_M \geq \rho^1_M \geq \cdots \geq \rho^{n(M)}_M = \rho^{n(M)+i}_M$$

and $n(M)$ is the least m such that $\rho^m_M = \rho^{m+i}_M$ for all $i \leq \omega$, and if $\rho^{i+1}_M < \rho^0_M$ then $\omega\rho^{i+1}_M$ is a cardinal in M for $i < \omega$.

(3) For all $1 \leq i < \omega$, $\omega\rho^i_M$ is the least $\tau \leq \nu$ such that $P(\tau) \cap \Sigma_1^{(i-1)}(M) \not\subseteq M$.

(4) Two sequences of families of functions: $f\Sigma_1^{(n)}(M)$ and $g\Sigma_1^{(n)}(M)$, the families of *good* $\Sigma_1^{(n)}(M)$–functions, as they are called, which are defined below.

For $i < \omega$, let $H^i_M = H^M_{\omega\rho^i_M}$.

Definition 1.3 Let $F : dom(F) \subseteq H^{j_1}_M \times \cdots \times H^{j_k}_M \rightarrow H^i_M$ be a partial function, where $i, j_1, \cdots, j_k \leq n$. Let G_l be partial functions with range $\subseteq H^{j_l}_M$ for $1 \leq l \leq k$ and with the same domain and the types of the arguments are of $\leq n$. Then we say that $F(G_1, \cdots, G_k)$ is a *type–matching composition* of these functions.

Definition 1.4 $F \in f\Sigma_1^{(n)}(M)$ if and only if $F \in \Sigma_1^{(i)}(M)$ and F is a function such that $range(F) \subseteq H^i_M$ and

$$dom(F) \subseteq H^{j_1}_M \times \cdots \times H^{j_k}_M$$

with $i, j_1, \cdots, j_k \leq n$.

$g\Sigma_1^{(n)}(M)$ is the type–matching composition closure of $f\Sigma_1^{(n)}(M)$.

$g\Sigma_1^{(-1)}(M) = \{f \in M \mid f \text{ is a function }\}$.

(5) For $1 \leq n < \omega$, $h^n_M \in g\Sigma_1^{(n-1)}(M)$ (light face), which are appropriate type–matching compositions of certain uniformly definable Σ_1–Skolem functions h^1_N's, and each of h^n_M is a binary function. Notice that in Zeman's book [8], they are denoted by $\tilde{h}^n_M$. We hope that some notational difference from [8] shall not cause the reader too much trouble.

The key properties of these functions are the following:

If $\sigma : M \rightarrow_{\Sigma_1^{(i)}} N$, then $\sigma(h^{i+1}_M(x,p)) = h^{i+1}_N(\sigma(x),\sigma(p))$.

If $\sigma : M \rightarrow_{\Sigma^*} N$, then for all $i < \omega$, $\sigma(h^{i+1}_M(x,p)) = h^{i+1}_N(\sigma(x),\sigma(p))$.

(6) Two sequences of parameter spaces: P^n_M, R^n_M for $1 \leq n \leq \omega$, which are subsets of J^A_ν. They are identified by the following:

(6a) For all $1 \leq i < \omega$, for $p \in M$, $p \in P^i_M$ if and only if there is a relation $B \subseteq \omega\nu$ which is $\Sigma_1^{(i-1)}(M)$ in p and such that $B \cap \omega\rho^i_M \notin M$.

(6b) $P^*_M = P^\omega_M = \bigcap_{1\leq n<\omega} P^n_M$, and $P^*_M \neq \emptyset$.

(6c) For $1 \leq i < \omega$, for $p \in M$, $p \in R^i_M$ if and only if $J^A_\nu = h^i_M(\omega\rho^i_M \cup \{p\})$.

(6d) $R^*_M = R^\omega_M = \bigcap_{1\leq n<\omega} R^n_M$.

(7) A *standard parameter* $p_M \in P^*_M$, which is the $<^*$ least member of $P^* \cap [\nu]^{<\omega}$, where $<^*$ is the canonical well ordering of finite sets of ordinals defined by, for two finite sets of ordinals a and b, $a <^* b \iff \max(\triangle(a,b)) \in b$.

We shall need to take cores of premice in our inner model construction. Let us recall relevant definitions here.

DEFINITION 1.5 Let $M = \langle J^A_\nu, B\rangle$ be an acceptable J–structure such that $n(M) > 0$.

(1) Let $core(M)$ be the transitive collapse of $h^{n(M)}_M(\omega\rho^\omega_M \cup \{p_M\})$. $core(M)$ is called the core of M. [Here we are using the convention that $h_M(\alpha\cup\{p\})$ to denote $h_M(\omega\times(\alpha\times\{p\}))$.]

(2) For $\alpha \in [\omega\rho^{i+1}_M, \omega\rho^i_M)$, let $core_\alpha(M)$ be the transitive collapse of $h^{i+1}_M(\alpha\cup\{p_M - \alpha\})$. $core_\alpha(M)$ is called the α–core of M.

(3) For $\alpha \in [\omega\rho^{i+1}_M, \omega\rho^i_M)$, let $core^-_\alpha(M)$ be the transitive collapse of $h^{i+1}_M(\alpha \cup \{p_M - (\alpha+1)\})$. $core^-_\alpha(M)$ is called the minus–α–core of M.

Let us run a little more fine structural illustrative construction to redefine these core's to give a slightly more information about these functions.

Let us assume that $n(M) > 0$. Let $\langle \phi_i \mid i < \omega\rangle$ be a recursive enumeration of Σ_1 formulae with one free variable in an appropriate language.

The first thing we do is to reconstruct the sequence of reducts determined by p_M. For $i < n(M)$, write

$$p_M(i) = p_M \cap [\omega\rho^{i+1}_M, \omega\rho^i_M).$$

Let

$$A_1 = \{(i,x) \in H^M_{\omega\rho^1_M} \mid M \models \phi_i(x, p_M(0))\},$$

and let $M_1 = \langle M|\rho^1_M, A_1\rangle$.

Inductively, for $i < n(M)$, let $M_0 = M$ and

$$A_{i+1} = \{(i,x) \in H^M_{\omega\rho^1_{M_i}} \mid M_i \models \phi_i(x, p_M(i))\},$$

and let $M_{i+1} = \langle M_i | \rho^1_{M_i}, A_{i+1}\rangle$. Notice that $\omega\rho^{i+1}_M = \omega\rho^1_{M_i}$ for $i < n(M)$.

For each of these reducts M_i, let h_{M_i} be the Σ_1–skolem function of M_i, which is uniformly Σ_1 definable over J–structures.

We can now define $h^{n(M)}_M(\omega\rho^\omega_M \cup \{p_M\})$ as the result of iterating the h_{M_i}'s along p_M as follows.

Let $X_1 = h_{M_{n(M)-1}}(\omega\rho^{n(M)}_M \cup \{p_M(n(M)-1)\})$. Inductively, for $1 \le i < n(M)$, let

$$X_{i+1} = h_{M_{n(M)-(i+1)}}(X_i \cup \{p_M(n(M)-(i+1))\}).$$

Then $core(M)$ is the transitive collapse of $X_{n(M)}$. $h^{n(M)}_M$ is the result taking compositions of $\langle h_{M_i} \mid i < n(M)\rangle$.

$core_\alpha(M)$ and $core^-_\alpha(M)$ can be redefined in the same way.

Lemma 1.1 Let $M = \langle J^A_\nu, B\rangle$ be an acceptable structure.

(1) Let σ be the inverse mapping of the transitive collapse in forming $core(M)$. Then

(1a) $\sigma : N = core(M) \to_{\Sigma^*} M$,

(1b) $\omega\rho^\omega_N = \omega\rho^\omega_M$,

(1c) $\sigma\restriction_{H^N_{\omega\rho^\omega_N}}$ is identity,

(1d) $p_M \in range(\sigma)$ and $\sigma^{-1}(p_M) \in R^*_N$,

(1e) $core(M)$ is sound $\iff \sigma(p_N) = p_M$.

(2) Let σ_α be the inverse mapping of the transitive collapse in forming $core_\alpha(M)$. Then

(2a) $\sigma_\alpha : N = core_\alpha(M) \to_{\Sigma^{(i)}_1} M$,

(2b) $\sigma_\alpha\restriction_\alpha$ is identity,

(2c) $(p_M - \alpha) \in range(\sigma)$ and $\sigma^{-1}(p_M - \alpha) \in R^*_N$.

(3) Let σ^-_α be the inverse mapping of the transitive collapse in forming $core^-_\alpha(M)$. Then

(3a) $\sigma^-_\alpha : N = core^-_\alpha(M) \to_{\Sigma^{(i)}_1} M$,

(3b) $\sigma^-_\alpha\restriction_\alpha$ is identity,

(3c) $(p_M - (\alpha+1)) \in range(\sigma)$ and $\sigma^{-1}(p_M - (\alpha+1)) \in R^*_N$.

(4) (Indiscernibility of p_M) Assume that $n(M) > 0$. Let $\alpha \in [\omega\rho^\omega_M, \omega\nu)$.

(4a) If $core^-_\alpha(M) \in M$, then $\alpha \in p_M$.

(4b) $core_\alpha(M) \notin M$.

(4c) $\alpha \in p_M$ if and only if $core_\alpha(M) \neq core^-_\alpha(M)$ if and only if $\sigma_\alpha\restriction_{\alpha+1} = id$ and $\sigma^-_\alpha(\alpha) > \alpha$.

Definition 1.6 Let $M = \langle J^A_\nu, B\rangle$ be an acceptable J–structure.

(1) M is n–sound if and only if $P^n_M = R^n_M$.

(2) M is sound if and only if $P^*_M = R^*_M$.

A basic fact of fine structure theory is that M is sound if and only if $p_M \in R^*_M$.

Let us prove one elementary lemma on soundness that we shall use in our application.

LEMMA 1.2 Assume that $M = \langle J^A_\alpha, B\rangle$ be an acceptable J–structure. Assume that $n(M) = 1$ and $\omega\rho^\omega_M$ is the largest cardinal in M. Then M is sound.

Proof Let $\kappa = \omega\rho^1_M < \omega\alpha$ be the largest cardinal in M. Let $X = h_M(\omega\rho^1_M \cup \{p_M\})$.

(1) $\kappa \in X$.

Assume not. Let γ be the least $\eta \in X - \kappa$. Then $\gamma > \kappa$. Since κ is the largest cardinal in M, M thinks the following Σ_1 property of γ, denote it by $\phi(\gamma)$, is true:

$$\exists\beta\exists f\beta < \gamma \wedge \ f : \beta \to \gamma \wedge \forall\eta \in \gamma\exists\xi \in \beta(\eta = f(\xi)).$$

By Σ_1–elementarity, $X \models \phi(\gamma)$. Let β and f be witnesses to the truth of $\phi(\gamma)$ in X. Then $\beta < \kappa$ since $\kappa \notin X$. But then in M the following Σ_0 property of f, β, κ and γ holds:

$$\beta < \kappa < \gamma \ \wedge f : \beta \to \gamma \wedge \forall\eta \in \gamma\exists\xi \in \beta(\eta = f(\xi)).$$

This is a contradiction since κ is a cardinal in M.

(2) X is transitive.

Let $u \in X$. In M, the following Σ_1 property $\phi(\kappa, u)$ holds:

$$\exists f : \kappa \to_{onto} u.$$

Hence $X \models \phi(\kappa, u)$. Let $f \in X$ be such that

$$X \models f : \kappa \to_{onto} u.$$

Then

$$M \models f : \kappa \to_{onto} u.$$

For $\gamma \in \kappa$, we have $f(\gamma) \in X$ since

$$X \models \exists x \in u(x = f(\gamma)).$$

Therefore, $u \subseteq X$.

(3) $X = J^A_{\overline{\alpha}}$.

This follows from Σ_1 elementarity.

(4) $\overline{\alpha} = \alpha$.

Let $N = \langle J^A_{\overline{\alpha}}, \overline{B}\rangle$, where $\overline{B} = B \cap X$. Then N is a Σ_1–elementary submodel of M with $p_M \in X$.

Let D be $\Sigma_1(M)$ in p_M be such that

$$A^{1,p_M} = D \cap J^A_\kappa$$

is the standard code of M. Let $\overline{D} = D \cap J^A_{\overline{\alpha}}$. Then $\overline{D}$ is $\Sigma_1(N)$ in p_M and

$$A^{1,p_M} = \overline{D} \cap J^A_\kappa.$$

If $\overline{\alpha} < \alpha$, then $\overline{D} \in J^A_\alpha$. Hence if $\overline{\alpha} < \alpha$ then $A^{1,p_M} \in J^A_\alpha$. Therefore, $\overline{\alpha} = \alpha$.

Hence, $p_M \in R^*_M$ and M is sound.

□

DEFINITION 1.7 Let $M = \langle J^A_\nu, B\rangle$ be an acceptable J–structure. M is *solid* if and only if for all $\alpha \in p_M$, $core^-_\alpha(M) \in M$.

Let us take a notice that in the literature of fine structure theory, these structures $core^-_\alpha(M)$ are called *witnesses*.

LEMMA 1.3 (Preservation Lemma) Let M and N be two acceptable J–structures.

(1) If $\sigma : N \rightarrow_{\Sigma^*} M$ and M is solid, then N is solid and $\sigma(p_N) = p_M$.

(2) If M is solid and $N = core(M)$, then N is both sound and solid.

Premice

DEFINITION 1.8 Let M be an acceptable J–structure. Let $\kappa < \lambda$ be primitive recursively closed and $\kappa \in M$.

$$F : P(\kappa) \cap M \rightarrow P(\lambda)$$

is a (κ, λ)–extender on M if and only if for all $v_1, \cdots, v_m < \kappa$, for all $A_1, \cdots, A_n \in P(\kappa) \cap M$, for all $B \in P(\kappa) \cap M$, if B is primitive recursive in $A_1, \cdots, A_n$ and $v_1, \cdots, v_m$, then $F(B)$ is primitive recursive in $F(A_1), \cdots, F(A_n)$ and $v_1, \cdots, v_m$ by the same definition.

For a (κ, λ)–extender F, λ is called the length of F, denoted by $\lambda = lh(F)$, and κ is called the critical point of F, denoted by $\kappa = crit(F)$.

A (κ, λ)–extender F on M is whole if and only if $\lambda = F(\kappa)$.

DEFINITION 1.9 $M = \langle J^E_\alpha, E_{\omega\alpha}\rangle$ is a *coherent structure* if J^E_α is acceptable and there is a unique triple (κ, τ, λ) such that

(i) $E_{\omega\alpha}$ is a (κ, λ)–extender on J^E_τ and $\tau = (\kappa^+)^M < \lambda < \omega\alpha$;

(ii) $\kappa = crit(E_{\omega\alpha})$ and $\lambda = E_{\omega\alpha}(\kappa)$;

(iii) $J^E_\alpha = ult(J^E_\tau, E_{\omega\alpha})$;

(iv) κ is the largest cardinal in J^E_τ and λ is the largest cardinal in M.

Definition of the restriction of an extender:

Let $M = \langle J^E_\nu, F\rangle$ be coherent. Let κ be the critical point of F. Assume that $(\kappa^+)^M \leq \eta \leq \nu$. Then define that $F||\eta$ with $dom(F||\eta) = dom(F)$ by $(F||\eta)(X) = \eta \cap F(X)$.

DEFINITION 1.10 $M = \langle J^E_\alpha, E_{\omega\alpha}\rangle$ is a prepremouse (ppm) iff the following four conditions hold:

(a) M is acceptable

(b) $E = \{(\nu, \eta, X) \mid \eta \leq \nu \leq \omega\alpha \ \wedge \ \eta \in E_\nu(X)\}$, where $E_\nu = \emptyset$ or E_ν is a whole extender on J^E_ν and $\langle J^E_\nu, E_\nu\rangle$ is coherent. (Hence, $lh(E_\nu) = \lambda = E_\nu(crit(E_\nu))$, which is the largest cardinal in the sense of J^E_ν.)

(c) If $\pi : J_\nu^E \to_{E_\nu} N$, then $E_\nu^N = \emptyset$ (taking well founded core of N as transitive in case that N is not well founded).

(d) The restriction of M to ν, denoted by $M||\nu =_{def} \langle J_\nu^E, E_{\omega\nu}\rangle$, must be sound for all $\nu < \alpha$.

DEFINITION 1.11 Let M be a prepremouse. For $\nu \leq ht(M)$, we say that E_ν is a total (or surviving) extender of M if $E_\nu \neq \emptyset$ and, letting $\kappa = crit(E_\nu)$,

$$(\kappa^+)^M = (\kappa^+)^{M||\nu}.$$

DEFINITION 1.12 A premouse is a prepremouse satisfying an initial segment condition (IS).

Initial Segment Condition

Let $\nu \leq ht(M)$, $E_\nu^M \neq \emptyset$, $M||\nu = \langle J_\nu^E, E_\nu\rangle$. $\kappa = crit(E_\nu)$, $\lambda = lh(E_\nu)$, $\tau = (\kappa^+)^{J_\nu^E}$ in M. C_ν^M is the set of all $\beta \in (\kappa, \lambda)$ such that if $\alpha < \beta, f \in ({}^\kappa\kappa)^{M||\nu}, \pi : J_\tau^E \to_{E_\nu} J_\nu^E$, then $\pi(f)(\alpha) < \beta$. C_ν^M is the set of *cutoff points* of E_ν over M.

If $\beta \in C_\nu^M$, then $E_\nu||\beta \in J_\nu^E$.

DEFINITION 1.13 Let A be a set of ordinals which is closed under Gödel pairing. A function $h : A \to Ord$ is a Gödel homomorphism if and only if for all α, β in A, (1) if $\alpha < \beta$ then $h(\alpha) < h(\beta)$; and (2) $h(\prec \alpha, \beta \succ) = \prec h(\alpha), h(\beta) \succ$.

DEFINITION 1.14 A *measure preserving mapping* is a pair

$$\langle \sigma, h\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$$

such that

(1) $\overline{F}$ is a $(\overline{\kappa}, \overline{\lambda})$–extender on $\overline{M}$ and F is a (κ, λ)–extender on M,
(2) $\sigma : \overline{M} \to_{\Sigma_0} M$,
(3) $\kappa = \sigma(\overline{\kappa})$,
(4) $h : \overline{\lambda} \to \lambda$ is a Gödel homomorphism, and
(5) for all $\alpha < \overline{\lambda}$ and for all $x \in P(\overline{\kappa}) \cap \overline{M}$, if $\alpha \in \overline{F}(x)$ then $h(\alpha) \in F(\sigma(x))$.

DEFINITION 1.15 A Σ_1–*measure preserving mapping* is a pair

$$\langle \sigma, h\rangle : \langle \overline{M}, \overline{F}\rangle \to^* \langle M, F\rangle$$

such that $\langle \sigma, h\rangle$ is a measure preserving mapping

$$\langle \sigma, h\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$$

and for each $\alpha < \overline{\lambda}$, there is a $p \in \overline{M}$ such that

$$\overline{F}_\alpha = \{x \in P(\overline{\kappa}) \cap \overline{M} \mid \alpha \in \overline{F}(x)\}$$

is $\Sigma_1(\overline{M})$ in p and

$$F_{h(\alpha)} = \{x \in P(\kappa) \cap M \mid h(\alpha) \in F(x)\}$$

is $\Sigma_1(M)$ in $\sigma(p)$ with the same definition.

We now turn to iteration.

First, we define a one step iteration, namely a fine structural ultrapower construction.

The $*$–ultrapower construction

Let M be a premouse and let $F = E^M_\nu$ be an extender on the extender sequence of M. Let $\kappa = crit(F)$ and let $\lambda = lh(F) = F(\kappa)$. Let $m(\kappa, M)$ be the unique n such that $\omega\rho^{n+1}_M \leq \kappa < \omega\rho^n_M$ if $\omega\rho^\omega_M \leq \kappa$. Otherwise, let $m(\kappa, M) = \omega$.

Define
$$\Gamma(\kappa, M) = \bigcup\{{}^\kappa M \cap g\Sigma_1^{(n)}(M) \mid -1 \leq n < m(\kappa, M)\}.$$

For $\alpha, \beta < \lambda$, for $f, g \in \Gamma(\kappa, M)$, define

$$(\alpha, f) =_F (\beta, g) \iff (\alpha, \beta) \in F(\{(\xi, \eta) \mid f(\xi) = g(\eta)\})$$

and
$$(\alpha, f) \in_F (\beta, g) \iff (\alpha, \beta) \in F(\{(\xi, \eta) \mid f(\xi) \in g(\eta)\}).$$

This defines an equivalence relation on $\lambda \times \Gamma(\kappa, M)$ and $\in_F$ respects the equivalence relation.

Let $ult^*(M, F)$ be the set of $=_F$ equivalence classes. Identify its $\in_F$–well founded part with its transitive collapse. Let

$$\pi : M \rightarrow^*_F ult^*(M, F)$$

be the induced canonical embedding.

DEFINITION 1.16 An (κ, λ)–extender E over M is *weakly amenable to* M if for every sequence $\langle A_\alpha \mid \alpha < \kappa\rangle \in M$ of length κ of subsets of κ and for every $\eta < \lambda$,

$$\{\alpha < \kappa \mid \eta \in E(A_\alpha)\} \in M.$$

DEFINITION 1.17 An (κ, λ)–extender E over M is Σ_1 *amenable to* M if for every $\eta < \lambda$, the ultrafilter E_η generated by η is $\Sigma_1(M)$, where

$$E_\eta = \{x \subseteq \kappa \mid x \in M \ \wedge \ \eta \in E(x)\}.$$

DEFINITION 1.18 An extender F is *close* to an acceptable structure M if F is both weakly amenable and Σ_1 amenable to M.

The fundamental fact is that if F is close to M then the *–ultrapower embedding is Σ^*–preserving.

THEOREM 1.1 Let $M = \langle J^E_\nu, F\rangle$ be an active premouse. Let κ be the critical point of F and $\lambda = F(\kappa)$. Let $\pi : M \rightarrow^*_F N$ be the $*$–ultrapower of M by F. Assume that this is well–founded (say, F is ω–complete).

(1) If $n < \min\{m(\kappa, M)+1, \omega\}$ then π is an $\Sigma_0^{(n)}$–embedding.

(2) If $n < m(\kappa, M)$ then π is an $\Sigma_2^{(n)}$–embedding and $\pi[P_M^n] \subseteq P_N^n$.

(3) If $\omega\rho_M^\omega > \kappa$, then π is fully Σ^* and $\pi[P_M^*] \subseteq P_N^*$.

(4) Assume that $m(\kappa, M) < \omega$ (i.e., $\omega\rho_M^\omega \leq \kappa$) and F is close to M (i.e., both weakly amenable and Σ_1–amenable to M). Then

(a) π is cofinal in N.

(b) for all $n > m(\kappa, M)$, $H_M^n = H_N^n$ and

$$P(H_M^n) \cap \Sigma_1^{(n)}(M) = P(H_N^n) \cap \Sigma_1^{(n)}(N)$$

and $\pi[P_M^n] \subseteq P_N^n$.

(c) π is fully Σ^*.

(d) $\pi[P_M^*] \subseteq P_N^*$.

(e) If M is solid, then so is N and $\pi(p_M) = p_N$.

(f) $P(\kappa) \cap \Sigma_1^{(n)}(M) = P(\kappa) \cap \Sigma_1^{(n)}(N)$ for all $n < \omega$.

(g) For all $n \geq m(\kappa, M)$, $\omega\rho_M^n = \omega\rho_N^n$.

□

Iteration Trees

Definition 1.19 Let $0 < \theta \leq Ord$. $T \subseteq \theta \times \theta$ is an *iteration tree of length* θ if and only if the following conditions are satisfied:

(1) T is a tree order with least point 0, and if $\alpha <_T \beta$ (i.e., $(\alpha, \beta) \in T$), then $\alpha < \beta$,

(2) if $\beta < \theta$ is a successor ordinal, then β has an immediate predecessor, denoted by $T(\beta)$, in T, (i.e., $T(\beta) <_T \beta$ and if $\xi <_T \beta$ then $\xi <_T T(\beta)$ or $\xi = T(\beta)$).

(3) if $\beta < \theta$ is a limit ordinal, then the following set

$$[0, \beta)_T = \{\xi \mid \xi <_T \beta\}$$

is $\in$–cofinal in β.

Definition 1.20 Let T be an iteration tree of length θ. We write $\alpha \leq_T \beta$ if and only if $\alpha <_T \beta$ or $\alpha = \beta$.

For $\alpha <_T \beta$, we define

$[\alpha, \beta]_T = \{\xi \mid \alpha \leq \xi \ \wedge \ \xi \leq_T \beta\}$,

$[\alpha, \beta)_T = \{\xi \mid \alpha \leq \xi \ \wedge \ \xi <_T \beta\}$,

$(\alpha, \beta]_T = \{\xi \mid \alpha < \xi \ \wedge \ \xi \leq_T \beta\}$, and

$(\alpha, \beta)_T = \{\xi \mid \alpha < \xi \ \wedge \ \xi <_T \beta\}$.

If $i < \theta$, $[0, i]_T$ is called the branch of T upto i and $[0, i)_T$ is called the branch of T to i.

$b \subseteq \theta$ is called a branch of T if and only if b is a maximal linearly ordered by T. T has a *cofinal branch* if there is a branch b such that b is $\in$–cofinal in θ.

Definition 1.21 A *generalized iteration of premice of length* θ is an iteration tree T on θ associated with a sequence of premice, called *iterates*, $\langle M_i \mid i < \theta\rangle$, two sequences of *indices*, $\langle \nu_i \mid i \in D\rangle$ and $\langle \eta_i \mid i+1 < \theta\rangle$, and a sequence of *iteration maps* $\langle \pi_{ij} \mid i \leq_T j\rangle$, denoted by

$$\mathcal{T} = \langle < M_i >, (< \nu_i >, D), < \eta_i >, < \pi_{ij} >, T\rangle,$$

satisfies the following requirements:

(a) each M_i is a premouse,

(b) each π_{ij} is a partial map from M_i to M_j, and if $i \leq_T h \leq_T j$, then $\pi_{ij} = \pi_{hj}\pi_{ih}$,

(c) if $i+1 < \theta$, then $\eta_i \leq ht(M_{T(i+1)})$,

(d) if $j < \theta$, then $\{i \mid i+1 \leq_T j \ \wedge \ \eta_i < ht(M_{T(i+1)})\}$ is finite,

(e) if $i \notin D$ and $i+1 < \theta$, then $i = T(i+1)$, $M_{i+1} = M_i||\eta_i$, and $\pi_{i,i+1}$ is the identity map,

(f) if $i \in D$, then $i+1 < \theta$ and $E^{M_i}_{\nu_i} \neq \emptyset$, letting $\xi = T(i+1)$, $\kappa_i = crit(E^{M_i}_{\nu_i})$, $\tau_i = (\kappa_i^+)^{M_i||\nu_i}$, then $\tau_i = (\kappa_i^+)^{M_\xi||\eta_i}$, $J^{M_i}_{\tau_i} = J^{M_\xi}_{\tau_i}$ and $\pi_{\xi,i+1} : M_\xi||\eta_i \rightarrow^*_{E^{M_i}_{\nu_i}} M_{i+1}$,

(g) if $j < \theta$ is a limit ordinal, then $(M_j, \langle \pi_{ij} \mid i <_T j\rangle)$ is the direct limit of $(\langle M_i \mid i <_T j\rangle, \langle \pi_{ik} \mid i \leq_T k <_T j\rangle)$.

Remark By (d), *the finiteness of truncation condition*, if j is limit ordinal, then for sufficiently large $i <_T k <_T j$, the maps $\pi_{i,k}$ are total maps from M_i to M_k, hence the direct limit is well defined and for sufficiently large $i <_T j$, π_{ij} is total.

DEFINITION 1.22 Let $\mathcal{T} = \langle < M_i >, (< \nu_i >, D), < \eta_i >, < \pi_{ij} >, T\rangle$ be a generalized iteration of premice of length θ.

(1) Let D be the domain of the ν–indices. For $i \in D$, let $\kappa_i = crit(E^{M_i}_{\nu_i})$, $\tau_i = (\kappa_i^+)^{M_i||\nu_i}$, and $\lambda_i = \sigma_i(\kappa_i)$ be the largest cardinal in $M_i||\nu_i$, where $\sigma_i : J^{M_i}_{\tau_i} \rightarrow M_i||\nu_i$ is the canonical embedding given by $E^{M_i}_{\nu_i}$, which agrees with $\pi_{T(i+1),i+1}$ on τ_i.

(2) $i+1$ is a *truncation point* of $\mathcal{T}$ if and only if $i+1 < \theta$ and $\eta_i < ht(M_{T(i+1)})$.

(3) A branch b in T is *simple in* $\mathcal{T}$ if and only if b has no truncation point. $i < \theta$ is *simple in* $\mathcal{T}$ if and only if the branch $[0,i]_T$ upto i has no truncation point.

(4) $\mathcal{T}$ is *direct* if and only if $i \in D$ for all $i+1 < \theta$.

(5) $\mathcal{T}$ is *smooth* if and only if for all $i+1 < \theta$, if $i \notin D$ then $\eta_i = ht(M_i)$; if $i \in D$ then $\xi = T(i+1) \in D$ and η_i is the maximal $\eta \leq ht(M_\xi)$ such that

$$(\kappa_i^+)^{M_\xi||\eta_i} = (\kappa_i^+)^{M_i||\nu_i}.$$

(6) $\mathcal{T}$ is *normal (with λ–index)* if and only if $\mathcal{T}$ is smooth and for all $i \in D$, $\nu_i > \nu_h$ for $h \in D \cap i$ and $T(i+1)$ is the least $\xi \in D$ such that $\kappa_i < \lambda_\xi$.

(7) $\mathcal{T}$ is a *piecewise normal iteration with a marking sequence* $< \alpha_i \mid i \leq \Gamma >$ if and only if the following hold:

(a) the marking sequence $< \alpha_i \mid i \leq \Gamma >$ is normal (continuous and strictly increasing) and $\alpha_0 = 0$ and $\alpha_\Gamma = \theta$.

(b) $\alpha_i \notin D$ for all $i < \Gamma$.

(c) If $\alpha_i < j < \alpha_{i+1}$ and $j \notin D$, then $\eta_j = ht(M_j)$.

(d) If $\alpha_i < j < \alpha_{i+1}$ and $j \in D$, then

(i) $\nu_j > \nu_h$ for all $h \in (\alpha_i, j) \cap D$,

(ii) $T(j+1) =$ the least $\xi > \alpha_i$ such that $\xi \in D$ and $\kappa_j < \lambda_\xi$, and

(iii) $\eta_j = \max \eta \leq ht(M_{T(j+1)})$ such that $(\kappa^+)^{M_{T(j+1)}||\eta} = (\kappa^+)^{M_j||\nu_j}$.

(8) $\mathcal{T}$ is a *smooth piecewise normal iteration with a marking sequence* $< \alpha_i \mid i \leq \Gamma >$ if and only if it is a piecewise normal iteration and $\eta_{\alpha_i} = ht(M_{\alpha_i})$ for all $i < \Gamma$, i.e, it is a piecewise normal iteration and it is smooth.

Remark For readers of [2], piecewise normal iterations are called *good iterations* there and smooth piecewise normal iterations are called *smooth iterations* there.

DEFINITION 1.23 Let $\mathcal{T}$ be a normal iteration of length θ. We say that $\mathcal{T}$ *can be uniquely continued* if and only if the following hold:

(1) If θ is a limit ordinal, then $\mathcal{T}$ has a unique well founded cofinal branch $b = [0, \theta)_T$ and $\mathcal{T}$ can be extended to $\mathcal{T}'$ in the following way: Set

$$T' = T \cup \{(i, \theta) \mid i \in b\}$$

and let $\langle M_\theta, \langle \pi_{i\theta} \mid i \in b \rangle\rangle$ be the direct limit along the branch b. We have a normal iteration $\mathcal{T}'$ of length $\theta + 1$.

(2) If $\theta = \xi + 1$ and $\nu \in M_\xi$ satisfying that $E_\nu^{M_\xi} \neq \emptyset$ and $\nu > \nu_j$ for all $j \in D$, then $\mathcal{T}$ can be extended to $\mathcal{T}'$ in the following way: Set $D' = D \cup \{\xi\}$, let $j \in D'$ be the least such that $\kappa = crit(E_\nu^{M_\xi}) < \lambda_j$ and let η_ξ be the maximum $\eta \leq ht(M_j)$ such that

$$(\kappa^+)^{M_j||\eta} = (\kappa^+)^{M_\xi||\nu},$$

and let

$$\pi_{j,\xi+1} : M_j||\eta_\xi \to M_{\xi+1} = ult^*(M_j||\eta_\xi, E_\nu^{M_\xi}),$$

and set $T' = T \cup \{(i, \xi + 1)\ i \leq_T j\}$ and $\nu_\xi = \nu$.

DEFINITION 1.24 Let $\mathcal{T}$ be a piecewise normal iteration of length θ with marking sequence $< \alpha_i \mid i \leq \Gamma >$. We say that $\mathcal{T}$ *can be uniquely continued* if and only if the following hold:

(1) If θ is a limit ordinal, then $\mathcal{T}$ has a unique well founded cofinal branch $b = [0, \theta)_T$ and $\mathcal{T}$ can be extended to $\mathcal{T}'$ in the following way: Set

$$T' = T \cup \{(i, \theta) \mid i \in b\}$$

and let $\langle M_\theta, \langle \pi_{i\theta} \mid i \in b \rangle\rangle$ be the direct limit along the branch b and let $\alpha'_i = \alpha_i$ for all $i \leq \Gamma$ and $\alpha'_{\Gamma+1} = \theta + 1$. We have a piecewise normal iteration $\mathcal{T}'$ of length $\theta + 1$.

(2) If $\theta = \xi + 1$, $\Gamma = h + 1$ and $\nu \in M_\xi$ satisfying that $E_\nu^{M_\xi} \neq \emptyset$ and $\nu > \nu_j$ for all $j \in (\alpha_h, \xi) \cap D$, then $\mathcal{T}$ can be extended to $\mathcal{T}'$ in the following way: Set $D' = D \cup \{\xi\}$, $\alpha'_i = \alpha_i$ for $i \leq h$ and $\alpha'_{h+1} = \xi + 2$ (we change the last marker from $\xi + 1$ to $\xi + 2$) and let $j \in (\alpha_h, \xi] \cap D'$ be the least such that $\kappa = crit(E_\nu^{M_\xi}) < \lambda_j$ and let η_ξ be the maximum $\eta \leq ht(M_j)$ such that

$$(\kappa^+)^{M_j||\eta} = (\kappa^+)^{M_\xi||\nu},$$

and

$$\pi_{j,\xi+1} : M_j||\eta_\xi \to M_{\xi+1} = ult^*(M_j||\eta_\xi, E_\nu^{M_\xi}),$$

and set $T' = T \cup \{(i, \xi+1)\ i \leq_T j\}$ and $\nu_\xi = \nu$.

(3) If $\theta = \xi + 1$, $\Gamma = h + 1$, and $\eta \leq ht(M_\xi)$, then $\mathcal{T}$ extends to $\mathcal{T}'$ in the following way: Set $\eta_\xi = \eta$ and $\alpha'_i = \alpha_i$ for all $i \leq h+1$ and $\alpha'_{h+2} = \xi + 2$, and $M_{\xi+1} = M_\xi || \eta_\xi$ and $\pi_{\xi,\xi+1}$ is the identity map and set $T' = T \cup \{(l, \xi+1) \mid l \leq_T \xi\}$, and $D' = D$.

Definition 1.25 (1) A premouse M is *(uniquely) normally iterable* if every normal iteration of M can be uniquely continued.

(2) A premouse M is *(uniquely) smoothly iterable* if every smooth piecewise normal iteration of M can be uniquely continued.

(3) A premouse M is *(uniquely) iterable* if every piecewise normal iteration of M can be uniquely continued.

(4) A premouse M is a *mouse* if it is uniquely iterable.

The following is a special form of a general theorem of Jensen which provides exact information on these iterabilities.

Theorem 1.2 (Jensen) Assume that there is an inaccessible cardinal. If a type 1 premouse is normally iterable, then it is smoothly piecewise normally iterable.

□

We believe that the assumption of existence of an inaccessible cardinal can be removed for type 1 premice. But at this point, we have not yet checked in detail. Hopefully, we shall report in our sequential work.

Theorem 1.3 (Jensen) A premouse is a mouse if and only if it is smoothly piecewise normally iterable.

□

Theorem 1.4 (Steel) If M is a mouse, then M is solid. Hence, the core of a mouse is always a sound and solid mouse, and if M is a mouse, then $core(core(M)) = core(M)$.

□

Theorem 1.5 (Condensation Lemma, Jensen)

Let M be a mouse and let $\nu \leq ht(M)$ be such that if $\tau < \nu$ is a cardinal in $M||\nu$ then τ is a cardinal in M. Let $\sigma : \overline{M} \to_{\Sigma_0} M$ be such that

$$\nu = \max\{\xi \mid \sigma\restriction_\xi = id\}$$

and σ is $\Sigma_0^{(n)}$–preserving whenever $\omega\rho^n_{\overline{M}} > \nu$. Then $\overline{M}$ is a mouse. Moreover, if $\omega\rho^\omega_{\overline{M}} \leq \nu$ and $\overline{M}$ is sound above ν, then one of the following holds:

(a) $\overline{M} = core_\nu(M)$ and σ is the core map;

(b) $\overline{M} = M||\eta$ for some $\eta < ht(M)$;

(c) $\pi : M||\eta \to^*_{E^M_\mu} \overline{M}$, where

(i) $\nu \leq \eta < ht(M)$ and $\omega\rho^\omega_{M||\eta} < \nu$,

(ii) $\mu \leq \omega\eta$,

(iii) $\nu = (\kappa^+)^{M||\eta}$, $\kappa = crit(E^M_\mu)$,

(iv) E^M_μ is generated by $\{\kappa\}$;

(d) $\overline{M} = M_1||\eta$, where $\eta < ht(M_1)$ and $\pi : M \to_{E^M_\nu} M_1$.

To study type 1 mice, we shall need a condition stronger than acceptability, *strong acceptability*, which provides us very nice reflection property.

DEFINITION 1.26 Let $M = \langle J^E_\alpha, F\rangle$ be a J–structure. M is *strongly acceptable* if and only if whenever $\tau < \alpha$, $\vec{\xi} < \omega\tau$, $J^E_{\tau+1} \models \phi(\vec{\xi})$ and $J^E_\tau \models \neg\phi(\vec{\xi})$ for a Σ_1 formula ϕ, then $Card(\tau) \leq \max(\vec{\xi}, \omega)$ in $J^E_{\tau+1}$.

LEMMA 1.4 Let $N = \langle J^E_\alpha, F\rangle$ be a J–structure. Assume that N is strongly acceptable. Then N is acceptable.

Proof Toward a contradiction, assume otherwise. Let $\tau < \alpha$ be the least counter example. Let $\xi < \omega\tau$ be the least witness. Then ξ must be a limit ordinal and $\xi \geq \omega$. Consider the following Σ_1 sentence $\phi(\xi, \omega)$:

$$\exists x\ \exists\gamma\ (x \subseteq \xi < \gamma \wedge x \notin J^E_\gamma \wedge \exists m < \omega\ x \in S^E_{\omega\gamma+m} \wedge \phi_1(\xi, \gamma))$$

where $\phi_1(\xi,\gamma)$ if and only if $\forall m < \omega \forall f \in S^E_{\omega\gamma+m} f$ is not a surjective function from ξ to $\omega\gamma$.

Then $J^E_{\tau+1} \models \phi(\xi, \omega)$ and by minimality, $J^E_\tau \models \neg\phi(\xi,\omega)$. But this contradicts to the fact that N is strongly acceptable.

□

Strongly acceptable structures have nice reflection properties as indicated by the following lemma.

LEMMA 1.5 Assume that $N = \langle J^E_\alpha, F\rangle$ is strongly acceptable. Let $\omega < \beta < \alpha$ be a cardinal in N. Then $J^E_\beta \prec_{\Sigma_1} J^E_\alpha$.

Proof Let $\vec{\xi} < \beta$. Let ϕ be a Σ_1 formula. Assume that $J^E_\alpha \models \phi(\vec{\xi})$ and $J^E_\beta \models \neg\phi(\vec{\xi})$.

Let $\beta \leq \tau < \alpha$ be maximum such that $J^E_\tau \models \neg\phi(\vec{\xi})$. Then $\tau + 1 \leq \alpha$ and $J^E_{\tau+1} \models \phi(\vec{\xi})$. By the strong acceptability of N, τ has cardinality at most $\max(\vec{\xi}, \omega)$ in N. But this contradicts to the fact that $\beta \leq \tau$ is a cardinal in N.

□

Another easy consequence of strong acceptability is the following preservation property.

LEMMA 1.6 Assume that both M and N are J–structures and $\sigma : M \to_{\Sigma_1} N$. Then M is strongly acceptable if and only if N is strongly acceptable.

Proof Notice that being not strongly acceptable is a Σ_1 statement.

□

Although we have disqualified a class of premice to be our objects of studying, we don't lose any information on our study of mice.

THEOREM 1.6 Mice are strongly acceptable.

We need to prove several lemmas first.

LEMMA 1.7 Let N be an acceptable J–structure. Let $p \in R^k_N$. Let $l \geq 1$ and ϕ be $\Sigma_l(N)$. Then ϕ is uniformly $\Sigma^{(k)}_l$ in p, i.e., there is $\overline{\phi} \in \Sigma^{(k)}_l$ such that for all $p \in R^k_N$ and for all $\vec{x} \in N$,

$$N \models \phi(\vec{x}) \iff N \models \overline{\phi}(\vec{x}).$$

Proof Assume that $k \geq 1$. For simplicity of notations, we assume that $l = 1$. Let $\phi = \exists z \phi_0(\vec{x}, z)$, where ϕ_0 is Σ_0. Let h^k be the canonical $\Sigma^{(k)}_1$ Skolem function for N. Set

$$f(\prec \xi, i \succ) = h^k(i, < \xi, p >)$$

for $\xi < \omega\rho^k_N$ and $i < \omega$. Then f is a partial good $\Sigma^{(k)}_1$ function from $\omega\rho^k_N$ to N. We take $\overline{\phi}$ as

$$\exists\, \xi^k_1 \cdots \xi^k_m (x_1 = f(\xi^k_1) \wedge \cdots x_m = f(\xi^k_m) \wedge \exists \eta^k \in D\ \phi_0(f(\vec{\xi}), f(\eta^k))),$$

where $D = dom(f)$ and $\phi_0(f(\vec{\xi}), f(\eta))$ is $\Delta^{(k)}_1$ in $\vec{x}, \eta$, and p.

□

LEMMA 1.8 Let $N = \langle J^E_\alpha, F \rangle$ be acceptable. Let $k = n(N)$. Let $p \in R^k_N$. Then every $\Sigma^{(k)}_l$–condition is uniformly $\Sigma^{(k+l)}_1$ in p.

□

LEMMA 1.9 Let $N = \langle J^E_\alpha, F \rangle$ be a sound mouse. Let $\vec{\xi} < \omega\alpha$. Let ϕ be Σ_ω such that $N \models \phi(\vec{\xi})$ and $N || \beta \models \neg\phi(\vec{\xi})$ whenever $\beta < \alpha$ and $\vec{\xi} < \omega\beta$. Then $\omega\rho^\omega_N \leq \max(\vec{\xi}, \omega)$.

Proof Suppose not. Let α be the least counter example. Assume without lose of generality that $\xi_0 = \max(\vec{\xi})$ is primitive recursive closed (hence $\xi_0 = \max(\vec{\xi}, \omega)$). Let $k = n(N)$, i.e., $\omega\rho^k_N = \omega\rho^{k+l}_N$ for all $l < \omega$ and is the least with this property. Let ϕ be $\Sigma_l (l \geq 1)$ and let $n >> l + k$ be sufficiently large.

Let $p = p_N$ be the standard parameter of N (hence $p \cap \omega\rho^k_N = \emptyset$). Let $N^{n,p}$ be the reduct. Let $X = h_{N^{n,p}}(\xi_0 + 1)$. Let $\overline{\sigma} : \overline{M} \to N^{n,p}$ be induced by the transitive collapsing of X. Let $\sigma : \overline{N} \to_{\Sigma^{(n)}_1} N$ be such that $\overline{\sigma} \subseteq \sigma$ and $\overline{M} = \overline{N}^{n,\overline{p}}$, and $\overline{p} \in R^n_{\overline{N}}$ and $p = \sigma(\overline{p})$.

We claim that for $k \leq h \leq n$, $\omega\rho^h_{\overline{N}} = \omega\rho^n_{\overline{N}}$.

To see this, suppose not. Then $h < n$. Let $x \in H^h_{\overline{N}} - H^n_{\overline{N}}$. Then $\sigma(x) \in H^h_N$ and

$$N \models \exists y^n (y^n = \sigma(x)).$$

Hence

$$\overline{N} \models \exists y^n (y^n = x).$$

We get a contradiction.

Now by Lemma 1.8, every $\Sigma^{(k)}_{n-k}$ condition in p is uniformly $\Sigma^{(n)}_1$ in p. Hence

(1) $\sigma : \overline{N} \to_{\Sigma_{n-k}} N$.

Since $n >> k$, we have

(2) $\sigma(\omega\rho^h_{\overline{N}}) = \omega\rho^h_N$, $(h \leq k)$.

Moreover,

(3) $\overline{p} = p_{\overline{N}} - \omega\rho^n_{\overline{N}}$ and $\overline{N}$ is solid above $\omega\rho^n_{\overline{N}}$.

To see this, let $\overline{\nu} \in p_{\overline{N}} - \omega\rho^n_{\overline{N}}$. We show that $\overline{\nu}$ has a witness in $\overline{N}$. Since $\overline{p} \cap \omega\rho^k_{\overline{N}} = \emptyset$, for some $h < k$, we have $\omega\rho^{h+1}_{\overline{N}} \leq \overline{\nu} < \omega\rho^h_{\overline{N}}$. Let $\nu = \sigma(\overline{\nu})$. Then $\omega\rho^{h+1}_N \leq \nu < \omega\rho^h_N$. The fact that ν has a witness in N is expressed by the following:

there is $W \in N, q \in W$ such that for all $\vec{\xi} < \nu$, and for all $\Sigma^{(h)}_1$–formula ϕ, we have

$$W \models \phi(\vec{\xi}, q) \iff N \models \phi(\vec{\xi}, p - \nu + 1).$$

By (1) and that $n >> k$, the same holds of $\overline{\nu}, \overline{p}$ in $\overline{N}$.

This shows (3).

Since for $\nu \leq \xi_0$, $\sigma(\nu) = \nu$ and $N \models \exists\, y^n(y^n = \nu)$, we have $\xi_0 < \omega\rho^n_{\overline{N}}$. It follows that $\overline{N}^{n,\overline{p}} = h_{\overline{N}^{n,\overline{p}}}(\xi_0 + 1)$ by the construction of $\overline{N}$. Since $\overline{p} \in R^n_{\overline{N}}$ and $\overline{p} = p_{\overline{N}} - (\xi_0 + 1)$, we conclude that

(4) $\overline{N}$ is sound above $\xi_0 + 1$.

Let $\nu = crit(\sigma)$. Then $\xi_0 + 1 < \nu$ and

(5) $\overline{N}$ is sound above ν.

Since $n >> k + l$ was chosen large enough to express mousehood in a Σ_{n-k} way, $\overline{N}$ is then a premouse. We now apply the Condensation Lemma to obtain a contradiction. According to the lemma, we consider each of the four cases (one of them must occur).

Case 1 $\overline{N} = core_\nu(N)$ and σ is the core map. Then $\omega\rho^\omega_{\overline{N}} = \omega\rho^\omega_N \leq \nu < \omega\rho^n_N$. Contradicts to that α is a counter example.

Case 2 $\overline{N} = N||\eta$ for $\eta < ht(N)$. By(1), $\overline{N} \models \phi(\vec{\xi})$. Hence $N||\eta \models \phi(\vec{\xi})$. But $\eta < \alpha$.

Case 3 $\pi : N||\eta \to^*_{E^N_\mu} \overline{N}$, where

(i) $\nu \leq \eta < ht(N)$ and $\omega\rho^\omega_{N||\eta} < \nu$,

(ii) $\mu \leq \omega\eta$,

(iii) $\nu = (\kappa^+)^{N||\eta}$, $\kappa = crit(E^N_\mu)$,

(iv) E^N_μ is generated by $\{\kappa\}$.

Then κ is a cardinal in N, since $\sigma : \overline{N} \to_{\Sigma_{n-k}} N$ and $\sigma\restriction_\nu = id$.

Let $\phi^*(\gamma_0, \cdots, \gamma_m)$ be the condition:

$\gamma_0 = \max(\vec{\gamma})$ is primitive recursive closed and $\phi(\vec{\gamma})$ and for all β, if $\omega\beta \in Ord$, then $\langle J^E_\beta, E_{\omega\beta}\rangle \models \neg\phi(\vec{\gamma})$.

Then $N \models \phi^*(\vec{\xi})$. Since $n >> l$, we have $\overline{N} \models \phi^*(\vec{\xi})$ by (1). Moreover, since $n >> l$, we know that the condition "$\overline{N} \models \phi^*(\vec{\gamma})$" is uniformly $\Sigma^{(n)}_0$ in $\overline{p}$ for $\vec{\gamma} < \omega\rho^n_{\overline{N}}$.

Also, $\omega\rho^{(n+1)}_{N||\eta} \leq \kappa < \omega\rho^n_{N||\eta}$ and $N||\eta$ is sound above $\omega\rho^n_{N||\eta}$. Hence, if $q = p_{N||\eta} - \omega\rho^n_{N||\eta}$, then $\pi(q) = \overline{p}$ and the condition "$(N||\eta) \models \phi^*(\vec{\gamma})$" is uniformly $\Sigma_0^{(n)}(N||\eta)$ in q by the same definition (since $N||\eta$ is sound above $\omega\rho^n_{N||\eta}$).

Set $X = \{\prec\vec{\gamma}\succ < \kappa \mid N||\eta \models \phi^*(\vec{\gamma})\}$. Then $X \in N||\eta$ since $\kappa < \omega\rho^n_{N||\eta}$.

Hence $\sigma(X) = \{\prec\vec{\gamma}\succ < \sigma(\kappa) \mid \overline{N} \models \phi^*(\vec{\gamma})\}$.

It follows that $X \neq \emptyset$ since $\prec\vec{\xi}\succ \in \sigma(X)$. Let $\prec\vec{\gamma}\succ \in X$. Then $N||\eta \models \phi(\vec{\gamma})$ and $N||\beta \models \neg\phi(\vec{\gamma})$ for all $\beta < \eta$ such that $\vec{\gamma} \in J^E_\beta$.

Since $\alpha > \eta$ was a minimal counter example, we have $\omega\rho^\omega_{N||\eta} \leq \gamma_0 < \kappa$, where $N||\eta$ is sound. Hence κ is not a cardinal in N. This is a contradiction.

Case 4 $\overline{N} = N_1||\eta$, where $\eta < ht(N_1)$ and $\pi : N \to_{E^N_\nu} N_1$.

Then $\nu = \lambda^{+N_1}$ is a cardinal in N_1, where $\lambda = \lambda_{E^N_\nu}$. However, $\overline{N} \models \phi(\vec{\xi})$ and

$$\overline{N}||\beta = N_1||\beta \models \neg\phi(\vec{\xi})$$

for all $\beta < \eta = ht(\overline{N})$ by (1) and that $n >> l$. But then $\omega\rho^\omega_{\overline{N}} \leq \xi_0$ by minimality of α since $\eta < \lambda^{+N} \leq \alpha$. But $\eta \geq \nu$. Hence ν is not a cardinal in N_1 since $\overline{N} = N_1||\eta$ is sound. We have our last contradiction.

□

Proof of the Theorem

Let $N = \langle J^E_\alpha, F\rangle$ be a mouse. We show that N is strongly acceptable.

Let $\tau + 1 \leq \alpha$ be such that $J^E_{\tau+1} \models \phi(\vec{\xi})$ and $J^E_\tau \models \neg\phi(\vec{\xi})$, where $\vec{\xi} < \omega\tau$ and ϕ is Σ_1. We assume that $\max(\vec{\xi}) = \max(\vec{\xi}, \omega)$ is primitive recursive closed. Let $\xi_0 = \max(\vec{\xi})$.

Let $\phi(\vec{x}) \iff \exists\, z\, \phi_0(z, \vec{x})$. Then

$$J^E_{\tau+1} \models \exists\, z \in J^E_\tau \phi_0(t(z, J^E_\tau, E_{\omega\tau}), \vec{\xi})$$

where t is rudimentary (since $J^E_{\tau+1} = \langle rud(N||\tau), E\rangle$). Hence

$$J^E_{\tau+1} \models \phi(\vec{\xi}) \iff N||\tau \models \phi_1(\vec{\xi})$$

uniformly, where ϕ_1 is Σ_ω.

Since $N||\tau$ is sound, it suffices to show that $\omega\rho^\omega_{N||\tau} \leq \xi_0$. Note that $N||\eta \models \neg\phi_1(\vec{\xi})$ for $\eta < \tau$ such that $\vec{\xi} < \omega\eta$. This is because otherwise one would have by the same reasoning that $J^E_{\eta+1} \models \phi(\vec{\xi})$ and hence $J^E_\tau \models \phi(\vec{\xi})$ since ϕ is Σ_1 and $\eta + 1 \leq \tau$.

Now we have set up to apply the previous lemma to conclude that $\omega\rho^\omega_{N||\tau} \leq \xi_0$.

□

To end this introduction, we wish to record our acknowledgements here. The first author wishes to thank DFG for providing a Mercator Guest Professorship, which allowed him to visit Humboldt University twice to work with the second author on this joint work, and to thank Humboldt University for its hospitality during his visits.

§2 Normal Iterations of Type 1 Premice

In this section, we carry out an analysis of iteration trees of type 1 premice. We shall prove that every iteration tree of type 1 premice of limit length has a unique cofinal branch.

First, let us give a definition of type 1 premice.

Definition 2.1 Let M be a premouse.

(1) Let $\tau < \kappa < ht(M)$. We say that τ *is strong upto* κ, denoted by $o^M(\tau) \geq \kappa$, if

$$\forall\, \beta < \kappa\, \exists\, \nu \leq ht(M)\ (\tau = crit(E^M_{\omega\nu})\ \wedge\ lh(E^M_{\omega\nu}) \geq \beta).$$

(2) κ is *of type* 0 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and $\{\tau < \kappa \mid o^M(\tau) \geq \kappa\}$ is bounded in κ.

(3) κ is *of type* ≥ 1 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and $\{\tau < \kappa \mid o^M(\tau) \geq \kappa\}$ is unbounded in κ.

(4) κ is *of type* ≥ 2 in M iff there is a $\nu \leq ht(M)$ such that $\kappa = crit(E^M_{\omega\nu})$ and

$$\{\tau < \kappa \mid o^M(\tau) \geq \kappa\ \wedge \tau \text{ is of type } \geq 1 \text{ in } M\}$$

is unbounded in κ.

(5) κ is *of type* 1 in M if κ is of type ≥ 1 and κ is not of type ≥ 2.

(6) For $\nu \leq ht(M)$, we say that $E^M_{\omega\nu} \neq \emptyset$ is of type 0 (of type 1, or of type ≥ 2) if $crit(E^M_{\omega\nu})$ is of type 0 (of type 1, or of type ≥ 2).

Definition 2.2 (1) A premouse M is *of type* 0 iff for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type 0 in M.

(2) A premouse M is *of type* 1 iff M is strongly acceptable and for all $\nu \leq ht(M)$ if $E^M_{\omega\nu} \neq \emptyset$ then $crit(E^M_{\omega\nu})$ is of type < 2 (i.e., not of type ≥ 2) in M.

Remark Type 0 mice are those iterable premice whose iterations shall never result to infinite branching iteration trees. Hence for type 0 premice, the iterations are almost linear iterations, as studied by Dodd and rediscovered by Schindler [5]. Type 1 mice are those iterable premice which may result to infinitely branching iteration trees.

In almost linear iterations, every iteration tree is finitely branched. Namely, in any iteration tree of type 0 premice, at any point of the tree, the number of immediate successors is finite. However, when we deal with type 1 premice iterations, infinite branching occurs, as indicated in the following example.

Example 2.1 Assume that $M =< J^E_\nu, E_\nu >$ is an iterable type 1 premouse. Assume that $E_\nu \neq \emptyset$. Let κ be the critical point of E_ν and let $\lambda = E_\nu(\kappa)$. Assume that κ is of type 1 in M. We define a normal iteration of M of length ω so that 0 is infinitely branching.

First, we observe a basic property which M has.

Define that

$$A(\alpha, \tau, \kappa) \iff \forall\, \beta < \kappa\, \exists\, \eta < \kappa(\beta < \eta \text{ and } crit(E_\eta) = \tau > \alpha),$$

and

$$B(\kappa,\lambda) \iff \forall\, \alpha < \kappa\, \exists\, \tau < \kappa\; A(\alpha,\tau,\lambda).$$

We observe that $M \models B(\kappa,\lambda)$.

Since κ is of type 1, M satisfies the following sentence:

$$\forall\, \alpha < \kappa\, \exists\, \tau < \kappa\; A(\alpha,\tau,\kappa),$$

Let $M_0 = M$. Let $M_1 = ult(M, E_\nu)$ and let π_{01} be the induced canonical map. Let $\alpha < \kappa$ and let $\tau < \kappa$ be such that $\alpha < \tau$ and $M_0 \models A(\alpha,\tau,\kappa)$. Then $M_1 \models A(\alpha,\tau,\lambda)$. By coherency, $M_0 \models A(\alpha,\tau,\lambda)$. Hence, $M_0 \models B(\kappa,\lambda)$.

By elementarity, $M_1 \models B(\lambda,\pi_{01}(\lambda))$. Applying this to the pair (κ,ν), we have an extender on the sequence E^{M_1}, indexed by ν_1, such that $\nu_0 = \nu < \nu_1 < \pi_{01}(\lambda)$ and

$$\kappa_0 = \kappa < \kappa_1 = crit(E^{M_1}_{\nu_1}) < \lambda_0 = \lambda.$$

Then we apply $E^{M_1}_{\nu_1}$ to M_0 and let M_2 be the ultrapower and let π_{02} be the induced canonical mapping.

Let $E^{M_2}_{\nu_2}$ be the top extender of M_2. Let κ_2 be the critical point of it. Then $\kappa_2 = \pi_{02}(\kappa_0)$. Let $\lambda_2 = \pi_{02}(\lambda_0)$ and let $\nu_2 = ht(M_2)$. Notice that $\kappa_2 = \kappa_0$ since $\kappa_1 > \kappa_0$.

Apply $E^{M_2}_{\nu_2}$ to M_0 to get an ultrapower M_3 and let π_{03} be the induced canonical mapping. Then

$$M_3 \models B(\pi_{03}(\kappa_0),\pi_{03}(\lambda_0)).$$

Since $\lambda_2 = \pi_{02}(\lambda_0) = \pi_{03}(\kappa_0)$ and $\nu_2 = \pi_{02}(\nu_0) = (\lambda_2^+)^{M_3}$, we have $\nu_1 < \lambda_2 < \nu_2 < \pi_{03}(\lambda_0)$. Applying $B(\lambda_2,\pi_{03}(\lambda_0))$ to the pair (ν_1,ν_2), we get an extender on the M_3 sequence, $E^{M_3}_{\nu_3}$ indexed by ν_3, such that $\nu_1 < \kappa_3 = crit(E^{M_3}_{\nu_3}) < \lambda_2$ and $\nu_2 < \nu_3 < \pi_{03}(\lambda_0)$.

Now $E^{M_3}_{\nu_3}$ must be applied to M_2 by the iteration rule of λ–indexing. Let M_4 be the ultrapower and let π_{24} be the induced canonical mapping. Let $E^{M_4}_{\nu_4}$ be the top extender of M_4 with $\nu_4 = ht(M_4)$ and $\kappa_4 = \kappa_0$ is the critical point and $\lambda_4 = \pi_{24}(\lambda_2)$.

Then $E^{M_4}_{\nu_4}$ must be applied to M_0. Let M_5 be the ultrapower and let π_{05} be the induced canonical mapping. Notice that $\lambda_4 = \pi_{05}(\kappa_0)$. So

$$M_5 \models B(\lambda_4,\pi_{05}(\lambda_0))$$

by elementarity. Apply this property to the pair (ν_3,ν_4). We get $E^{M_5}_{\nu_5}$ with $\nu_3 < \kappa_5 < \lambda_4$ and $\nu_4 < \nu_5 < \pi_{05}(\lambda_0)$. Then this extender must apply to M_4.

Inductively, at stage $n = 2k+2$, we apply $E^{M_{2k+1}}_{\nu_{2k+1}}$ to M_{2k} to get M_{2k+2} and the top extender $E^{M_{2k+2}}_{\nu_{2k+2}}$ with critical point $\kappa_{2k+2} = \kappa_0$ etc. Then we apply it to M_0 to get M_{n+1}. Using $\pi_{0,2k+3}$ to $B(\kappa_0,\lambda_0)$, by elementarity, we get $E^{M_{n+1}}_{\nu_{n+1}}$ which must be applied to M_{2k+2}.

By this way, we have seen that 0 is an infinite branching point.

□

We shall eventually prove that every normal iteration tree of type 1 premice of limit length has a unique cofinal branch. We need to establish some basic facts first.

Let T be a normal iteration tree with λ–indexing.

LEMMA 2.1 For $i+1 < lh(T)$, $T(i+1) \notin [0,i]_T$ if and only if there is a unique h such that $h+1 \leq_T i$ and $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$.

In case of that $T(i+1) \notin [0,i]_T$, we let $h(i)$ be the unique h to witness this fact and we say that $h(i)$ is defined.

Proof ($\Leftarrow$) Let h be such that $0 <_T h+1 \leq_T i$ and $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$. Then $T(h+1) < T(i+1) \leq h < h+1$. Hence $T(i+1) \notin [0,i]_T$.

($\Rightarrow$) Assume that $T(i+1) \notin [0,i]_T$. Then $0 < T(i+1) < i$.

Case1. $i = j+1$ and $\kappa_j \leq \kappa_i$.

If $T(i+1) < T(i)$, then $\kappa_j \leq \kappa_i < \lambda_{T(i+1)}$. But we have $\lambda_{T(i+1)} \leq \kappa_j < \lambda_{T(i)}$. Hence, $T(i+1) \geq T(i)$. Since $T(i) \leq_T i$, we have that $T(i) < T(i+1) \leq j$. Hence, $\lambda_{T(j+1)} \leq \kappa_i < \lambda_{T(i+1)} \leq \lambda_j$ and $j+1 \leq_T i$. Take $h = j$. We have our desired conclusion.

Case2. Otherwise.

(a) There is an m such that $m+1 \leq_T i$ and $\kappa_i < \lambda_{T(m+1)}$.

If i is a limit ordinal, then let $m_1 <_T i$ be such that $T(i+1) < m_1$. Then let m be such that $m+1 <_T i$ and $m_1 = T(m+1)$.

If $i = j+1$, then $\kappa_i < \kappa_j$ (otherwise we are back to the previous case). Hence $\kappa_i < \kappa_j < \lambda_{T(j+1)}$. Let $m = j$.

From (a), let m be the least n such that $n+1 \leq_T i$ and $\kappa_i < \lambda_{T(n+1)}$.

(b) $T(m+1)$ is not a limit ordinal.

First, $T(m+1) > 0$. Secondly, if $T(m+1)$ is a limit ordinal, then $[0, T(m+1))_T$ is cofinal in $T(m+1)$ and for all $h < T(m+1)$, letting $l+1 > h$ be such that $l+1 <_T T(m+1)$, then $\lambda_h < \lambda_{l+1} \leq \kappa_i$. Hence $T(i+1) = T(m+1)$ since $T(i+1) \leq T(m+1)$. But $T(m+1) \in [0,i]_T$.

Let h be such that $T(m+1) = h+1$. Then

$$T(h+1) <_T T(m+1) <_T m+1 \leq_T i,$$

and

$$\lambda_{T(h+1)} \leq \kappa_i < \lambda_{T(i+1)} \leq \lambda_h$$

since $\kappa_i < \lambda_{T(m+1)}$ and $T(i+1) \notin [0,i]_T$, and $T(m+1) \in [0,i]_T$ and hence $T(i+1) < T(m+1)$ and $T(i+1) \leq h$.

□

Let M be a type 1 premouse. Let $\mathcal{T} = \langle \langle N_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$ be a normal iteration of M.

LEMMA 2.2 For all i, if there is some $j > i$ such that

$$\kappa_i \leq \kappa_j < \lambda_i < \lambda_j$$

then κ_i or E_{ν_i} is of type 1.

Proof We prove Lemma 2.2 by induction on the length of normal iteration trees.

Let θ be an ordinal and let $\mathcal{T}$ be a normal iteration tree of length θ. Our induction hypothesis reads that for all normal iteration trees of length smaller than θ, lemma 2.2 is true.

Start with $\theta = 2$.

Let $\pi : N_0 \to N_1$ be given by E_{ν_0} and let E_{ν_1} be the next extender on N_1. Assume that $\kappa_0 \leq \kappa_1 < \lambda_0 < \lambda_1$. We want to show that E_{ν_0} is of type at least 1.

Claim E_{ν_1} cannot be a top extender of N_1.

Toward a contradiction, we assume that E_{ν_1} is a top extender of N_1.

By elementarity of π, let F be a top extender of N_0 and $\overline{\kappa}$ be the critical point of F such that $\pi(\overline{\kappa}) = \kappa_1$.

If $\overline{\kappa} \geq \kappa_0$, then $\kappa_1 = \pi(\overline{\kappa}) \geq \pi(\kappa_0) = \lambda_0$, contradicting to the fact that $\kappa_1 < \lambda_0$.

If $\overline{\kappa} < \kappa_0$, then $\kappa_1 = \pi(\overline{\kappa}) = \overline{\kappa} < \kappa_0$, contradicting to the fact that $\kappa_1 \geq \kappa_0$.

This shows the claim.

We then must have that $E_{\nu_1} \in N_1$. It follows that

$$N_1 \models \kappa_1 \text{ is strong upto } \lambda_0.$$

Let $\beta < \kappa_0$. Then

$$N_1 \models \exists\, \kappa < \lambda_0 (\kappa > \beta \;\&\; \kappa \text{ is strong upto } \lambda_0).$$

Hence,

$$N_0 \models \exists\, \kappa < \kappa_0 (\kappa > \beta \;\&\; \kappa \text{ is strong upto } \kappa_0).$$

Hence E_{ν_0} is of type at least 1.

Assume that $\theta > 2$ and assume that every normal iteration tree whose length is strictly smaller than θ satisfies the Lemma.

Let $\mathcal{T}$ be a normal iteration tree of length θ having an overlapping pair $i < j < \theta$, namely $\kappa_i \leq \kappa_j < \lambda_i < \lambda_j$.

If θ is a limit ordinal, then it follows by our induction hypothesis that $\mathcal{T}$ must satisfies the lemma.

If $\theta = \gamma + 1$ and that γ is a limit ordinal, then $j \leq \gamma$. If $j < \gamma$, then it follows from induction hypothesis. So let us assume that $j = \gamma$.

First let us assume that E_{ν_j} is the top extender of N_j.

Since N_j is the direct limit along the branch $[0, j]_T$, let $m < j$ be such that $i < m$ and $[m, j]_T$ is simple and $\overline{\kappa} \in N_m$ and $\overline{\lambda} \in N_m$ and $\pi_{mj}(\overline{\kappa}) = \kappa_j$ and $\pi_{mj}(\overline{\lambda}) = \lambda_j$.

Let $n + 1 \in [0, j]_T$ be such that $m = T(n + 1)$.

We then have $\lambda_i \leq \kappa_n$. Let $\pi = \pi_{m,j}$ be the embedding from N_m to N_j. Notice that $\kappa_i \leq \overline{\kappa}$.

We change the iteration to $\mathcal{T}'$ of length $m + 1$ by setting $\mathcal{T}'\restriction_m = \mathcal{T}\restriction_m$ and set $\nu'_m = ht(N_m)$ and $E'_{\nu'_m}$ to be the top extender of N_m. This shorter iteration tree allows us to apply induction hypothesis to finish this case.

Secondly, we assume that $\nu_j < ht(N_j)$. Let $m > i$ be sufficiently large such that $[m, j]_T$ is simple and $E_{\nu_j} = \pi_{mj}(E_{\overline{\nu}})$.

Let $n+1 \in [0,j]_T$ be such that $m = T(n+1)$.

Since $crit(\pi_{m,j}) = \kappa_n$, $\pi_{m,j}(\overline{\kappa}) = \kappa_j$, and $\pi_{m,j}(\overline{\lambda}) = \lambda_j$, we have $\overline{\kappa} = \kappa_j$ and $\overline{\lambda} \geq \kappa_n \geq \lambda_i$.

Let $F = \pi_{m,n+1}(E_{\overline{\nu}})$. Let $\nu = \pi_{m,n+1}(\overline{\nu})$.

If $\nu < \nu_n$, then by agreements, F is on E^{N_n} with an index $\nu > \lambda_n$. Hence $\nu_n \leq \nu$. If $\nu_n = \nu$, then ν is a cardinal in N_{n+1} and F is on $E^{N_{n+1}}$ and $F \in N_{n+1}$ whose index is a cardinal. Therefore, $\nu_n < \nu$.

We can now change the iteration tree $\mathcal{T}$ to $\mathcal{T}'$ of length $n+2$ by setting $\mathcal{T}'\restriction_{n+1} = \mathcal{T}\restriction_{n+1}$ and setting ν'_{n+1} to be ν and $E'_{n+1} = F$. This shorter iteration tree allows us to apply induction hypothesis to conclude the lemma in this case.

Assume now that $\theta = j+2$.

If $i < j$ is an overlapping pair, then induction hypothesis applies.

Let $i \leq j$ be such that $(i, j+1)$ be an overlapping pair.

Let $m = T(j+1)$ and let

$$\pi : N_m || \eta_j \to^*_{E_{\nu_j}} N_{j+1}.$$

If $i = j$, then similar to the case of $\theta = 2$ we conclude that E_{ν_i} is of type at least 1.

We assume that $i < j$ and (i,j) is not an overlapping pair. So either $\kappa_j < \kappa_i$ or $\lambda_i \leq \kappa_j$.

Case 1 $E_{\nu_{j+1}}$ is a top extender.

In this case, we show that the induction hypothesis can be applied to a tree $\mathcal{T}^*$ of length $m+1$ which agrees with the tree $\mathcal{T}$ upto m.

Let F be a top $(\overline{\kappa}, \overline{\lambda})$–extender on $N_m||\eta_j (\eta_j \geq \nu_m)$ corresponding to $E_{\nu_{j+1}}$ such that $\pi(\overline{\kappa}) = \kappa_{j+1}$, $\pi(\overline{\lambda}) = \lambda_{j+1}$.

Notice that $\pi(\kappa_j) = \lambda_j$.

First, $\overline{\kappa} < \kappa_j$. Otherwise,

$$\lambda_i > \kappa_{j+1} = \pi(\overline{\kappa}) \geq \pi(\kappa_j) = \lambda_j > \lambda_i.$$

Secondly, $\overline{\lambda} > \kappa_j$. $[\pi(\kappa_j) = \lambda_j < \lambda_{j+1} = \pi(\overline{\lambda}).]$

Since either $\kappa_j < \kappa_i$ or $\kappa_j \geq \lambda_i$, we show that $\kappa_j < \kappa_i$ cannot occur. Otherwise, we have

$$\overline{\kappa} < \kappa_j < \kappa_i \leq \kappa_{j+1}$$

and $\overline{\kappa} = \pi(\overline{\kappa}) = \kappa_{j+1}$. Contradiction.

Hence we have $\kappa_j \geq \lambda_i$ and $m > i$. Then we have

$$\kappa_i \leq \overline{\kappa} = \kappa_{j+1} < \lambda_i \leq \kappa_j < \overline{\lambda}.$$

We then change the iteration by setting $\nu_m = \eta_j$, etc., from $\mathcal{T}$ to get a normal iteration tree of length $m+1$ which agrees with $\mathcal{T}$ up to m. Hence E_{ν_i} is of type at least 1.

This finishes the argument for Case 1.

Case 2 $E_{\nu_{j+1}} \in N_{j+1}$.

[Notice that the extender may not be in the range of π. If it is in the range of π, we then could proceed as in the previous case.]

Subcase 1 $\kappa_j \geq \lambda_i$. Hence $i < m \leq j$.

Then we have

$$\kappa_i \leq \kappa_{j+1} < \lambda_i < \kappa_j < \lambda_j < \lambda_{j+1}.$$

Since $\pi(\kappa_{j+1}) = \kappa_{j+1}$, N_{j+1} thinks that $\pi(\kappa_{j+1}) < \lambda_j$ is the critical point of an extender on the sequence whose length is strictly larger than λ_j. It follows that $N_m||\eta_j$ thinks that $\kappa_{j+1} < \kappa_j$ is the critical point of an extender on the sequence whose length is strictly larger than κ_j.

Since $l < m$ implies that $\lambda_l \leq \kappa_j$, we have there is an $E_\nu^{N_m||\eta_j}$ whose critical point is κ_{j+1} and such that $\nu > \nu_l$ for all $l < m$.

We can then change the iteration using this extender, replacing the original E_{ν_m} in the construction of the tree $\mathcal{T}$, by setting $\kappa_m = \kappa_{j+1}$, etc., to get an iteration tree of length $m+1$ which agrees with $\mathcal{T}$ upto m. We then apply the induction hypothesis to conclude that E_{ν_i} is of type at least 1.

Subcase 2 $\kappa_j < \kappa_i$.

Let $\sigma : N_{T(i+1)} \to N_{i+1}$ be given by the iteration with extender E_{ν_i} and κ_i is the critical point.

Since $i < j$, $i+1 \leq j$ and $J_{\lambda_i}^{N_{i+1}} = J_{\lambda_i}^{N_{j+1}}$. We have

$$\kappa_j < \kappa_i \leq \kappa_{j+1} < \lambda_i < \lambda_{i+1} \leq \lambda_j < \lambda_{j+1}.$$

Let $N_{j+1} = \langle J_\alpha^E, F\rangle$. Since $E_{\nu_{j+1}} \in N_{j+1}$,

$$J_\alpha^E \models \forall\beta < \lambda_i \exists\nu(\kappa_{j+1} = crit(E_\nu) \wedge lh(E_\nu) > \beta.$$

Since λ_i is inaccessible in N_{j+1}, by strong acceptability,

$$J_{\lambda_i}^{E^{N_{j+1}}} \prec_{\Sigma_1} J_\alpha^{E^{N_{j+1}}}.$$

Hence κ_{j+1} is strong upto λ_i in $J_{\lambda_i}^{E^{N_{j+1}}}$.

By the agreements of N_{j+1} and N_{i+1}, κ_{j+1} is strong upto λ_i in N_{i+1}.

It follows that for $\gamma < \kappa_i$, $N_{T(i+1)}$ thinks that there is τ such that $\gamma < \tau < \kappa_i$ and τ is strong upto κ_i. Therefore, E_{ν_i} is of type at least 1.

This finishes the proof of Lemma 2.2.

□

We are going to prove eventually the following structure theorem of normal iteration trees of type 1 premice. We prove it first under the assumption that the iteration tree has no truncation and then later remove this assumption. This theorem is needed in proving our iterability theorem.

THEOREM 2.1 Let $\mathcal{T}$ be a normal iteration tree of type 1 premice. If κ_i is of type 1, then $T(i+1) \leq_T i$ and $\kappa_i < crit(\pi_{T(i+1)i})$.

We first prove a weaker version of this theorem with the assumption that there is no truncation in the tree. We split the proof into three lemmas. Let M be a type 1 premouse. Let $\mathcal{T} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal iteration of M.

LEMMA 2.3 If $T(j+1) \leq_T j$ and E_{ν_j} is a top extender, and $\kappa_j \leq crit(\pi_{T(j+1),j})$, then $\kappa_j < crit(\pi_{T(j+1),j})$.

Proof To see this, assume that $\kappa_i = crit(\pi_{T(j+1),j})$. Let $\overline{\kappa}$ be the critical point of $E_{top}^{M_{T(j+1)}}$. Then $\pi_{T(j+1),j}(\overline{\kappa}) = \kappa_j =$ the critical point of $E_{top}^{M_j}$. If $\overline{\kappa} \geq \kappa_j$ then

$$\kappa_j = \pi_{T(j+1),j}(\overline{\kappa}) \geq \pi_{T(j+1),j}(\kappa_j) > \kappa_j.$$

If $\overline{\kappa} < \kappa_j$ then

$$\overline{\kappa} = \pi_{T(j+1),j}(\overline{\kappa}) = \kappa_j.$$

So in any case, we have a contradiction.

□

LEMMA 2.4 Assume that there is no truncation. If there is some $i < j$ such that $\kappa_i \leq \kappa_j < \lambda_i$ and E_{ν_j} is of type 1, then the following holds:

(a) E_{ν_j} is a top extender (i.e., $\nu_j = ht(M_j)$);

(b) $T(j+1) \leq_T j$ and $T(j+1)$ is the least m such that $m \leq_T j$ and $crit(\pi_{m,j}) \geq \kappa_j$;

(c) $crit(\pi_{T(j+1),j}) > \kappa_j$.

Proof First we show that (a) $\nu_j = ht(M_j)$.

Suppose not. Since λ_i is a cardinal in M_j, $\lambda_i \leq \omega\rho^1_{M_j||\nu_j}$, κ_j is strong to λ_i in $M_j||\nu_j$, by strong acceptability, it follows that for all $\xi < \kappa_i$, $J_{\lambda_i}^{E^{M_i}} = J_{\lambda_i}^{E^{M_j}}$ satisfies the sentence $\theta(\xi)$ that there is a $\kappa > \xi$ such that $o(\kappa) = \infty$ and there are unbounded many $\tau < \kappa$ with $o(\tau) \geq \kappa$.

By elementarity, $J_{\kappa_i}^{E^{M_i}}$ satisfies the same sentence $\theta(\xi)$ for each $\xi < \kappa_i$. This contradicts to the fact that E_{ν_i} is not of type 2.

This shows (a).

To see (b), let $h =$ the least $m \leq_T j$ such that $\kappa_j \leq crit(\pi_{m,j})$. We show that $T(j+1) = h$.

(i) $h \leq T(j+1)$. Namely, if $l < h$ then $\lambda_l \leq \kappa_j$.

Otherwise, there is an $l < h$ such that $\kappa_j < \lambda_l$. Then h must be a successor ordinal since otherwise

$$\sup_{m<h}\lambda_m = \sup_{l+1<_T h}\kappa_l \leq \kappa_j$$

by definition of h. Let $h = l+1$. By minimality of h again, $\kappa_l < \kappa_j$ since $\kappa_l = crit(\pi_{T(h),h})$. By our assumption, $T(j+1) < h$. So $\kappa_j < \lambda_l$.

Let $\overline{\kappa} = crit(E_{top}^{M_{T(h)}})$.

If $\overline{\kappa} < \kappa_l$, then

$$\overline{\kappa} = \pi_{T(h),j}(\overline{\kappa}) = \kappa_j = crit(E_{\nu_j}) < \kappa_l.$$

If $\kappa_l \leq \overline{\kappa}$, then

$$\kappa_j = \pi_{T(h),j}(\overline{\kappa}) \geq \lambda_l > \kappa_j.$$

This contradiction shows that $T(j+1) \geq h$.

(ii) $h \geq T(j+1)$. Namely, $\kappa_j < \lambda_h$.

Otherwise, we have $\lambda_h \leq \kappa_j$. So $h <_T j$ by definition of h. Let l be such that $h = T(l+1)$ and $l+1 \leq_T j$. Then

$$crit(\pi_{h,j}) = \kappa_l < \lambda_h \leq \kappa_j.$$

Contradicts to the definition of h.

Hence we have (b) and (c).

□

LEMMA 2.5 Assume that there is no truncation. If E_{ν_j} is of type 1, then $T(j+1) \leq_T j$ and $\kappa_j < crit(\pi_{T(j+1),j})$.

Proof First, $T(j+1) \leq_T j$. If not, by (1), $h(j)$ is defined and $h(j) < j$ and $\kappa_{h(j)} < \kappa_j < \lambda_{h(j)} < \lambda_j$. By Lemma 2.1, we have $T(j+1) \leq_T j$.

Secondly, $\kappa_j < crit(\pi_{T(j+1),j})$.

If $T(j+1) = j$, it's trivial. So we assume that $T(j+1) <_T j$. Let l be such that $l+1 \leq_T j$ and $T(j+1) = T(l+1)$. If $\kappa_l \leq \kappa_j$, then we have $l < j$ and $\kappa_l \leq \kappa_j < \lambda_l < \lambda_j$. But by Lemma 4.4, $\kappa_j < \kappa_l$. So $\kappa_j < \kappa_l = crit(\pi_{T(j+1),j})$.

Now we want to prove the full structure theorem by removing the assumption that the iteration has no truncation. To carry out our analysis in case of truncations, it is necessary to develop some technical objects first.

DEFINITION 2.3 For a normal iteration $\mathcal{T} = \langle\ \langle N_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ of a type 1 premice, we define (uniformly with respect to iterations) the following sequences λ_{ij}, U_j and κ_{ij}, which are naturally associated to iteration trees, as follows.

(1) For $i < j$, we define $\lambda_{ij} = \min\{\kappa_h \mid i \leq h < j\ \wedge \kappa_h < \lambda_i \wedge\ \kappa_h \text{ is type } 0\}$ if there is some h such that $i \leq h < j$ and $\kappa_h < \lambda_i$ and κ_h is of type 0; and define $\lambda_{ij} = \lambda_i$ if otherwise.

(2) Let $U_j = \{i < j \mid \lambda_{ij} > \sup_{l<i}\lambda_l\}$.

(3) For each i, let $\kappa_{ii} = \lambda_i$.

Induction on j, we define κ_{ij} for $i \in U_j$ so that (*) if $i < h <_T j$ then $\kappa_{ih} = \kappa_{ij}$ as follow:

For $j = h+1$, let $\xi = T(h+1)$. If $\xi \in U_j$, then set $\kappa_{\xi j} = \kappa_h$; if $i \in U_j \cap \xi$, then set $\kappa_{ij} = \kappa_{i\xi}$; and if $i \in U_j - (\xi+1)$, then set $\kappa_{ij} = \min(\kappa_h, \kappa_{ih})$.

The property that we demand is satisfied: if $i \in U_j$ and $l <_T j$ and $i < l$, then $l \leq \xi$. Hence $\kappa_{il} = \kappa_{i\xi} = \kappa_{ij}$.

For limit j, if $i \in U_j$, let $h <_T j$ be the least such that $i < h$, then we set $\kappa_{ij} = \kappa_{ih}$.

We list some basic properties of these sequences here for later on usage.

LEMMA 2.6 Let the notations of the previous definition stand. Then
(1) $\lambda_{ij} \leq \lambda_i$.
(2) $\lambda_{ij} < \lambda_i$ iff there is an index m such that $i \leq m < j$ and $\kappa_m < \lambda_i$ and κ_m is of type 0.
(3) If $i < j \leq k$, then $\lambda_{ij} \geq \lambda_{ik}$. Also, if $i < j$ and κ_j is of type 1, then $\lambda_{ij} = \lambda_{ij+1}$.
(4) If $i < j$ and $i \in U_j$, then $\kappa_{ij} < \lambda_i$.
(5) If $i < j$ and there is some $k \geq j$ such that $i = T(k+1)$, then $i \in U_j$. In particular, if $T(h+1) < l \leq h$, then $T(h+1) \in U_l$.
(6) If $h < j$, then $U_j \cap h \subseteq U_h$.
(7) If κ_h is of type 0, then for all $i \in [T(h+1), h]$ (in the sense of ordinal interval), $\lambda_{ih+1} = \kappa_h$, and $U_{h+1} \cap (T(h+1), h] = \emptyset$, and for all $j > h$, $\max\{\lambda_{hj}, \lambda_{T(h+1)j}\} \leq \kappa_h$. In particular, if there is some l such that $T(h+1) < l \leq h$ and $l \in U_{h+1}$, then κ_h must be of type 1.
(8) If κ_h is of type 1, then for all $i < h$, $\lambda_{ih} = \lambda_{ih+1}$ and hence $U_{h+1} \cap h = U_h$.
(9) If $i \in U_j$ and $i <_T j$, then κ_{ij} is the critical point of the canonical mapping π_{ij}. I.e., let $k+1 > i$ be the least such that $k+1 \leq_T j$, then $\kappa_{ij} = \kappa_k$.

[By induction on j. Assume that $i \in U_j$ and $i <_T j$. When j is a limit ordinal, it follows by induction hypothesis and the eventual constant value property of the κ_{hj} along the branch to j. Let $j = h+1$. Since $i <_T j$, and $i \in U_j$, either $i < T(j)$ and $\kappa_{ij} = \kappa_{iT(j)}$, reducing to induction hypothesis, or $i = T(j)$ and $\kappa_{ij} = \kappa_h$, which is the critical point of π_{ij}.]
(10) For $h < i$, $[(T(i+1) \in U_{h+1} \text{ and } T(h+1) < T(i+1)) \iff (\lambda_{T(h+1)} \leq \kappa_i < \lambda_h)]$.

[Let $h < i$. Assume that $T(i+1) \in U_{h+1}$ and $T(h+1) < T(i+1)$. Then $T(i+1) \leq h$ and $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$. Assume that $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$. Then $T(h+1) < T(i+1)$ and since $h+1 \leq i$, $\sup_{l<T(i+1)} \lambda_l \leq \kappa_i < \lambda_{T(i+1)h+1}$. Hence $T(i+1) \in U_{h+1}$.]
(11) Let $j = h+1$ and $\xi = T(j)$. If κ_h is of type 0, then $\lambda_{\xi j} = \kappa_h = \kappa_{\xi j}$.

[Notice that if $\xi \leq l < h$ and κ_l is of type 0, then $\kappa_h < \kappa_l$.]

LEMMA 2.7 (Continuity Lemma) Let $\mathcal{T}$ and $\mathcal{T}'$ be two iteration trees of type 1 premice. Let $\eta + 1 < \min\{lh(\mathcal{T}), lh(\mathcal{T}')\}$. Assume that $\mathcal{T}\restriction_\eta = \mathcal{T}'\restriction_\eta$. Then

(1) For all $i < \eta$, $\lambda^{\mathcal{T}}_{i\eta} = \lambda^{\mathcal{T}'}_{i\eta}$.

(2) $U^{\mathcal{T}}_\eta = U^{\mathcal{T}'}_\eta$.

(3) If $i \in U^{\mathcal{T}}_\eta$, then $\kappa^{\mathcal{T}}_{i\eta} = \kappa^{\mathcal{T}'}_{i\eta}$.

Proof Both (1) and (2) follow from the definition and the agreements of the two trees.

(3) is proved by induction on η. If η is a limit ordinal, then it follows from the basic property of the κ_{mn} sequences and induction hypothesis. If $\eta = \xi + 1$, then it follows from induction hypothesis and the fact that κ_ξ is the same on both trees.

The following is the key lemma in proving the structure theorem. This lemma is actually a weaker version of the theorem, as we shall see later.

LEMMA 2.8 Let $j = h+1$ and $\xi = T(j)$. If κ_h is of type 1 and $i \in U_h - \xi$, then $\kappa_h < \kappa_{ih}$.

Proof Suppose the lemma is false. Let $\mathcal{T}$ be a counter example of minimal length. Then $j+1 = lh(\mathcal{T})$ and $j = h+1$ and $h = k+1$. Let ξ be the least such that $\kappa_h < \lambda_\xi$ and let $\eta = T(h)$.

(1) There is some $i \in [\xi, h)$ such that $\kappa_i \leq \kappa_h$.

Otherwise, for every $i \in [\xi, h)$, $\kappa_i > \kappa_h$. We claim that if $\xi \leq i < l \leq h$ and $i \in U_l$, then $\kappa_{il} > \kappa_h$. This shall give us the desired contradiction.

Let $i < l$ be a minimal pair of counter example to the claim. Then $l = m+1$, and $\xi \leq T(l) \leq m$. If $i < T(l)$, then $\kappa_{il} = \kappa_{iT(l)} > \kappa_h$. If $i = T(l)$, then $\kappa_{il} = \kappa_m > \kappa_h$. If $i > T(l)$, then $\kappa_{il} = \min(\kappa_m, \kappa_{im})$. If $i = m$, then $\kappa_{im} = \lambda_m$. If $i < m$, then $i \in U_m$. Hence, $\kappa_{im} > \kappa_h$ and $\kappa_m > \kappa_h$. Therefore, our claim follows.

(2) E_{ν_h} is a top extender.

Let $i \in [\xi, h)$ be such that $\kappa_i \leq \kappa_h$. Then κ_i and κ_h are both of type 1. If E_{ν_h} is not a top extender, then $\lambda_i \leq \omega\rho^1_{N_h||\nu_h}$. Hence κ_h is strong up to λ_i. Therefore, there are arbitrarily larger $\tau < \kappa_i$ of type 1 such that τ is strong up to κ_i. It then follows that κ_i is of type 2. We get a contradiction.

(3) Let

$$\pi_{\eta,h} : N_\eta||\eta_k \to^*_{E_{\nu_k}} N_h.$$

Then $\kappa_k > \kappa_h$.

Let F be the top extender of $N_\eta||\eta_k$ be such that $\kappa_h = \pi_{\eta,h}(\kappa)$ with $\kappa = crit(F)$. Then $\kappa < \kappa_k$. Hence, $\kappa = \kappa_h$.

(If $\kappa \geq \kappa_k$, then $\kappa_h = \pi_{\eta,h}(\kappa) \geq \lambda_k$. Hence, $\xi = h$. But $\xi < h$ is assumed.)

(4) If $k \in U_h$, then $\kappa_{kh} = \kappa_k$.

If $k = T(h)$, then $k \in U_h$ implies that $\kappa_{kh} = \kappa_k$. If $T(h) < k$, then $k \in U_h$ implies that $\kappa_{kh} = \min\{\kappa_k, \kappa_{kk}\} = \kappa_k$.

Now let $i \in U_h$ be such that $\xi \leq i < h$ such that $\kappa_{ih} \leq \kappa_h$. Then $i < k$ by (4) above.

(5) $\eta \leq i$.

Otherwise, then $i < \eta < h$ and $\kappa_{ih} = \kappa_{i\eta}$. Hence $\kappa_{i\eta} \leq \kappa_h$. By (3), κ_h is the critical point of the top extender F on $N_\eta||\eta_k$. We define a new iteration tree of shorter length as follows:

If $T(h) = k$, then replace E_{ν_k} by the top extender F of $N_\eta||\eta_k$. We then have a tree of length h. If $T(h) < k$, then we take the initial part of the tree $\mathcal{T}$ up to $\eta+1$ and change the extender at the index ν_η to take the F. We also get a tree of length $\eta+2$.

Let $\mathcal{T}'$ be the new tree. Then $lh(\mathcal{T}') < lh(\mathcal{T})$.

By the Continuity Lemma, U_η is invariant. So $i \in U_\eta$ since it is true for $\mathcal{T}$. Also $\kappa_{i\eta}$ is invariant by passing from $\mathcal{T}$ to the new tree $\mathcal{T}'$.

But, $T'(\eta+1) = \xi$ and $\kappa_\eta = \kappa_h$. So we have that κ_η is of type 1 and $\xi \leq i < \eta$ and $i \in U_\eta$. By minimality of $lh(\mathcal{T})$, we have $\kappa_\eta < \kappa_{i\eta}$. However, $\kappa_{i\eta} = \kappa_{ih} < \kappa_h$. We have a contradiction.

(6) κ_k is of type 0.

Assume otherwise. Then $\lambda_{ik} = \lambda_{ih}$. It follows that $i \in U_k$ since $i \in U_h$. By the minimality of j, $\kappa_k < \kappa_{ik}$. If $i = T(h)$, then $\kappa_{ih} = \kappa_k$. If $i > T(h)$, then $\kappa_{ih} = \min(\kappa_k, \kappa_{ik}) = \kappa_k$. In any case, $\kappa_{ih} = \kappa_k > \kappa_h$. We have a contradiction.

(7) $i = T(h)$. This is because $i \in U_h$ and κ_k is of type 0 and $\kappa_k < \lambda_i$ and hence $\kappa_k = \lambda_{ih} > sup_{l<i}\lambda_l$.

It now follows that $\kappa_{ih} = \lambda_{ih} = \kappa_k > \kappa_h$. We get a contradiction.

This finishes the proof of the lemma.

□

THEOREM 2.2 Let $\mathcal{T}$ be a normal iteration tree of type 1 premice. If κ_i is of type 1, then $T(i+1) \leq_T i$ and $\kappa_i < crit(\pi_{T(i+1)i})$.

Proof We prove the theorem by induction on i. We just need to show that if κ_i is of type 1, then $T(i+1) \leq_T i$. The second part of the conclusion follows from the fact that $T(i+1) \in U_i$ (in the nontrivial case) and the previous lemma and the fact (9) of Lemma 2.6 above.

Let us assume that i is a minimal counter example. Then $T(i+1) \notin [0,i]_T$ and for all $h < i$, if κ_h is of type 1, then $T(h+1) \leq_T h$. By Lemma 2.1, we have a unique h such that $T(h+1) < T(i+1) \leq h$ and $h+1 \leq_T i$.

Claim 1 κ_h is of type 1.

This follows from that $T(h+1) < T(i+1) \leq h < i$ and $T(i+1) \in U_{h+1}$ and the basic fact 7 above.

Hence by our induction hypothesis, $T(h+1) <_T h$. Also

$$\kappa_h < \kappa_{T(h+1)h} = crit(\pi_{T(h+1)h})$$

by the previous lemma and the basic fact Lemma 2.6(9) above.

Claim 2 If $l \in [h+1, i]_T$, then $T(i+1) \in U_l$ and $\kappa_{T(i+1)l} = \kappa_h$.

We just need to check that $\kappa_{T(i+1)h+1} = \kappa_h$.

By definition, $\kappa_{T(i+1)h+1} = \min\{\kappa_h, \kappa_{T(i+1)h}\}$.

If $T(i+1) = h$, then $\kappa_{hh} = \lambda_h$. If $T(i+1) < h$, then by the previous lemma, $\kappa_h < \kappa_{T(i+1)h}$. Hence, $\kappa_{T(i+1)h+1} = \kappa_h$.

The rest of the claim follows from the basic property of the κ_{mn} sequences abstractly.

Claim 3 $T(h+1) <_T T(i+1)$.

Granting this claim, we finish our induction proof as follows. First, $\kappa_h < \lambda_{T(h+1)} \leq \kappa_i$. Secondly, since κ_i is of type 1 and $T(i+1) \in U_i$, by our previous lemma, $\kappa_i < \kappa_{T(i+1)i}$. Thirdly, by claim 2 above, $\kappa_{T(i+1)i} = \kappa_h$. We have a contradiction.

Now let us proceed to prove Claim 3.

First let us observe the following fact: if l is such that

$$T(h+1) \leq T(l+1) < T(i+1) \leq l \leq h,$$

then $T(l+1) <_T l$, and $\kappa_l < crit(\pi_{T(l+1),l})$.

To see this, for such l, we have that $T(l+1) < T(i+1) \leq l \leq h < i$ implies that $T(i+1) \in U_{l+1}$. Applying the basic fact Lemma 2.6(7) above, we conclude that κ_l is of type 1. By minimality of i, we have that $T(l+1) <_T l$ and $\kappa_l < crit(\pi_{T(l+1),l})$.

We now define a minimal walk from i to $T(i+1)$ as follows.

Let $s(0) =$ the least $m \leq_T i$ such that $T(i+1) \leq m$. (Hence $s(0) = h(i) + 1$.)

If $s(0) = T(i+1)$, then we stop.

If not, let $t(0) = h(i)$ and so $s(0) = t(0)+1$. We now move to the branch from $T(h+1)$ to $t(0)$, $[T(h+1), t(0)]_T$. We have that $T(h+1) <_T t(0)$, $T(h+1) < T(i+1) \leq t(0)$ and $\kappa_h = \kappa_{t(0)} < crit(\pi_{T(h+1),t(0)})$.

Let $s(1) =$ the least $m \leq_T t(0)$ be such that $T(i+1) \leq m$.

If $s(1) = T(i+1)$, then we stop. Otherwise, $T(i+1) < s(1)$. Since $s(1) \leq_T t(0)$, $T(h+1) < T(i+1)$ and $T(h+1) <_T t(0)$, by minimality of $s(1)$, $s(1)$ must be a successor ordinal. Let $s(1) = t(1)+1$. Then we have

$$T(s(0)) = T(h+1) \leq_T T(s(1)) < T(i+1) \leq t(1).$$

Hence, by the first observation above, we have

$$T(s(1)) <_T t(1)$$

and

$$\kappa_{t(0)} < crit(\pi_{T(s(0)),t(0)}) \leq \kappa_{t(1)} < crit(\pi_{T(s(1)),t(1)}).$$

Now by induction, assume that $s(n)$ has been defined and we have the following:
$s(0) = t(0) + 1 \leq_T i$,
$s(1) \leq_T t(0), s(1) = t(1) + 1$, $T(s(0)) \leq_T T(s(1)) <_T t(1)$,
$s(n) \leq_T t(n-1), s(n) = t(n) + 1$, and
$T(s(0)) \leq_T T(s(1)) \leq_T \cdots \leq_T T(s(n)) < T(i+1) \leq t(n)$ and
$T(s(n)) <_T t(n)$, and
$\kappa_{t(0)} < crit(\pi_{T(s(0)),t(0)}) \leq \cdots \leq \kappa_{t(n)} < crit(\pi_{T(s(n)),t(n)})$.
Let $s(n+1)$ be the least $m \leq_T t(n)$ such that $T(i+1) \leq m$.
If $s(n+1) = T(i+1)$, then stop.

Otherwise, $s(n+1) > T(i+1)$ and by minimality, $s(n+1)$ must be a successor ordinal and let $s(n+1) = t(n+1)+1$. Then we are in the situation as above to define $s(n+2)$ while maintaining our induction hypothesis. Namely,

$$T(s(n)) \leq_T T(s(n+1)) < T(i+1) \leq t(n+1),$$

$$T(s(n+1)) <_T t(n+1),$$

and

$$crit(\pi_{T(s(n)),t(n)}) \leq \kappa_{t(n+1)} < crit(\pi_{T(s(n+1)),t(n+1)}).$$

Since $s(0) > s(1) > s(2) > \cdots s(n) > s(n+1) > \cdots$, we must stop at some stage, i.e., we must reach a stage $m+1$ such that $s(m+1) = T(i+1)$. When we stop at stage $m+1$, we have that

$$T(t(m)+1) < s(m+1) = T(i+1) \leq_T t(m)$$

and

$$T(t(m)+1) <_T t(m) + 1 \leq_T t(m-1).$$

Since $T(i+1) \in U_{t(m)+1}$ and $T(t(m)+1) < T(i+1) \leq_T t(m)$, $\kappa_{t(m)}$ is of type 1. Hence by induction hypothesis, we have $T(t(m)+1) \leq_T t(m)$. Therefore, $T(t(m)+1) <_T T(i+1)$. This gives us that $T(h+1) <_T T(i+1)$.

□

We abstract the following useful fact from the previous proof.

LEMMA 2.9 (Minimal Walk to $T(i+1)$) Assume that $T(i+1) \notin [0,i]_T$. Then there is a sequence $< t(0), t(1), t(2), \cdots, t(m) >$ such that

(1) $t(0)+1 \leq_T i$ and $T(i+1) \leq_T t(m)$,

(2) $t(n+1)+1 \leq_T t(n)$ for all $n < m$, (hence, $t(0) > t(1) > \cdots > t(m)$),

(3) $T(t(n)+1) \leq_T T(t(n+1)+1) < T(i+1)$ for $n < m$, and

(4) each $\kappa_{t(n)}$ is of type 1 for $n \leq m$.

Proof Assume that $T(i+1) \notin [0,i]_T$. We now define a minimal walk from i to $T(i+1)$ as what we did in the previous proof.

Set $n = 0$ and $t(-1) = i$. Start the Minimal Walk to $T(i+1)$ process.

Step 1 Let $s(n) =$ the least $m \leq_T t(n-1)$ such that $T(i+1) \leq m$.

Step 2 If $s(n) = T(i+1)$, then we stop and output $t : n \to i$ with success. Otherwise, continue to Step 3 to define $t(n)$.

Step 3 We have $T(i+1) < s(n) \leq_T t(n-1)$ and $s(n)$ is a successor ordinal. Let $t(n)$ be the ordinal predecessor of $s(n)$. Then $T(t(n)+1) < T(i+1) \leq t(n)$, $t(n)+1 \leq_T t(n-1)$ and if $n > 0$ then $T(t(n-1)+1) \leq_T T(t(n)+1)$. It also follows that $T(i+1) \in U_{t(n)+1}$ and $\kappa_{t(n)}$ is of type 1 and $T(t(n)+1) <_T t(n)$. (Notice that if $T(i+1) = t(n)$, then in the next round of the process, $s(n+1) = t(n) = T(i+1)$ and we exit in Step 2 of next round.) Continue to Step 4 to increase the counter n.

Step 4 Set $n = n+1$ and go to Step 1 to repeat the process one more round.

This gives the description of the process.

Since $t(0) > t(1) > t(2) > \cdots \geq T(i+1)$, there must be a stage m for the process to stop. When it stops at m, we have the following:

$T(t(0)+1) \leq_T T(t(1)+1) \leq_T \cdots \leq_T T(t(m)+1) < T(i+1) \leq_T t(m)$ and

$t(0)+1 \leq_T i$ and each $\kappa_{t(n)}$ is of type 1 for $n \leq m$. It also follows that $T(t(m)+1) <_T T(i+1)$.

□

DEFINITION 2.4 Let $\mathcal{T}$ be an iteration tree. For $i < j < lh(\mathcal{T})$, we define the *point of joint* of i and j on the tree, $T_\wedge(i,j)$, to be the maximum of the intersection of the two intervals $[0,i]_T$ and $[0,j]_T$, i.e., $T_\wedge(i,j) = \max\{m \mid m \leq_T i \text{ and } m \leq_T j\}$.

COROLLARY 2.1 Let $\mathcal{T}$ be a normal iteration of type 1 premice. If $T(i+1) \notin [0,i]_T$, then $T_\wedge(T(i+1), i) = T(h(i)+1)$, where $h(i)$ is the unique h such that $\lambda_{T(h+1)} \leq \kappa_i < \lambda_h$.

We now give a geometric explanation of the κ_{ij} sequence. This will make it clear why our key lemma is indeed a weaker version of the structure theorem.

THEOREM 2.3 Assume that $i < j$ are two ordinals less than the length of a normal iteration tree $\mathcal{T}$ of type 1 premice.

(1) If $i \in U_j$, then $T_\wedge(i,j) \in U_j$.
(2) If $i \in U_j$, then $\kappa_{ij} = \min\{\kappa_{T_\wedge(i,j),i}, \kappa_{T_\wedge(i,j),j}\}$, hence,

$$\kappa_{ij} = \min\{crit(\pi_{T_\wedge(i,j),i}), crit(\pi_{T_\wedge(i,j),j})\}.$$

(3) If $j = h+1$ and $T(j) < i < h$ and $i \in U_j$, then $T_\wedge(i,j) = T(j)$.

Proof We prove (1) and (2) by induction on j. We prove (1) first.
Case 1 j is a limit ordinal.

Assume that $i \in U_j$. Let $h <_T j$ be such that $i < h$. Then $T_\wedge(i,j) = T_\wedge(i,h) <_T h$. Notice that if $T_\wedge(i,j) <_T h_1 <_T h_2 <_T j$, then $\lambda_{T_\wedge(i,j)j} \le \lambda_{T_\wedge(i,j)h_2} \le \lambda_{T_\wedge(i,j)h_1}$. Therefore, we can choose our h so large that, in addition, $\lambda_{T_\wedge(i,j)j} = \lambda_{T_\wedge(i,j)h}$. Then for such an h, by induction hypothesis, $T_\wedge(i,j) = T_\wedge(i,h) \in U_h$ since $i \in U_h$. Hence, $sup_{l<T_\wedge(i,j)}\lambda_l < \lambda_{T_\wedge(i,j)h} = \lambda_{T_\wedge(i,j)j}$. Therefore, $T_\wedge(i,j) \in U_j$.

Case 2 $j = h+1$.

Let $i \in U_j$. If $i <_T j$, then $T_\wedge(i,j) = i$. Nothing needs to be proved. So we assume that $T_\wedge(i,j) <_T i$.
Subcase 1 κ_h is of type 1.

Then by Theorem 2.2, $T(j) \le_T h$. Hence $T_\wedge(i,j) = T_\wedge(i,h)$.

If $i < h$, then $i \in U_h$. By induction hypothesis, $T_\wedge(i,j) = T_\wedge(i,h) \in U_h$. Since $T_\wedge(i,j) < i < h$ and κ_h is of type 1, $U_j \cap h = U_h$. Hence, $T_\wedge(i,j) \in U_j$.

If $i = h$, then $T_\wedge(i,j) = T(j) <_T i = h$. Hence, $T(j) \in U_h = h \cap U_j$.
Subcase 2 κ_h is of type 0.

We are in that $i \in U_j$ and $T_\wedge(i,j) <_T i$. Since κ_h is of type 0, $i \le T(j)$. Since $T_\wedge(i,j) <_T i$ and $T_\wedge(i,j) \le_T T(j) <_T j$, we must have $i < T(j)$, $T_\wedge(i,j) <_T T(j)$ and $T_\wedge(i,j) = T_\wedge(i,T(j))$. It follows that $\lambda_{T_\wedge(i,j)j} = \lambda_{T_\wedge(i,j)h}$ since $\lambda_{T_\wedge(i,j)} \le \kappa_h$.

Since $i < T(j)$ and $i \in U_j$, $i \in U_{T(j)}$. By induction hypothesis, $T_\wedge(i,j) = T_\wedge(i,T(j)) \in U_{T(j)}$.

Since $T(j) \le h$, we consider two cases.

(a) $T(j) = h$. Then $T_\wedge(i,j) \in U_h$. Hence,

$$\sup_{l<T_\wedge(i,j)}\lambda_l < \lambda_{T_\wedge(i,j)h} = \lambda_{T_\wedge(i,j)j}.$$

Hence, $T_\wedge(i,j) \in U_j$.

(b) $T(j) < h$.

If $\lambda_{T_\wedge(i,j),h} = \lambda_{T_\wedge(i,j),T(j)}$, then $\lambda_{T_\wedge(i,j),j} = \lambda_{T_\wedge(i,j),T(j)}$. Since $T_\wedge(i,j) \in U_{T(j)}$, we have $T_\wedge(i,j) \in U_j$.

So let us assume that $\lambda_{T_\wedge(i,j),h} < \lambda_{T_\wedge(i,j),T(j)}$. In this case, there is an m such that $T(j) \le m < h$ and κ_m is of type 0 and $\kappa_m < \lambda_{T_\wedge(i,j)}$ and $\kappa_m = \lambda_{T_\wedge(i,j),h}$ and if $l \in [T_\wedge(i,j), T(j))$, in the ordinal sense, and κ_l is of type 0 and $\kappa_l < \lambda_{T_\wedge(i,j)}$ then $\kappa_m < \kappa_l$. This gives us that

$$\lambda_{T_\wedge(i,j),h} \ge \lambda_{T(j)h}.$$

Since $T(j) < h$, $T(j) \in U_h$. Hence

$$\sup_{l<T_\wedge(i,j)}\lambda_l < \sup_{l<T(j)} < \lambda_{T(j)h} \le \lambda_{T_\wedge(i,j),h} = \lambda_{T_\wedge(i,j),j}.$$

Hence $T_\wedge(i,j) \in U_j$.

This finishes the proof of (1).

We now move to prove (2).

To keep certain uniformity, let us make a convention that $crit(\pi_{ii}) = \lambda_i$.

From (1), we conclude that if $i \in U_j$, then $\kappa_{T_\wedge(i,j),i} = crit(\pi_{T_\wedge(i,j),i})$ and $\kappa_{T_\wedge(i,j),j} = crit(\pi_{T_\wedge(i,j),j})$. (In case that $T_\wedge(i,j) = i$, we use our convention above. If $T_\wedge(i,j) < i$, then $T_\wedge(i,j) \in U_i$ since $i \cap U_j \subseteq U_i$.)

Therefore, we prove (2) by showing that if $i \in U_j$, then

$$\kappa_{ij} = \min\{crit(\pi_{T_\wedge(i,j),i}), crit(\pi_{T_\wedge(i,j),j})\}.$$

We prove this by induction on j.

Let us define that $\overline{\kappa}_{ii} = \lambda_i$ and for $i < j$, we define that

$$\overline{\kappa}_{ij} = \min\{crit(\pi_{T_\wedge(i,j),i}), crit(\pi_{T_\wedge(i,j),j})\}.$$

Notice that $\overline{\kappa}_{ij}$ is defined for all i, j within the length of the iteration tree, including those $i < j$ but $i \notin U_j$ in particular.

We proceed now by induction to show that if $i \in U_j$, then $\kappa_{ij} = \overline{\kappa}_{ij}$.

Case A $j = h + 1$.

Let $i \in U_j$.

Subcase 1 $i < T(j)$.

Then $\kappa_{ij} = \kappa_{iT(j)}$ by definition. Also $T_\wedge(i,j) = T_\wedge(i, T(j))$ and $\overline{\kappa}_{ij} = \overline{\kappa}_{iT(j)}$. By induction hypothesis, we have $\kappa_{iT(j)} = \overline{\kappa}_{iT(j)}$.

Subcase 2 $i = T(j)$.

Then $T(j) \in U_j$. By definition, $\kappa_{ij} = \kappa_h$ and $\overline{\kappa}_{ij} = \kappa_h$.

Subcase 3 $T(j) < i \leq h$.

By definition, $\kappa_{ij} = \min\{\kappa_h, \kappa_{ih}\}$.

Subcase 3.1 $i = h$.

Then $\kappa_{ij} = \kappa_h$. Since $h = i \in U_j$ and $j = h+1$, κ_h is of type 1. Hence $T(j) <_T h = i$ and $\kappa_h < crit(\pi_{T(j)h})$ by our Lemma. Therefore, $\overline{\kappa}_{ij} = \kappa_h$ since $T_\wedge(i,j) = T(j)$. So we have $\kappa_{ij} = \overline{\kappa}_{ij}$.

Subcase 3.2 $T(j) < i < h$.

Then $i \in U_h$ and κ_h is of type 1. By Lemma and Lemma, $\kappa_h < \kappa_{ih}$ and $T(j) <_T h$ and $crit(\pi_{T(j)h}) > \kappa_h$. Hence, $\kappa_i j = \kappa_h$. By induction hypothesis, $\overline{\kappa}_{ih} = \kappa_{ih}$.

Subcase 3.2.1 $T(j) = T_\wedge(i,j) <_T T_\wedge(i,h)$.

Then $\kappa_h < \overline{\kappa}_{ih} = \kappa_{ih}$ and $\overline{\kappa}_{ij} = \kappa_h = \kappa_{ij}$.

Subcase 3.2.2 $T_\wedge(i,j) <_T T(j)$.

This cannot happen. Since if this were the case, then we would have had that

$$\overline{\kappa}_{ih} = \min\{crit(\pi_{T_\wedge(i,j),i}), crit(\pi_{T_\wedge(i,j),h})\} < \kappa_h$$

and $\kappa_{ih} = \overline{\kappa}_{ih}$, by induction hypothesis, and $\kappa_h < \kappa_{ih}$ since κ_h is of type 1 and $i \in U_h$ and $T(j) < i$.

Subcase 3.2.3 $T(j) = T_\wedge(i,j) = T_\wedge(i,h)$.

By definition, $\overline{\kappa}_{ij} = \min\{crit(\pi_{T(j)i}), crit(\pi_{T(j)j})\}$. Since κ_h is of type 1, $\kappa_h < crit(\pi_{T(j)h})$.

If $crit(\pi_{T(j)i}) < \kappa_h$, then

$$crit(\pi_{T(j)i}) = \overline{\kappa}_{ih} = \kappa_{ih} < \kappa_h.$$

Hence, $\overline{\kappa}_{ij} = \kappa_{ih} = \kappa_{ij}$.

If $crit(\pi_{T(j)i}) \geq \kappa_h$, then $\overline{\kappa}_{ij} = \kappa_h$ and $\overline{\kappa}_{ih} \geq \kappa_h$ (since $\kappa_h < crit(\pi_{T(j)h})$). Since $\kappa_{ih} = \overline{\kappa}_{ih}$ by induction hypothesis, $\kappa_{ij} = \kappa_h$. Therefore, $\kappa_{ij} = \overline{\kappa}_{ij}$.

This finishes the Case A.

Case B j is a limit ordinal.

Let $i \in U_j$. Let $h <_T j$ be large enough such that $i < h$. Then $i \in U_h$. Hence,

$$\overline{\kappa}_{ij} = \overline{\kappa}_{ih} = \kappa_{ih} = \kappa_{ij}.$$

This finishes the proof of (2).

(3) follows from the proof of (2) that Subcase 3.2.2 cannot happen.

□

LEMMA 2.10 Assume that $i < j$ and both E_{ν_i} and E_{ν_j} are of type 1 and $\kappa_i < \kappa_j < \lambda_i$. Then E_{ν_j} is a top extender and there is some h such that $i < h < j$, $h+1 \leq_T j$, and $\kappa_j < \kappa_h < \lambda_i < \lambda_h < \lambda_j$ and $\kappa_j = crit(E_{top}^{M_h^*})$.

Proof By the first part of the proof of the previous lemma, E_{ν_j} is a top extender.

Let $\xi = T(h+1) <_T h+1 \leq_T j$ be the least such that either

(a) $\xi > i$ and $M_\xi \neq M_h^*$, or

(b) $\xi \leq i < h+1$.

Let $\overline{\kappa} = crit(\pi_{\xi,j})$.

(1) $\overline{\kappa} > \kappa_j$.

Otherwise, let $\kappa_0 = crit(E_{top}^{M_h^*})$. Then $\pi_{\xi,j}(\kappa_0) = \kappa_j$. Hence $\kappa_0 \geq \overline{\kappa}$ by our assumption. Then

$$\kappa_j \geq \pi_{\xi,h+1}(\kappa_0) \geq \pi_{\xi,h+1}(\overline{\kappa}) = \lambda_h.$$

But $\kappa_j < \lambda_i \leq \lambda_h$.

(2) Hence $\kappa_j = crit(E_{top}^{M_h^*}) < \kappa_h$.

(3) The first alternative doesn't hold. Otherwise, $E_{top}^{M_h^*} \in M_\xi$. If $\alpha < \lambda_i$ is a limit cardinal in $J_{\lambda_\xi}^{E^{M_i}}$, then $E_{top}^{M_h^*}||\alpha \in M_i$ and is indexed below λ_i and is of type 1. This shall give that κ_i is of type 2.

Therefore, $i \neq h$ since $\kappa_h \neq \kappa_i$. Hence $i < h$.

This finishes the proof of the lemma.

LEMMA 2.11 Let $\mathcal{T}$ be a normal iteration of type 1 premice. Then

(1) There is no infinite sequence $\langle i_m \mid m < \omega\rangle$ such that $i_m < i_{m+1}$ and $\kappa_{i_m} < \kappa_{i_{m+1}} < \lambda_{i_0}$ and all of these κ_{i_m} are of type 1.

(2) For each i, the set $\{\kappa_j \mid i < j \ \wedge\ \kappa_j < \lambda_i\}$ is finite.

Proof (1) First we notice that there are only finitely many $h > i$ such that $\kappa_h < \lambda_i$ and κ_h is of type 0.

Suppose that we had such an infinite sequence

$$i_0 < i_1 < \cdots < i_l < i_{l+1} < \cdots$$

such that

$$\kappa_{i_0} < \kappa_{i_1} < \cdots < \kappa_{i_l} < \kappa_{i_{l+1}} < \cdots < \lambda_{i_0}$$

and all κ_{i_l} are of type 1. Hence for all $0 < l < \omega$, $E^{M_{i_l}}_{\nu_{i+l}}$ is the top extender of M_{i_l}.

Now for each $l > 0$, we repeatedly apply the previous lemma finitely many times to get a unique sequence

$$i_l > h(l,0) > h(l,1) > \cdots > h(l,k_l - 1) > h(l,k_l) > i_0$$

such that

1) $\kappa_{i_l} < \kappa_{h(l,0)} < \kappa_{h(l,1)} < \cdots < \kappa_{h(l,k_l)} < \lambda_{i_0}$,

2) $\kappa_{h(l,m)}$ is of type 1 and $E^{M_{h(l,m)}}_{\nu_{h(l,m)}}$ is the top extender of $M_{h(l,m)}$ for $m < k_l$,

3) $\kappa_{h(l,k_l)}$ is of type 0,

4) $T(h(l,m)+1) \leq_T T(h(l,m+1)+1)$ for $m < k_l$,

5) $h(l,m)+1 \leq_T h(l,m-1)$ for $m \leq k_l$, where $h(l,-1) = i_l$,

6) $\kappa_{h(l,m)} < crit(\pi_{T(h(l,m)+1),h(l,m)})$ for $m < k_l$,

7) if $T(h(l,m)+1) <_T T(h(l,m+1)+1)$, letting $h^*(l,m)+1 \leq_T T(h(l,m+1)+1)$ be such that $T(h(l,m)+1) = T(h^*(l,m)+1)$, then

$$dom(\pi_{T(h(l,m)+1),h(l,m)+1}) \neq dom(\pi_{T(h(l,m)+1),h^*(l,m)+1}),$$

i.e., a truncation must have occurred when $E^{M_{h^*(l,m)}}_{\nu_{h^*(l,m)}}$ is applied to $M_{T(h(l,m)+1)}$.

Since the set

$$\{h \mid h > i_0 \ \wedge\ \kappa_h < \lambda_{i_0} \ \wedge\ \kappa_h \text{ is of type } 0\}$$

is finite, by passing to an infinite subsequence if necessary, we may assume without loss of generality that $h(l,k_l) = h$ for all $1 \leq l < \omega$.

Let $\xi = T(h+1)$.

Claim There is an infinite subset $H \subseteq \omega$ and some $\gamma \leq_T \xi$ such that $T(h(l,0)+1) = \gamma$ for all $l \in H$.

To see this, for $l < m$, define $p(\{l,m\}) = 0$ if $T(h(l,0)+1) = T(h(m,0)+1)$, and $p(\{l,m\}) = 1$ if $T(h(l,0)+1) < T(h(m,0)+1)$, and $p(\{l,m\}) = 2$ if $T(h(l,0)+1) > T(h(m,0)+1)$.

Let $H \subseteq \omega$ be an infinite homogeneous set for this partition.

We show that $p''[H]^2 = \{0\}$. Assume not. Then it must be 1–homogeneous since there is no infinite decreasing sequence of ordinals. Then for $l < m$ in H, we have $T(h(l,0)+1) < T(h(m,0)+1)$ and hence $T(h(l,0)+1) <_T T(h(m,0)+1) \leq_T \xi$. But this means that

$[0,\xi]_T$ has infinitely many places of truncations. This is impossible. Therefore, for $l < m$ in H, we must have $T(h(l,0)+1) = T(h(m,0)+1)$.

So without loss of generality, we may assume that for all $1 \leq l < \omega$, $T(h(l,0)+1) = \gamma$, i.e., $H = \omega - \{0\}$.

Again, since there are only finitely many places in $[0,\xi]_T$ where a truncation may have occurred, we have that the following set

$$\{T(h(l,m)+1) \mid T(h(l,m)+1) <_T \xi \ \wedge \ m < k_l \ \wedge \ l \in H\}$$

is finite.

Let $\xi > \xi_1 > \cdots > \xi_n > \gamma$ be an enumeration of these points.

In M_ξ, let

$$A_0 = \{\kappa \mid E_\nu^{M_\xi} \text{ is of type 1 } \wedge \ \kappa = crit(E_\nu^{M_\xi}) < \kappa_h < lh(E_\nu^{M_\xi})\}.$$

Then A_0 is a finite set since M_ξ has no type 2 extender on its sequence and hence there are no overlapping pairs of the form $\kappa_i < \kappa_j < \lambda_i < \lambda_j$ with both κ_i and κ_j are of type 1 in the extender sequence of M_ξ.

It follows that $\{\kappa_{h(l,k_l-1)} \mid l \in H\} \subseteq A_0$ is finite. By a simple induction, we have that for each $1 \leq k \leq n$,

$$\{\kappa_{h(l,m)} \mid \xi_k = T(h(l,m)+1) \ \wedge \ l \in H \ \wedge \ m < k_l\}$$

is finite. Hence we conclude that $\{\kappa_{h(l,0)} \mid l \in H\}$ is finite since there are only finitely many places to apply extenders to stretch to generate these critical points.

Therefore, there is an infinite subset $H_1 \subseteq H$ such that $\kappa_{h(l,0)} = \kappa$ for all $l \in H_1$.

Now look at the extender sequence of M_γ, for each $l \in H_1$, let E_l be the extender to be stretched by π_{γ,i_l} to produce the top extender $E_{top}^{M_{i_l}}$. Notice that $crit(\pi_{\gamma,i_l}) = \kappa$, which is larger than the critical point of E_l, for $l \in H_1$.

Define a partition of $[H_1]^2$ by $p(\{l,m\}) = 0$ if $dom(\pi_{\gamma,h(l,0)+1}) = dom(\pi_{\gamma,h(m,0)+1})$, and $p(\{l,m\}) = 1$ if $lh(E_l) < lh(E_m)$ and $p(\{l,m\}) = 2$ otherwise.

Let H_2 be an infinite homogeneous set. If it has value 0, then the two top extenders must have the same critical point, which is a contradiction. If it has value 1, then there is an overlapping pair of two type 1 extenders on the sequences of M_γ, which is impossible. If it has value 2, then we have an infinite decreasing sequence of ordinals, again which is impossible.

Therefore, there is no infinite sequence as stated at the beginning of the proof. Hence (1) is proved.

(2) follows from (1). Assume that $\{\kappa_j \mid j > i \ \wedge \ \kappa_j < \lambda_i\}$ is infinite. Let X_i be this infinite set. For each $\kappa \in X_i$, let $j(\kappa)$ be the least $j > i$ such that $\kappa = \kappa_j$. Since there are only finitely many type 0 $\kappa_h < \lambda_i$ for $h > i$, by removing these finitely many objects, we may assume without loss of generality that every $\kappa \in X_i$ is of type 1. We may also assume that X_i has order type ω, by taking the first ω many elements if necessary.

Now define $p(\{\kappa,\tau\}) = 0$ if $j(\kappa) < j(\tau)$ and define $p(\{\kappa,\tau\}) = 1$ if $j(\kappa) > j(\tau)$ for $\kappa < \tau$ in X_i.

Let $H \subseteq X_i$ be an infinite homogeneous set. Since there are no infinite decreasing sequences of ordinals, $p''[H]^2 = \{0\}$. Hence for $\kappa < \tau$ in H, we have $j(\kappa) < j(\tau)$. Let $i_m = j(\kappa)$ if κ is the m–th element of H. This gives us a sequence which should not exist according to (1). So we have a contradiction.

□

We now prove the uniqueness and the existence of a cofinal branch in a normal iteration tree of iterating a type 1 premouse.

THEOREM 2.4 Let $\mathcal{T} = \langle < M_i >, < \nu_i >, < \eta_i >, < \pi_{ij} >, T \rangle$ be a normal iteration of a type 1 premouse of limit length θ. Then

(a) $\mathcal{T}$ has at most one cofinal branch. In fact, set

$$b = b_{\mathcal{T}} = \{ i \mid \forall k < \theta \, \exists \, j > k \, (i <_T j) \}.$$

Then b is a chain under the tree ordering and if $\mathcal{T}$ has a cofinal branch, then b is the unique cofinal branch of the tree.

(b) $\mathcal{T}$ has a cofinal branch.

Proof Let $b = b_{\mathcal{T}}$ be as given in the theorem. Let us prove several general facts about b first.

LEMMA 2.12 For $l \in b$, if $k \geq l$ and $T(k+1) < l$, then κ_k is of type 1.

To see this, let $l \in b$ and let $k \geq l$ be such that $T(k+1) < l$. Let $\eta \geq k$ be the least such that $l <_T \eta + 1$ (since $l \in b$). Then $\eta > k$ and $l \leq_T T(\eta + 1)$.

We claim that $T(\eta + 1) \leq k$ by minimality of η. Let us assume that $T(\eta + 1) > k$. If $T(\eta + 1) = m + 1$, then $k \leq m < \eta$ and $l <_T m + 1$. If $T(\eta + 1)$ is a limit ordinal, let m be such that $l <_T m + 1 <_T T(\eta + 1)$ and $m + 1 > k$, then $m < \eta$. This contradicts to the minimality of η.

Hence $\kappa_k < \lambda_{T(k+1)} \leq \kappa_\eta < \lambda_{T(\eta+1)} \leq \lambda_k < \lambda_\eta$.

This gives the first general fact.

LEMMA 2.13 Assume that $i = h + 1 \in b$. Let $i \leq k$ and $l \in i \cap U_k$. Then $\kappa_{lk} \leq \kappa_h$.

Assume otherwise. Let μ be the least counterexample.

Case 1. $\mu = i$.

For $l \in T(i) \cap U_\mu$, $\kappa_{li} \leq \lambda_l \leq \kappa_h$. For $l \in U_\mu - T(i)$, $\kappa_{li} = \min(\kappa_h, \kappa_{lh}) \leq \kappa_h$.

Case 2. $i < \mu$. By minimality of μ, $\mu = \gamma + 1$.

Assume that $T(\mu) \geq i$ and $l \in i \cap U_\mu$. Then $\kappa_{l\mu} = \kappa_{l,T(\mu)}$. Since $l < T(\mu)$, $l \in U_{T(\mu)}$. By minimality of μ, we have $\kappa_{l,T(\mu)} \leq \kappa_h$. Hence $\kappa_{l\mu} \leq \kappa_h$.

Assume that $T(\mu) < i$. By the above general fact, κ_γ is of type 1 and $T(\mu) \in U_\gamma$. Then by the previous lemma, $\kappa_\gamma < \kappa_{T(\mu),\gamma}$. By minimality of μ, we also have $\kappa_{T(\mu),\gamma} \leq \kappa_h$.

Hence $T(\mu) \leq T(i)$.

Now for $l \in T(i) \cap U_\mu$, $\kappa_{l\mu} \leq \lambda_l \leq \kappa_h$. For $l \in i \cap U_\mu - T(i)$, we have $l \geq T(\mu)$, and hence either $\kappa_{l\mu} = \kappa_\gamma$ or $\kappa_{l\mu} = \min(\kappa_\gamma, \kappa_{l\gamma})$. In any case, $\kappa_{l\mu} \leq \kappa_\gamma < \kappa_h$.

This shows the second lemma.

We now proceed to prove (a), the uniqueness.

Let d be a cofinal branch of $\mathcal{T}$. Then $d \subseteq b$.

We prove that $b_{\mathcal{T}}$ is a chain.

Suppose not. Let $\mathcal{T}$ be a counterexample of minimal length. Let $b = b_{\mathcal{T}}$. Let $i \in b$ be the least such that there is some $j \in i \cap b$ such that $j \not<_T i$.

(1) $i = h + 1$ for some h. This follows from the minimality of i.

(2) $i \cap b$ is a chain by minimality of i again.

Let $C = \{k \in b \mid k < i \wedge; k \not<_T i\}$. Then C is not empty. Let $j = \sup(C)$.

(3) $j < i$ and $i \cap b \subseteq j + 1$.

If $j = i$, then j is a successor ordinal and hence $j = \max C \in C$. This is a contradiction.

If $i \cap b - (j+1)$ is not empty, let k be the minimal member of this set. Then $j < k < i$ and $k \in b$. But now $k <_T i$ and by minimality of i, each $l \in C$ must satisfy that $l <_T k$.

Let $\xi = T(i)$. Then $\xi <_T \min(C)$.

(4) κ_h is of type 1 and if $i \leq k$ and $T(k+1) < i$, then κ_k is of type 1.

We now finish the proof by deriving a final contradiction.

Case1. $j \in C$.

Hence $j = \max C = \max(i \cap b)$.

Let η be the least such that $\eta + 1 > i$ and $j \leq_T \eta + 1$ (since $j \in b$). Let $\gamma = T(\eta + 1)$. Then $\gamma \in U_\eta$ and $\gamma < i \leq \eta$. Hence κ_η is of type 1 and $\kappa_\eta < \kappa_{\gamma\eta}$. By the second lemma above, $\kappa_{\gamma\eta} \leq \kappa_h$. Hence, $\kappa_\eta < \lambda_{T(i)}$. Therefore, $\gamma \leq T(i) < j$. But $j <_T \eta + 1$ and $\gamma \geq j$. This is a contradiction.

Case2. $j \notin C$.

In this case, j must be a limit ordinal and C is cofinal in j and $C \subseteq i \cap b = j \cap b$.

Let $d = \{k < j \mid k <_T j\}$. Then both d and $j \cap b$ are cofinal in j.

For $k < j$, let

$$k \in b_{\mathcal{T}\restriction j} \iff \forall m < j\ \exists\, l > m(l < j \ \wedge \ k <_T l).$$

Then $d \cup (j \cap b) \subseteq b_{\mathcal{T}\restriction j}$. Hence $b_{\mathcal{T}\restriction j}$ is cofinal in j. By minimality of θ, $b_{\mathcal{T}\restriction j}$ must be a cofinal branch of the tree $T \cap (j \times j)$. Therefore, $d = b_{\mathcal{T}\restriction j}$. Since $b \cap j = b \cap i$ is also a cofinal branch of the tree $T \cap (j \times j)$, $d = j \cap b$.

Since $j \notin C$, $j \notin b$. Since $T(i) \in i \cap b$, $T(i) \in d$. Hence $T(i) <_T j$ and $T(i) <_T \min(C)$ and $C = (T(i), j)_T$.

Let $\mu = \min(C)$. Then $\mu \in b$ and $\mu < i$ and $\mu \not<_T i$. Let $k < \theta$ be the least such that for all $m \in (k, \theta)$, $j \not<_T m$. Let $m > k$ be such that $m > i$ and $\mu <_T m$.

Let $\overline{\eta}$ be the least such that $\overline{\eta} \in (\mu, m]_T$. Then $\overline{\eta} = \eta + 1$ for a unique η by minimality. Hence $\eta + 1 > i$ and $\mu <_T \eta + 1$ and $T(\eta + 1) \in i \cap U_\eta$. By (4), κ_η is of type 1. Let $\gamma = T(\eta + 1)$. Then by previous lemma, $\kappa_\eta < \kappa_{\gamma\eta}$. By the second lemma above, $\kappa_{\gamma\eta} \leq \kappa_h$. So, $\kappa_\eta < \lambda_{T(i)}$. Therefore, $\gamma \leq T(i) <_T \mu <_T \eta + 1$. But $\gamma = T(\eta + 1)$ and hence $\mu \leq \gamma$. This is a contradiction.

This finishes the proof of the uniqueness.

We now prove that $b_{\mathcal{T}}$ is cofinal.

For $i < \theta$, let $S(i) = \{j < \theta \mid i \leq_T j\}$. Hence $i \in b$ if and only if $S(i)$ is unbounded in θ.

We decompose $S(i)$ into disjoint union of maximal intervals of ordinals as follows:
Let $\gamma_i(0) = i$ and $\xi_i(0) = \sup\{\xi < \theta \mid [i, \xi] \subseteq S(i)\}$.
Assume that $(\gamma_i\restriction_\beta, \xi_i\restriction_\beta)$ has been defined.
Let $\eta = \sup\{\xi_i(j) \mid j < \beta\}$.
If $S(i) \subseteq \eta$, then we stop.
Otherwise, let $\gamma_i(\beta) = \min(S(i) - \eta)$ and let $\xi_i(\beta) = \sup\{\xi < \theta \mid [\gamma_i(\beta), \xi] \subseteq S(i)\}$.

LEMMA 2.14 Let $j \in dom(\gamma_i)$.
(a) If $\xi_i(j) < \theta$, then $\xi_i(j) \in S(i)$ and $T(\xi_i(j) + 1) \notin S(i)$.
(b) If $j > 0$ and $\gamma_i(j)$ is a successor ordinal, letting $\gamma_i(j) = \overline{\gamma}_i(j) + 1$, then $\kappa_{\overline{\gamma}_i(j)}$ is of type 0.
(c) If $j = h + 1$, then $\gamma_i(j)$ is a successor ordinal and $T(\gamma_i(j)) \leq \xi_i(h) < \gamma_i(j)$.
(d) If $j = h + 1$ and $T(\xi_i(h) + 1) < \gamma_i(0)$, then $\kappa_{\xi_i(h)}$ is of type 1. In particular, if $j = 1$, then $\kappa_{\xi_i(0)}$ is of type 1.

COROLLARY 2.2 If $S(i)$ is bounded in θ, then

$$S(i) = \bigcup\{[\gamma_i(k), \xi_i(k)] \mid k \in dom(\gamma_i)\}$$

and if $dom(\gamma_i) > 1$ then $\kappa_{\xi_i(0)}$ is of type 1 and for all $k + 1 \in dom(\gamma_i)$, $\gamma_i(k + 1)$ is a successor ordinal and $\kappa_{\overline{\gamma}_i(k+1)}$ is of type 0.

LEMMA 2.15 If $S(i)$ is bounded then $S(i)$ has a maximum.

Proof Let $\gamma < \theta$ be the least upper bound of $S(i)$. We claim that γ must be a successor ordinal.

Suppose not. γ is a limit ordinal. Let $\beta = dom(\gamma_i)$. Then $\langle \xi_i(l) \mid l < \beta \rangle$ is an increasing cofinal sequence of γ from $S(i) \subseteq \gamma$.

Let $\xi = T_\wedge(i, \gamma)$. Then $\xi <_T i$ and $\xi <_T \gamma$.

Let $\eta_1 + 1 \leq_T \gamma$ be such that $\xi = T(\eta_1 + 1)$ and let $\eta_2 + 1 \leq_T i$ be such that $\xi = T(\eta_2 + 1)$. Then $\eta_1 \neq \eta_2$.

Since $\gamma < \theta$, $[\eta_1, \gamma)_T$ is cofinal in γ. Consider the tree $\mathcal{T}\restriction_\gamma$. Since $S(i) \subseteq \gamma$, $S(i)$ is the same as computed in $\mathcal{T}\restriction_\gamma$. But then $S(i) \subseteq \gamma \cap S(\eta_2 + 1)$ and hence $b_{\mathcal{T}\restriction_\gamma}$ contains two incompatible elements $\eta_1 + 1$ and $\eta_2 + 1$. This contradicts the uniqueness.

Therefore, $S(i)$ has a maximum.

LEMMA 2.16 Assume that $i \in b$ and $\eta + 1 \not<_T i$ and $T(\eta + 1) <_T i$. Assume that $S(\eta + 1) - i \neq \emptyset$. Then $\eta \geq i$ and κ_η is of type 1.

Proof Let m be the least $k \in S(\eta + 1) - i$. Then $m > i$ and m is a successor ordinal. Let $m = \overline{m} + 1$. Then $T(m) < i$. Since $i \in b$, by our first general fact about b, $\kappa_{\overline{m}}$ is of type 1. Therefore, m cannot start any maximal interval of $S(\eta + 1)$ but the first one. Hence $m = \eta + 1$ and $\overline{m} = \eta$.

COROLLARY 2.3 Assume that $i \in b$ and $m > i$. Then there is some $\eta \geq i$ such that $\eta + 1 \leq_T m$ and $T(\eta + 1) \leq_T i$.

To see this, let $\xi = T_\wedge(i, m)$. If $\xi = i$, then we are done. If $\xi <_T i$, then let $\eta+1 \leq_T m$ be such that $T(\eta+1) = \xi$. Then $\eta \geq i$.

LEMMA 2.17 Assume that $i < \theta$ is a limit ordinal and $i \cap b$ is cofinal in i. Assume that $i \notin S(\eta+1)$ and $T(\eta+1) <_T i$. Assume that $S(\eta+1) - i \neq \emptyset$. Then $\eta \geq i$ and κ_η is of type 1.

Proof Let m be the least $k \in S(\eta+1) - i$. Then $m > i$ and m is a successor ordinal and $T(m) < i$. Let $j \in b \cap i$ be such that $T(m) < j$. Let $m = \overline{m} + 1$. Then $\overline{m} = \eta$ and κ_η is of type 1.

COROLLARY 2.4 Assume that $i < \theta$ is a limit ordinal and that $i \cap b$ is cofinal in i. If $m > i$ then there is some $\eta \geq i$ such that $\eta + 1 \leq_T m$ and $T(\eta+1) \leq_T i$.

We now proceed to prove the existence.

Since $S(0) = \theta$, $0 \in b$. In order to show that b is cofinal, we simply need to show that b has no maximum element and that b is closed.

We first show that b is closed.

Let $i < \theta$ be a limit ordinal and $b \cap i$ is cofinal in i. We show that $i \in b$.

Let $C = \{\kappa_\eta \mid \eta \geq i \ \wedge \ T(\eta+1) <_T i \ \wedge \ \kappa_\eta \text{ is of type } 1\}$. Then C is a finite set.

Assume that $S(i)$ is bounded.

First we conclude that $C \neq \emptyset$.

Let η_0 be the least upper bound of $S(i)$. Let $\eta \geq i$ be such that $\eta + 1 \leq_T \eta_0$ and $T(\eta+1) \leq_T i$. Then $T(\eta+1) <_T i$. Let $j \in b \cap i$ be such that $T(\eta+1) < j$. It follows that κ_η is of type 1. Hence $C \neq \emptyset$.

Let $\kappa = \max C$. Let $\eta_0 \geq i$ be such that $\kappa_{\eta_0} = \kappa$ and $T(\eta_0+1) <_T i$. Let $j <_T i$ be such that $T(\eta_0+1) <_T j$. Notice that $j \in b$.

Let η be such that $j <_T \eta$ and $S(i) \subseteq \eta$.

Let $\xi \in (j, \eta]_T$ be least such that $i < \xi$. Then ξ is a successor ordinal and $T(\xi) \leq i < \xi$ and $T(\xi) \notin S(i)$. Also $j \leq T(\xi) < i$.

Let $m <_T i$ be such that $T(\xi) < m$. Then $j \leq_T T(\xi) <_T m \in b$. Let $\xi = \overline{\xi} + 1$. Then $\kappa_{\overline{\xi}}$ is of type 1. Hence $\kappa_{\overline{\xi}} \in C$ and $\kappa_{\overline{\xi}} \leq \kappa$. This is a contradiction since then $T(\xi) \leq_T T(\eta_0+1) <_T j$.

Therefore, $S(i)$ is unbounded in θ. Hence $i \in b$ and b is closed.

Claim b has no maximum.

Let $i \in b$. We show that there is $\eta \geq i$ such that $i = T(\eta+1)$ and $\eta+1 \in b$.

Let $t^+(i) = \{j+1 \mid i = T(j+1)\}$.

Let $t_0^+(i) = \{j+1 \mid i = T(j+1) \ \wedge \ \kappa_j \text{ is of type } 0\}$.

Let $t_1^+(i) = \{j+1 \mid i = T(j+1) \ \wedge \ \kappa_j \text{ is of type } 1\}$.

Case1 There is some $j+1 \in t_1^+ - (i+2)$ such that $S(j+1)$ is unbounded.

We are done in this case.

Case2 Otherwise.

If $t_0^+(i) \neq \emptyset$, then let $j_0 + 1 = \max(t_0^+(i))$. Otherwise, let $j_0 = i$.

Let $A_0 = \{j_0 + 1\}$. Let $\kappa_0 = \kappa_{j_0}$.

Let $A = t_1^+(i) - (j_0 + 2)$.

First let us notice that if $j+1 \in A$, then $\kappa_j < \kappa_0$. This follows from the following fact.

Assume that $t_0^+(i) = \emptyset$. Then $i+1 \in t_1^+(i)$ and for all $i <_T j$, $crit(\pi_{ij}) \leq \kappa_i$.

To see that $i = T(i+1)$, let $j+1 = \min(t^+(i))$. Then κ_j is of type 1. So $i \leq_T j$ by our structure theorem since $i = T(j+1)$. By minimality, $j = i$.

We prove the second part by induction on j. When $j = i+1$ or j is a limit ordinal, the conclusion is clear.

Assume that $i <_T j = h+1$ and $i < h$. If κ_h is of type 0, then $i <_T T(j) <_T j$ and $crit(\pi_{ij}) = crit(\pi_{iT(j)}) \leq \kappa_i$. If κ_h is of type 1, then either $i <_T T(j)$ or $i = T(j)$. In case that $i <_T T(j)$, induction hypothesis applies. In case that $i = T(j)$, then $i <_T h$ and $\kappa_h < crit(\pi_{ih}) \leq \kappa_i$.

Case2.1 $A = \emptyset$.

If $t_0^+(i) = \emptyset$, then $j_0 = i$ and $t^+(i) = \{i+1\}$. Hence $S(i+1)$ is unbounded in θ since $S(i)$ is unbounded.

If $t_0^+(i) \neq \emptyset$, then for all $j+1 \in t^+(i) \cap (j_0+1)$, we have $S(j+1) \subseteq j_0+1$ and hence $S(j_0+1)$ is unbounded in θ.

Case2.2 $A \neq \emptyset$.

Let

$$d(j_0+1) = \bigcup\{S(\eta+1) \mid \eta \geq j_0+1 \ \wedge\ T(\eta+1) <_T j_0+1 \ \wedge\ \kappa_\eta \text{ is of type } 1\}.$$

Case2.2.1 $d(j_0+1)$ is bounded.

If κ_0 is of type 1, then $j_0 = i$ and $S(i) \subseteq S(j_0+1) \cup d(j_0+1)$. Hence $S(j_0+1)$ is unbounded.

If κ_0 is of type 0, then $j_0+1 = \max(t_0^+(i))$ and $S(i) \subseteq S(j_0+1) \cup d(j_0+1) \cup j_0+1$. Hence $S(j_0+1)$ is unbounded.

Case2.2.2 $d(j_0+1)$ is unbounded.

We have now that for all $j+1 \in A$, $S(j+1)$ is bounded and $\kappa_j < \kappa_0$, and there is some $\eta+1 \in t^+(i)$ such that $j_0+1 \leq \eta+1 \leq j$ and $j \in S(\eta+1)$. Also for all $j+1 \in t^+(i) \cap (j_0+1)$, $S(j+1) \subseteq j_0+1$. And $d(j_0+1)$ is unbounded.

Let

$$\kappa_1 > \kappa_2 > \cdots > \kappa_m > \kappa_{m+1} > \cdots > \kappa_{m+l}$$

be an enumeration in strict decreasing order of the set

$$\{\kappa \mid \exists\, \eta > j_0\ (T(\eta+1) \leq_T i \ \wedge\ \kappa = \kappa_\eta)\}$$

and κ_m is the least such that there is some η with $i = T(\eta+1)$ and $\kappa_m = \kappa_\eta$.

Let $A_0 = \{j_0+1\}$.

For $1 \leq n \leq m+l$, let $A_n = \{\eta+1 \mid \eta > j_0 \ \wedge\ T(\eta+1) \leq_T i \ \wedge\ \kappa_\eta = \kappa_n\}$.

Notice that $A = \cup\{A_n \mid 1 \leq n \leq m\}$. Also notice that for all $1 \leq n \leq m+l$, for all $\eta+1 \in A_n$, $S(\eta+1)$ is bounded.

For $\eta+1 \in A_n$, we call $S(\eta+1)$ a subtree of level n.

Lemma 2.18 Assume that $\eta+1 \in A_n$ and $[\gamma, \xi] \subseteq S(\eta+1)$ is a maximal closed interval.

(1) If $n < m+l$ and κ_ξ is of type 1, then for some $\overline{n} > n$, $\xi+1 \in A_{\overline{n}}$.

(2) If $n < m+l$ and κ_ξ is of type 1, then there are two sequences $n_0 = n < n_1 < n_2 < \cdots < n_k$ and $\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_k$ such that $\xi_h + 1 \in A_{n_{h+1}}$ for $h < k$ and κ_{ξ_k} is of type 0 and $S(\xi_{k-1}+1) = [\xi_{k-1}+1, \xi_k]$.

(3) If $0 < n < m+l$ and $\xi = \max S(\eta+1)$ and κ_ξ is of type 1, and there is some $\eta_1 + 1 \in A_n$ such that $\xi < \eta_1$, then there are $k > \xi$, $\overline{n} < n$ and $j+1 \in A_{\overline{n}}$ such that $T(k+1) < \xi$, $k+1 \in S(j+1)$ and κ_k is of type 0.

(4) If $0 < n \leq m+l$ and κ_ξ is of type 0, then there are some $\overline{n} < n$ and $\overline{\xi}+1 \in A_{\overline{n}}$ such that $\xi+1 \in S(\overline{\xi}+1)$.

Proof (1) Since κ_ξ is of type 1, we have $T(\xi+1) \leq_T \xi$ and hence $T(\xi+1) <_T \eta+1$ and $\kappa_\xi < \kappa_\eta$.

(2) Start with $\xi_0 = \xi$ and $n_0 = n$. Let ξ_h and n_h be defined. Assume that κ_{ξ_h} is of type 1 and that ξ_h ends a maximal closed interval of a subtree of level n_h. By (1), let $n_{h+1} > n_h$ be such that $\xi_h + 1 \in A_{n_{h+1}}$. Let ξ_{h+1} be such that $[\xi_h+1, \xi_{h+1}]$ is the first maximal closed interval of $S(\xi_h+1)$. If $\kappa_{\xi_{h+1}}$ is of type 0, we stop. Otherwise, we continue. Since the κ_{ξ_h}'s form a decreasing sequence of ordinals, we must reach a stage at which we stop.

To see (3), let $\eta_1+1 \in A_n$ be such that $\eta_1 > \xi$. Then there are $\overline{n} < n$ and $j+1 \in A_{\overline{n}}$ such that $\eta_1 \in S(j+1)$ since κ_{η_1} is of type 1.

We claim that there are $k > \xi$, $\overline{n} < n$, and $j+1 \in A_{\overline{n}}$ such that $T(k+1) < \xi$, $k+1 \in S(j+1)$ and κ_k is of type 0.

Let $\overline{n}$ be the least such that there are $j+1 \in A_{\overline{n}}$ and $k \in S(j+1)$ such that $k > \xi$.

Let $j+1 \in A_{\overline{n}}$ be a minimal witness and let $k \in S(j+1)$ be a minimal witness.

Then k is a successor ordinal. Let $k = \overline{k}+1$. We have $\overline{k} > \xi$. Since $T(k) \in S(j+1)$, $T(k) < \xi$. Also $\overline{k} \notin S(j+1)$.

If $k > j+1$, then k starts a maximal closed interval of $S(j+1)$ other than the first one. Hence $\kappa_{\overline{k}}$ is of type 0. We are done in this case.

If $k = j+1$, then $\overline{k} = j$. If κ_j is of type 1, then there are $n^* < \overline{n}$ and $j^*+1 \in A_{n^*}$ such that $j \in S(j^*+1)$ and $j > \xi$. This contradicts to the minimality of $\overline{n}$. Therefore, κ_j must be of type 0.

This shows (3).

(4) follows from Minimal Walk to $T(\xi+1)$ Lemma. Since $\xi+1 \notin S(\eta+1)$, $T_\wedge(\xi, \xi+1) <_T \eta+1$. Hence $T_\wedge(\xi,\xi+1) \leq_T T(\eta+1)$. Since $i \in b$, we have actually $T(\eta+1) = T_\wedge(\xi,\xi+1)$. Then apply the Minimal Walk to $T(\xi+1)$ Lemma to conclude (3) since $crit(\pi_{T(\eta+1)T(\xi+1)}) > \kappa_\eta$.

For each $1 \leq n \leq m+l$ and for each $\eta+1 \in A_n$, let $\overline{\eta}$ be the maximum of $S(\eta+1)$.

For each $1 \leq n \leq m+l$, let

$$A_n^0 = \{\eta+1 \in A_n \mid \kappa_{\overline{\eta}} \text{ is of type } 0\}$$

and let $A_n^1 = A_n - A_n^0$.

By (2) of the lemma above, since $d(j_0+1)$ is unbounded, we have that

$$\{\overline{\eta} \mid \eta+1 \in \bigcup_{1 \leq n \leq m+l} A_n^0\}$$

is unbounded.

Let $\overline{n}$ be least such that

$$\{\overline{\eta} \mid \eta+1 \in A^0_n\}$$

is unbounded.

Case2.2.2.1 $\{\overline{\eta} \mid \eta+1 \in \bigcup_{1\leq n<\overline{n}} A^1_n\}$ is bounded.

Let γ be an upper bound of this set. Then

$$\{\overline{\eta} \mid \eta+1 \in A^0_{\overline{n}} \ \wedge \ \overline{\eta} > \gamma\}$$

is unbounded. For any such $\overline{\eta}$, we have $j_0+1 \leq_T \overline{\eta}+1$. Hence $S(j_0+1)$ is unbounded.

Case2.2.2.1 $\{\overline{\eta} \mid \eta+1 \in \bigcup_{1\leq n<\overline{n}} A^1_n\}$ is unbounded.

Then $\overline{n} > 1$. Let $n^* \geq 1$ be the least n such that

$$\{\overline{\eta} \mid \eta+1 \in A^1_n\}$$

is unbounded. Then $n^* < \overline{n}$.

Let $X = \{\overline{\eta} \mid \eta+1 \in A^1_{\overline{n}}\}$.

Let γ be an upper bound of

$$\{\overline{\eta} \mid \eta+1 \in \bigcup_{1\leq n<n^*} A_n\}.$$

Then for each $\overline{\eta} \in X-\gamma$, let $k > \overline{\eta}$ be such that $T(k+1) < \overline{\eta}$ and there are $n^{**} < n^*$ and $j+1 \in A_{n^{**}}$ such that $k+1 \in S(j+1)$ and κ_k is of type 0, given by (3) of the lemma above. Then n^{**} must be 0 since γ is a bound and $k > \overline{\eta} > \gamma$. Therefore, $j_0+1 <_T k+1$.

Since X is unbounded, $S(j_0+1)$ is unbounded.

This proves that b has no maximum.

Actually, the proof shows something slightly more. Let

$$d(i) = \bigcup\{S(\eta+1) \mid \eta \geq i \ \wedge \ T(\eta+1) <_T i \ \wedge \ \kappa_\eta \text{ is of type } 1\}$$

and

$$B(i) = \{\eta \mid i <_T T(\eta+1) \ \wedge \ \kappa_\eta \text{ is of type } 0\}.$$

Then the proof shows that for all $i \in b$, either $d(i)$ is bounded or $B(i)$ is unbounded.

□

§3 Normal Iterability

In this section, we prove our iterability theorem for type 1 premice with supercomplete surviving extenders.

Definition 3.1 Let F be an extender on $M = J^A_\alpha$. Let $\kappa = crit(F)$ and $\tau = (\kappa^+)^M$. Let $\pi : J^A_\tau \to_F J^{A'}_{\tau'}$. Let t_ξ be the ξ–th element of $J^{A'}_{\tau'}$. Let $\alpha(F, M)$ be the largest cardinal of M below $\pi(\kappa) + 1$.

(1) We say that F is *supercomplete on* M if and only if for every countable $X \subseteq \pi(\kappa)$, and every countable $W \subseteq P(\kappa) \cap J^A_\tau$, there is a $\delta : X \to \kappa$ such that

(a) $\prec \delta(\vec{\xi}) \succ \in Z \iff \prec \vec{\xi} \succ \in F(Z)$ for $Z \in W$ and $\xi_1, \cdots, \xi_n \in X$, and

(b) if $Y \subseteq X$ and $\bigcup_{\xi \in Y} t_\xi$ is a well–founded relation, then so is $\bigcup_{\xi \in Y} t_{\delta(\xi)}$.

Any δ as above will be called a *strong connection* with respect to (X, W). If $\delta : X \to \kappa$ satisfies only (a) with respect to (X, W), we say that δ is a *weak connection*.

(2) We say that F is *supercomplete with respect to* M if and only if for every countable $X \subseteq \pi(\kappa)$, and every countable $W \subseteq P(\kappa) \cap J^A_\tau$, there is a $\delta : X \to \kappa$ such that

(a) $\prec \delta(\vec{\xi}) \succ \in Z \iff \prec \vec{\xi} \succ \in F(Z)$ for $Z \in W$ and $\xi_1, \cdots, \xi_n \in X$, and

(b) if $Y \subseteq X \cap \alpha(F, M)$ and $\bigcup_{\xi \in Y} t_\xi$ is a well–founded relation, then so is $\bigcup_{\xi \in Y} t_{\delta(\xi)}$.

Notice that if F is the top extender of M, then F is supercomplete on M if and only if F is supercomplete with respect to M. But in general, the later is weaker. [Notice that if $\nu < ht(M)$ and $E_\nu \neq \emptyset$, then $\lambda = \lambda(E_\nu)$ is a cardinal in M if and only if E_ν is superstrong in M. This cannot occur in type 1 mice.]

Example 3.2 Assume that $0^\#$ exists. Let $\kappa = \aleph_1^V$ and let λ be the next Silver indiscernible of L above κ. Let $j : L \to L$ be the elementary embedding generated by an order preserving map that sends κ to λ. Let $\tau = (\kappa^+)^L$ and let $\nu = (\lambda^+)^L$. Let

$$F : J_\tau \cap P(\kappa) \to J_\nu \cap P(\lambda)$$

be such that $F(x) = j(x)$ for all $x \in dom(F)$. Let $M = (J_\nu, F)$. Then M is a fully iterable premouse and $core(M) = 0^\#$. F is a supercomplete (κ, λ)–extender on M.

The key point here is that if $\alpha < \lambda$, then there is a Skolem term t such that for some finite set indiscernibles $\vec{\gamma}$ from κ, $\alpha = t(\vec{\gamma}, \kappa, \aleph_2^V, \cdots, \aleph_m^V)$ by Remarkability of Silver indiscernibles. In what follows, we shall write $\aleph_m$ simply for $\aleph_m^V$.

Also let us observe that if $x \in P(\kappa) \cap L$ and $x = t(\vec{\gamma}, \kappa, \lambda, \aleph_2, \cdots, \aleph_m)$, then $x = t(\vec{\gamma}, \kappa, \aleph_2, \aleph_3, \cdots, \aleph_{m+1})$.

Assume otherwise. Let x be a counter example. Let $\alpha < \kappa$ be the least in the symmetric difference of $t(\vec{\gamma}, \kappa, \lambda, \aleph_2, \cdots, \aleph_m)$ and $t(\vec{\gamma}, \kappa, \aleph_2, \aleph_3, \cdots, \aleph_{m+1})$. Let $\vec{\eta}$ be a finite set of indiscernibles below κ such that $\alpha = s(\vec{\eta})$ (again, by Remarkability). Then we have

$$s(\vec{\eta}) \in t(\vec{\gamma}, \kappa, \lambda, \aleph_2, \cdots, \aleph_m) \iff s(\vec{\eta}) \in t(\vec{\gamma}, \kappa, \aleph_2, \aleph_3, \cdots, \aleph_{m+1})$$

by indiscernibility. This is a contradiction.

Now let $W \subseteq P(\kappa) \cap L$ be countable and let $A \subseteq \lambda$ be countable. Let $W = \{x_i \mid i < \omega\}$ and let $A = \{\alpha_i \mid i < \omega\}$.

Let $\{u_i, s_i \mid i < \omega\}$ be a countable set of Skolem terms and let $C \subseteq \kappa$ be a countable set of indiscernibles such that $\alpha_i = u_i(\vec{\gamma}_i, \kappa, \aleph_2, \cdots, \aleph_{m_i})$ and $x_i = s_i(\vec{\eta}_i, \kappa, \aleph_2, \cdots, \aleph_{n_i})$ for each $i < \omega$, where each $\vec{\gamma}_i$ and $\vec{\eta}_i$ is from the set C.

Let $\beta < \kappa$ be a Silver indiscernible such that $C \subseteq \beta$ and β is a limit of smaller Silver indiscernibles.

We then define $h : A \to \kappa$ by setting

$$h(u_i(\vec{\gamma}_i, \kappa, \aleph_2, \cdots, \aleph_{m_i})) = u_i(\vec{\gamma}_i, \beta, \aleph_2, \cdots, \aleph_{m_i})$$

for each $i < \omega$. By indiscernibility, we have

$$u_i(\vec{\gamma}_i, \kappa, \aleph_2, \cdots, \aleph_{m_i}) \in s_k(\vec{\eta}_k, \lambda, \aleph_2, \cdots, \aleph_{n_k}) \iff u_i(\vec{\gamma}_i, \beta, \aleph_2, \cdots, \aleph_{m_i}) \in s_k(\vec{\eta}_k, \kappa, \aleph_2, \cdots, \aleph_{n_k})$$

for all $i, k < \omega$.

This shows that F is ω–complete with respect to the pair (W, A).

Let $Y \subseteq \omega$ be such that $\bigcup_{i \in Y} t^L_{u_i(\vec{\gamma}_i, \kappa, \aleph_2, \cdots, \aleph_{m_i})}$ is a well founded relation. We need to show that $\bigcup_{i \in Y} t^L_{u_i(\vec{\gamma}_i, \beta, \aleph_2, \cdots, \aleph_{m_i})}$ is also a well founded relation.

Suppose for the contrary that it is not. Let R denote the above relation. Let $b_k \in J_\kappa$ $(k < \omega)$ be such that $b_{k+1} R b_k$ for all $k < \omega$. For each $n < \omega$, let

$$R_n = \bigcup_{i \in Y \cap n} t^L_{u_i(\vec{\gamma}_i, \beta, \aleph_2, \cdots, \aleph_{m_i})}.$$

Then $R_n \in J_\kappa$.

Let $\mu < \kappa$ be the next indiscernible above β. We claim that

$$u_i(\vec{\gamma}_i, \beta, \aleph_2, \cdots, \aleph_{m_i}) < \mu$$

for all $i < \omega$. Otherwise, let $n < \omega$ be the least counter example. Then

$$u_n(\vec{\gamma}_n, \beta, \aleph_2, \cdots, \aleph_{m_n}) \geq \mu.$$

By indiscernibility, we have that

$$u_n(\vec{\gamma}_n, \kappa, \aleph_2, \cdots, \aleph_{m_n}) \geq \lambda.$$

This is a contradiction.

Since μ is a strong inaccessible cardinal in L, each $b_k \in J_\mu$ for $k < \omega$. Therefore, each b_k is defined by a skolem term c_k and a finite set $\vec{\xi}_k$ of indiscernibles from β, β, μ and some larger $\aleph$'s. Namely,

$$b_k = c_k(\vec{\xi}_k, \beta, \mu, \aleph_2, \cdots, \aleph_{l_k})$$

for $k < \omega$.

Let $b^*_k = c_k(\vec{\xi}_k, \kappa, \lambda, \aleph_2, \cdots, \aleph_{l_k})$ for each $k < \omega$.

Let

$$R_n^* = \bigcup_{i \in Y \cap n} t^L_{u_i(\vec{\gamma}_i, \kappa, \aleph_2, \cdots, \aleph_{m_i})}$$

for each $n < \omega$.

Then we have

(*) $\forall\, k < \omega\; \exists\, n < \omega\; \forall\, l < \omega (l \geq n \rightarrow b_{k+1} R_l b_k)$.

We want to show that

(**) $\forall\, k < \omega\; \exists\, n < \omega\; \forall\, l < \omega (l \geq n \rightarrow b^*_{k+1} R^*_l b^*_k)$.

This shall give us a contradiction to that $R^* = \bigcup_{l<\omega} R^*_l$ is well founded.

Fix $k < \omega$. Let $n < \omega$ be given by (*). Let $l \geq n$ be in ω. Since $b_{k+1} R_l b_k$ holds, by indiscernibility, we must have $b^*_{k+1} R^*_l b^*_k$ holds.

Therefore, (**) holds.

This shows that h is a witness to that F is supercomplete with respect to the pair (W, A).

□

Example 3.3 Assume that there is a measurable cardinal in V and let κ be the first measurable cardinal. Let U be a normal κ–complete ultrafilter on κ. Let

$$j : V \rightarrow ult(V, U) = M$$

be the canonical elementary embedding given by the ultrapower of V by U. Let $\tau = (\kappa^+)^L$ and let $\mu = (j(\kappa)^+)^L$. Let $F^* = U \cap L$. Then F^* is an L–ultrafilter. Let

$$i : J_\tau \rightarrow N = ult(J_\tau, F^*).$$

Since $L^M = L$ and $j\restriction_{J_\tau} : J_\tau \rightarrow J_\mu$, setting $k([f]) = j(f)(\kappa)$ for all $f \in J_\tau$ and $f : \kappa \rightarrow J_\tau$, we have that

$$k : N \rightarrow J_\mu$$

and $j(x) = k(i(x))$ for all $x \in J_\tau$. Hence N must be an initial segment of J_μ by condensation of L. Let ν be such that $N = J_\nu$. Let $\lambda = i(\kappa) = crit(k)$. Let $F(X) = i(X) = j(X) \cap \lambda$ for all $X \in P(\kappa) \cap L$. Then F is a (κ, λ)–extender on J_ν.

Claim (J_ν, F) is a premouse and F is a supercomplete extender on J_ν.

We just need to check the supercompleteness.

Let $W \subseteq P(\kappa) \cap J_\tau$ be countable and let $B \subseteq \lambda$ be countable. Let $h : \omega \rightarrow B$ be an enumeration of B. Let $\langle X_n \mid n < \omega \rangle = W$. We may assume that B is closed under Gödel pairing.

First we show that there is an Gödel homomorphism $\delta : B \rightarrow \kappa$ such that

$$h(n) \in F(X_m) \iff \delta(h(n)) \in X_m$$

for all $n, m < \omega$.

Let $f_n \in J_\tau$ be such that $f_n : \kappa \to \kappa$ and $h(n) = [f_n]_{F^*}$. Then $h(n) = k(h(n)) = j(f_n)(\kappa)$ for all $n < \omega$ and hence we have

$$h(n) \in F(X_m) \iff h(n) \in k(F(X_m)) \iff j(f_n)(\kappa) \in j(X_m)$$

for all $n, m < \omega$. We also have that

$$j(f_n)(\kappa) \in j(X_m) \iff \{\beta < \kappa \mid f_n(\beta) \in X_m\} \in U$$

for all $n, m \in \omega$.

For $n, m < \omega$, if $\{\beta < \kappa \mid f_n(\beta) \in X_m\} \in U$, then let

$$A_{nm} = \{\beta < \kappa \mid f_n(\beta) \in X_m\};$$

otherwise, let

$$A_{nm} = \kappa - \{\beta < \kappa \mid f_n(\beta) \in X_m\}.$$

For $n, m < \omega$, if $\{\beta < \kappa \mid f_n(\beta) < f_m(\beta)\} \in U$, then let

$$D_{nm} = \{\beta < \kappa \mid f_n(\beta) \in X_m\};$$

otherwise, let

$$D_{nm} = \kappa - \{\beta < \kappa \mid f_n(\beta) < f_m(\beta)\}.$$

For $n, m, s < \omega$, if $\{\beta < \kappa \mid \prec f_n(\beta), f_m(\beta) \succ = f_s(\beta)\} \in U$, then let

$$B_{nms} = \{\beta < \kappa \mid \prec f_n(\beta), f_m(\beta) \succ = f_s(\beta)\};$$

otherwise, let

$$B_{nms} = \kappa - \{\beta < \kappa \mid \prec f_n(\beta), f_m(\beta) \succ = f_s(\beta)\}.$$

Let

$$C = \bigcap_{\prec n,m,i,j,k,s,t \succ \in \omega} (A_{nm} \cap D_{ij} \cap B_{kst}).$$

Then $C \in U$. Let $\beta \in C$. Let $\delta(h(n)) = f_n(\beta)$. Then $\delta : B \to \kappa$ is a Gödel homomorphism showing that F is ω–complete with respect to the pair (W, B).

To see that F is supercomplete with respect to the pair (W, B), we argue as follows.

Claim There must be a $\beta \in C$ such that $\delta(h(n)) = f_n(\beta)$ is a strong connection. (In fact there is a measure one subset of C of such β's.)

Suppose not. For each $\beta \in C$, there is some $Y_\beta \subseteq \omega$ such that $\bigcup_{n \in Y_\beta} t^L_{h(n)}$ is a well founded relation but $\bigcup_{n \in Y_\beta} t^L_{\delta_\beta(h(n))}$ is not a well founded relation. Let $C_0 \in U$ and $Y \subseteq \omega$ be such that $C_0 \subseteq C$ and for all $\beta \in C_0$, $Y_\beta = Y$.

Hence in V, we have $\bigcup_{n\in Y} t^L_{h(n)}$ is a well founded relation but for all $\beta \in C_0$, $\bigcup_{n\in Y} t^L_{f_n(\beta)}$ is not a well founded relation.

It follows now that in M, $\bigcup_{n\in Y} t^L_{j(f_n)(\kappa)}$ is not a well founded relation.

But, $h(n) = j(f_n)(\kappa)$ for all $n < \omega$. By absoluteness, we get a contradiction.

This finishes the proof of the above claim.

Notice that $|F| = \kappa$. Hence $(J_\nu, F) \in M$. It follows that if there is a measurable cardinal, then there are many premice $(J_{\nu'}, F')$ with supercomplete top extenders F' below the first measurable cardinal.

□

Definition 3.2 We say that a premouse M is simply normally iterable if every normal iteration of M without truncation can be continued.

Theorem 3.1 Let M be a type 1 premouse such that every surviving extender is supercomplete with respect to M. Then M is uniquely simply normally iterable.

First we prove the following realization theorem.

Theorem 3.2 Let $M = \langle J^E_\alpha, F\rangle$ be a type 1 premouse. Let $\sigma : N \to_{\Sigma^*} M$ be such that N is countable. Let $\mathcal{T} = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal countable iteration of N. Assume that either $\mathcal{T}$ has no truncation and every surviving extender is supercomplete with respect to M or $\mathcal{T}$ has truncations and every surviving extender is supercomplete on M. Then there are $\sigma_i : N_i \to M$ and $\delta_i : \lambda_i \to \sigma_{T(i+1)}(\kappa_i)$ such that

(a) $\sigma_0 = \sigma$, $\sigma_j \pi_{ij} = \sigma_i$ for $i \leq_T j$;

(b) $\sigma_i(\kappa_i) \leq \sigma_{T(i+1)}(\kappa_i)$;

(c) if $\sigma_i(\kappa_i) = \sigma_{T(i+1)}(\kappa_i)$, then

$$\delta_i : \lambda_i \to \sigma_{T(i+1)}(\kappa_i)$$

is a strong connection in that $\delta_i = g_i \sigma_i{\restriction}_{\lambda_i}$ and $g_i : \sigma_i[\lambda_i] \to \sigma_i(\kappa_i)$ is to witness the super completeness, and

(d) $\sigma_{i+1}(\pi_{T(i+1),i+1}(f)(a)) = \sigma_{T(i+1)}(f)(\delta_i(a))$, where $f \in \Gamma(\kappa_i, N_{T(i+1)}||\eta_i)$, $a \in \lambda_i^{<\omega}$.

(e) Set $\overline{n}(0) = \omega$,

$$\overline{n}(i+1) = \begin{cases} \omega & \text{if } \sigma_{T(i+1)}(\kappa_i) < \omega\rho^\omega_M \\ n & \text{if } \omega\rho^{n+1}_M \leq \sigma_{T(i+1)}(\kappa_i) < \omega\rho^n_M, \end{cases}$$

$\overline{n}(i) = \min\{\overline{n}(j) \mid j <_T i\}$ for limit ordinal i.

Then σ_i is $\Sigma_0^{(\overline{n}(i))}$–preserving and if $\overline{n}(i) = 0$, then, in addition, σ_i is cardinal preserving.

Proof Suppose that we have σ_i. We need to define δ_i and σ_{i+1}. All we need is a right connection δ_i since σ_{i+1} is canonically determined by δ_i and $\pi_{T(i+1),i+1}$.

Case1 $T(i+1) <_T i$.

Let l be such that $T(i+1) = T(l+1)$ and $l+1 \leq_T i$.

Case1(a) $\kappa_i < \kappa_l$.

Then $\sigma_{T(i+1)}(\kappa_i) = \sigma_i(\kappa_i)$ and $\sigma_{T(i+1)}\restriction_{\kappa_l} = \sigma_i\restriction_{\kappa_l}$. In particular, if $x \subseteq \kappa_i$, and $x \in M_{T(i+1)}$, then $\sigma_{T(i+1)}(x) = \sigma_i(x)$.

Let $g_i : \sigma_i[\lambda_i] \to \sigma_i(\kappa_i)$ be a witness to the supercompleteness with respect to the range of σ_i. Then define $\delta_i(\alpha) = g_i(\sigma_i(\alpha))$ for $\alpha < \lambda_i$.

This gives us a strong connection.

We then define that

$$\sigma_{i+1}(\pi_{T(i+1),i+1}(f)(a)) = \sigma_{T(i+1)}(f)(\delta_i(a))$$

for $f \in \Gamma(\kappa_i, N_{T(i+1)}||\eta_i)$ and $a \in \lambda_i^{<\omega}$.

Case1(b). $\kappa_l \leq \kappa_i$.

Notice that in this case κ_i must be of type 0 and $E_{\nu_i}^{N_i} \in N_i$. Also, κ_l is of type 1 and hence $\kappa_l < \kappa_i$.

In this case we have the following:

$\sigma_{T(i+1)}(\kappa_i) \geq \sigma_{T(i+1)}(\kappa_l) = \sigma_{l+1}(\pi_{T(l+1),l+1}(\kappa_l)) = \sigma_{l+1}(\lambda_l)$ and $\sigma_{l+1}(\lambda_l) \geq \sigma_i(\lambda_l) \geq \sigma_i(\lambda_{T(i+1)}) > \sigma_i(\kappa_i)$.

Let $j = T(i+1)$.

Since $j < i$, $N_i \models \lambda_j$ is inaccessible, and $J_{\nu_j}^{N_j} = J_{\nu_j}^{N_i}$, we have that for all $a \in \lambda_i^{<\omega}$, $E_{\nu_i,a}^{N_i} \in J_{\lambda_j}^{E^{N_j}}$.

We fix two 1–1 enumerations:

$$\langle \alpha_m \mid m < \omega \rangle = \lambda_i$$

and

$$\langle x_m \mid m < \omega \rangle = \bigcup_{n<\omega} (P(\kappa_i^n))^{N_i}.$$

For $n < \omega$, we define $\overline{D}_n$ and $\overline{R}_n$ as follows:

$$\overline{D}_n = \{s \mid s : n \to \kappa_i \ \wedge \ \forall\, b \in [n]^{<n} \forall\, k < n(x_k \in E_{\nu_i,\vec{\alpha}\restriction_b} \iff s\restriction_b \in x_k)\}.$$

$$\overline{R}_n = \{(t,s) \mid t, s \in \bigcup_{m \leq n} \overline{D}_m \ \wedge \ t = s\restriction_{|t|}\}.$$

Then $\overline{D}_n$ and $\overline{R}_n$ are all in $J_{\lambda_j}^{E^{M_j}}$ and

$$\sigma_{T(i+1)}(\overline{R}_n) = \sigma_i(\pi_{T(i+1),i}(\overline{R}_n)).$$

Let $R_n = \sigma_{T(i+1)}(\overline{R}_n)$ and $D_n = \sigma_{T(i+1)}(\overline{D}_n)$. Then
$s \in D_n \iff s : n \to \sigma_{T(i+1)}(\kappa_i)$ and $\forall b \in [n]^{<n} \forall k < n$ we have

$$\sigma_{T(i+1)}(x_k) \in \sigma_{T(i+1)}(E_{\nu_i,\vec{\alpha}\restriction_b}) \iff s\restriction_b \in \sigma_{T(i+1)}(x_k).$$

If $S = \bigcup_{n<\omega} R_n$ is ill–founded, letting f be a branch of the tree S, then we define

$$\delta_i(\alpha_m) = f(m)$$

for $m < \omega$. We check that

$$\sigma_{i+1}(\pi_{T(i+1),i+1}(F)(a)) = \sigma_{T(i+1)}(F)(\delta_i(a))$$

works, where $F \in \Gamma(\kappa_i, N_{T(i+1)}||\eta_i)$ and $a \in \lambda_i^{<\omega}$. [Notice that in this case, κ_i is of type 0 and δ_i is only a weak connection.]

Now we proceed to show that the tree S is indeed ill–founded.

Let $R_n^* = \sigma_i(\overline{R}_n)$ and $D_n^* = \sigma_i(\overline{D}_n)$. By ω–completeness, $\bigcup_{n<\omega} R_n^*$ is ill founded. We need to make a connection of the two trees, $\bigcup_{n<\omega} R_n$ and $\bigcup_{n<\omega} R_n^*$ in such a way that the ill foundedness of the later shall guarantee the ill foundedness of the former. This is where the super completeness and the strong connection property of $\delta_{T(i+1)}$ is applied.

Let us compare $\sigma_{T(i+1)}[J^{E^{N_i}}_{\lambda_{T(i+1)}}]$ and $\sigma_i[J^{E^{N_i}}_{\lambda_{T(i+1)}}]$. We have a canonical connection of the two structures:

$$\overline{\delta}_i(\sigma_{T(i+1)}(x)) = \sigma_i(x).$$

Since $j = T(i+1) \leq l < i$, and $\kappa_l \leq \kappa_i$, we have

$$\kappa_l \leq \kappa_i < \lambda_l < \lambda_i.$$

By Lemma 2.2, κ_l is of type 1.

In summary, we have that $j <_T l+1 \leq_T i$, $j = T(l+1) = T(i+1)$, $j \leq l$ and κ_l is of type 1. By Theorem 2.2, $j \leq_T l$ and $\kappa_l < crit(\pi_{j,l})$. Hence

$$\sigma_l(\kappa_l) = \sigma_{T(l+1)}(\kappa_l) = \sigma_j(\kappa_l).$$

By induction hypothesis, δ_l must be a strong connection:

$$\delta_l = g_l \sigma_l \restriction_{\lambda_l}.$$

We now prove a minimal walk lemma which allows us to successfully perform a serious reductions to get our desired ill–foundedness.

LEMMA 3.1 (Minimal Walk Around $T(i+1)$) Let $j = T(i+1) < \alpha \leq i$. Assume that $j <_T \alpha$. Then there is a sequence $t(0) = \alpha > t(1) > \cdots > t(n)$ of ordinals such that $j = T(t(m)+1) <_T t(m)+1 \leq_T t(m-1)$ for $1 \leq m \leq n$, and $\kappa_{t(1)} < \kappa_{t(2)} < \cdots < \kappa_{t(n)}$ are all of type 1, and either $t(n) = j$ and $\kappa_{t(n)} \leq \kappa_i$ or $\kappa_{t(n-1)} \leq \kappa_i < \kappa_{t(n)}$.

(Remark: When there is no truncation, the second alternative cannot happen and we must have searched successfully $j+1$. When there are truncations, the second alternative may happen and we may have that $T(j+1) < j$.)

Proof of the lemma.

Let $t(0) = \alpha$.

Set $k = 1$ and start the **Minimal Walk Around** $T(i+1)$ **Process.**

Step 1 Let $s(k)$ be the least m such that $j <_T m \leq_T t(k-1)$. By minimality, $s(k)$ must be a successor ordinal. Let $t(k)$ be the ordinal predecessor of $s(k)$.
Step 2 If $\kappa_i < \kappa_{t(k)}$, then we stop with success. Otherwise continue to Step 3.
Step 3 Now $\kappa_i \geq \kappa_{t(k)}$, and hence $\kappa_{t(k)}$ is of type 1. By our lemma, $j = T(t(k)+1) \leq_T t(k)$ and $\kappa_{t(k)} < crit(\pi_{jt(k)})$.
Step 4 If $j = t(k)$, then we stop with success. Otherwise continue to Step 5.
Step 5 Currently we have $j <_T t(k)$. We then increase value of the counter k by 1, i.e., set $k = k+1$, and go to Step 1 to repeat the Minimal Walk Around $T(i+1)$ Process one more round.

Since $t(0) = \alpha > t(1) > t(2) > \cdots$, the process must stop after finitely many rounds. We have our Minimal Walk Around $T(i+1)$ Lemma proved.

□

We now apply the Minimal Walk Around $T(i+1)$ Lemma to derive that $\bigcup_{n<\omega} \sigma_j(\overline{R}_n)$ is ill founded from our assumption that $\bigcup_{n<\omega} \sigma_\alpha(\overline{R}_n)$ is ill founded with $\alpha = i$.

Let $< t(0), t(1), t(2), \cdots, t(a) >$ be the trace of our minimal walk, where $0 < a < \omega$. First notice that for $1 \leq n \leq a$,

$$\kappa_{t(n)} = crit(\pi_{jt(n-1)}) < \lambda_j \leq \lambda_{t(n)} \leq crit(\pi_{t(n)+1,t(n-1)}).$$

Hence for $1 \leq n \leq a$, if $\bigcup_{m<\omega} \sigma_{t(n-1)}(\overline{R}_m)$ is ill founded then $\bigcup_{m<\omega} \sigma_{t(n)+1}(\overline{R}_m)$ is ill founded since all the parameters are fixed by $\pi_{t(n)+1,t(n-1)}$.

Now we show that for $1 \leq n \leq a$ if $\bigcup_{m<\omega} \sigma_{t(n)+1}(\overline{R}_m)$ is ill founded then $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{R}_m)$ is ill founded.

We assume that $\bigcup_{m<\omega} \sigma_{t(n)+1}(\overline{R}_m)$ is ill founded.

We need to show that $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{R}_m)$ is ill founded.

Since $\kappa_{t(n)}$ is of type 1, and $\kappa_{t(n)} < crit(\pi_{j,t(n)})$,

$$\sigma_j(\kappa_{t(n)}) = \sigma_{t(n)}(\kappa_{t(n)}).$$

Hence $\delta_{t(n)}$ is a strong connection by induction hypothesis,

$$\delta_{t(n)} = g_{t(n)}\sigma_{t(n)}\restriction_{\lambda_{t(n)}}.$$

If $j < t(n)$, then $\lambda_j < \lambda_{t(n)}$. It follows that λ_j is a cardinal in $N_{t(n)}$ and hence $\sigma_{t(n)}(\lambda_j)$ is a cardinal in M.

If $j = t(n)$, then a difference shows up. If $\eta_i < ht(N_j)$, then we must use the stronger assumption on extenders of M, i.e., every surviving extender is supercomplete on M. If $\eta_i = ht(N_j)$, then τ_i is a cardinal in N_j and hence $\sigma_j(\tau_i)$ is a cardinal in M, so the weaker requirement is sufficient.

Let $r_{t(n)} : \lambda_{t(n)} \to J^{E^{N_{t(n)}}}_{\lambda_{t(n)}}$ and $\overline{r}_{t(n)} : \kappa_{t(n)} \to J^{E^{N_{t(n)}}}_{\kappa_{t(n)}}$ be the respective canonical enumerations. Let $s(n) = t(n-1)$. Then

$\sigma_{s(n)}(\overline{R}_m) = \sigma_{s(n)}(r(\xi_m))$
$= \sigma_{t(n)+1}(r(\xi_m))$
$= \sigma_{t(n)+1}(r)(\sigma_{t(n)+1}(\xi_m))$
$= \sigma_{t(n)+1}(\pi_{j,t(n)+1}(\overline{r}_{t(n)}(\xi_m))$
$= \sigma_j(\overline{r}_{t(n)})(g_{t(n)}(\sigma_{t(n)}(\xi_m)))$
$= \sigma_{t(n)}(\overline{r}_{t(n)})(g_{t(n)}(\sigma_{t(n)}(\xi_m)))$. [Notice that $\kappa_{t(n)} < crit(\pi_{j,t(n)})$ and σ_j agrees with $\sigma_{t(n)}$ on this critical point.]

Since $\bigcup_{m<\omega} \sigma_{s(n)}(\overline{R}_m)$ is ill founded, we have that $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{r}_{t(n)})(g_{t(n)}(\sigma_{t(n)}(\xi_m)))$ is ill founded.

Since $g_{t(n)}$ is a strong connection, $\bigcup_{m<\omega} \sigma_{t(n)}(r_{t(n)})(\sigma_{t(n)}(\xi_m))$ is ill founded.

Therefore, $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{R}_m)$ is ill founded.

This finishes the induction step.

Now we have that $\bigcup_{m<\omega} \sigma_{t(a)}(\overline{R}_m)$ is ill founded.

We need to derive finally that $\bigcup_{m<\omega} \sigma_j(\overline{R}_m)$ is ill founded.

If $j = t(a)$, then we have finished.

Otherwise, we must have $\kappa_i < \kappa_{t(a)} = crit(\pi_{jt(a-1)})$. Since $\bigcup_{n<\omega} \sigma_{t(a-1)}(\overline{R}_n)$ is ill founded, the parameters in the definition of $\overline{R}_n$ and $\overline{D}_n$ are fixed by $\pi_{jt(a-1)}$, $\sigma_j(\overline{R}_n) = \sigma_{t(a-1)}(\overline{R}_n)$. We are done.

This finishes the ill foundedness proof. Case 1 is completed.

Case2. $i = T(i+1)$.

Then $\sigma_i(\kappa_i) = \sigma_{T(i+1)}(\kappa_i)$. Let $g_i : \sigma_i[\lambda_i] \to \sigma_i(\kappa_i)$ be given by the super completeness and let $\delta_i = g_i \sigma_i \restriction_{\lambda_i}$. Then we define

$$\sigma_{i+1}(\pi_{i,i+1}(f)(a)) = \sigma_i(f)(\delta_i(a)),$$

where $f \in \Gamma(\kappa_i, N_i)$ and $a \in \lambda_i^{<\omega}$.

This takes care **Case2**.

Case3. $T(i+1) \notin [0,i]_T$.

Let $< t(0), t(1), \cdots, t(k) >$ be the Minimal Walk to $T(i+1)$ sequence given by the Minimal Walk to $T(i+1)$ Lemma. Let $h = t(0)$.

Then $T(h+1) <_T T(i+1)$ and $T(h+1) <_T h+1 \leq_T i$ and let $\kappa_l = crit(\pi_{T(h+1),i+1})$ and $\kappa_h = crit(\pi_{T(h+1),i})$. Then we have $\lambda_h > \kappa_i \geq \kappa_l \geq \kappa_h$. It follows then

$\sigma_{T(i+1)}(\kappa_i) \geq \sigma_{T(h+1)}(\kappa_l)$
$\geq \sigma_{T(h+1)}(\kappa_h)$
$= \sigma_{h+1}(\lambda_h)$

$\geq \sigma_{h+1}(\lambda_{T(i+1)})$
$\geq \sigma_i(\lambda_{T(i+1)})$
$> \sigma_i(\kappa_i)$.

We will get a weak connection in this case. Notice that in this case κ_i is of type 0 and there will be no $m > i$ such that $i = T(m+1)$. So a weak connection is all we need.

Let $j = T(i+1)$. We can then define $\overline{D}_n$ and $\overline{R}_n$ in $J_{\lambda_j}^{E^{N_j}}$ as before and let $R_n = \sigma_j(\overline{R}_n)$ and $R_n^* = \sigma_i(\overline{R}_n)$. We have that $\bigcup_{n<\omega} R_n^*$ is ill founded. Also, for $l \in (j,i]$, λ_j is a cardinal in N_l, and hence $\sigma_l(\lambda_j)$ is a cardinal in M.

We need to show that all $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{R}_m)$ are ill founded for $n \leq k$.

This is done by induction on $n \leq k$.

We are given that, letting $t(-1) = i$, $\bigcup_{m<\omega} \sigma_{t(-1)}(\overline{R}_m)$ is ill founded. Since the critical point of $\pi_{t(0)+1,i}$ is larger than κ_i, it follows that $\bigcup_{m<\omega} \sigma_{t(0)+1}(\overline{R}_m)$ is ill founded.

We show that $\bigcup_{m<\omega} \sigma_{t(0)}(\overline{R}_m)$ is ill founded.

Since $\kappa_{t(0)}$ is of type 1, $\delta_{t(0)}$ is a strong connection ($\kappa_{t(0)} < crit(\pi_{T(t(0)+1)t(0)})$ and hence $\sigma_{T(t(0)+1)}(\kappa_{t(0)}) = \sigma_{t(0)}(\kappa_{t(0)})$), where

$$\delta_{t(0)} = g_{t(0)}\sigma_{t(0)}\restriction \lambda_{t(0)},$$

and $g_{t(0)}$ is a witness to the supercompleteness of the target extender with respect to the appropriate family in consideration.

Let $\eta = t(0)$. Let $r_\eta : \lambda_\eta \to J_{\lambda_\eta}^{E^{N_\eta}}$ and $\overline{r}_\eta : \kappa_\eta \to J_{\kappa_\eta}^{E^{N_\eta}}$ be the respective canonical enumerations.

Then $\sigma_i(\overline{R}_m) = \sigma_i(r(\xi_m))$
$= \sigma_{\eta+1}(r(\xi_m))$
$= \sigma_{\eta+1}(r)(\sigma_{\eta+1}(\xi_m))$
$= \sigma_{\eta+1}(\pi_{j,\eta+1}(\overline{r}_\eta)(\xi_m))$
$= \sigma_j(\overline{r}_\eta)(g_\eta(\sigma_\eta(\xi_m)))$
$= \sigma_\eta(\overline{r}_\eta)(g_\eta(\sigma_\eta(\xi_m)))$.

Hence $\bigcup_{m<\omega} \sigma_\eta(\overline{r}_\eta)(g_\eta(\sigma_\eta(\xi_m)))$ is ill founded.

It follows that $\bigcup_{m<\omega} \sigma_\eta(r_\eta)(\sigma_\eta(\xi_m)) = \bigcup_{m<\omega} \sigma_\eta(\overline{R}_m)$ is ill founded.

In general, letting $s(n) = t(n)+1$, we have that $T(s(n)) <_T t(n)$ and $T(s(n)) <_T s(n) <_T t(n-1)$ and $\kappa_{t(n)} < crit(\pi_{T(s(n)),t(n)})$ and $\kappa_{t(n)}$ is of type 1. Therefore exactly as in the case that $n = 0$, we can argue that $\bigcup_{m<\omega} \sigma_{t(n)}(\overline{R}_m)$ is ill founded since $\bigcup_{m<\omega} \sigma_{t(n-1)}(\overline{R}_m)$ is ill founded and $\delta_{t(n)}$ is a strong connection.

Now let us consider what could happen when we finish our minimal walk from i to $j = T(i+1)$.

There are two cases. And the first case that $T(i+1) = t(k)$ is the same as in the last step of reductions of Case 1b when $j = t(a)$. So let us assume that $j <_T t(k)$. Then we have that $T(t(k)+1) <_T j <_T t(k)$ and $T(t(k)+1) <_T t(k)+1 <_T t(k-1)$. If $crit(\pi_{j,t(k)}) > \kappa_i$, then by the agreements of the two mappings σ_j and $\sigma_{t(k)}$, we are done. Otherwise, We can now apply the Minimal Walk Around $T(i+1)$ Lemma as in the Case 1(b) (the part following the Lemma) to derive that $\bigcup_{m<\omega} \sigma_j(\overline{R}_m)$ is ill founded from the fact that $\bigcup_{m<\omega} \sigma_{t(k)}(\overline{R}_m)$ is ill founded.

This finishes the subcase.

We then let e be a branch of the searching tree and let $\delta_i(\alpha_m) = e(m)$ for $m < \omega$. Then we check that

$$\sigma_{i+1}(\pi_{j,i+1}(f)(a)) = \sigma_j(f)(\delta_i(a))$$

works, where $j = T(i+1), f \in \Gamma(\kappa_i, N_j||\eta_i)$, and $a \in \lambda_i^{<\omega}$.

This finishes the Case 3.

So we are done with the successor step.

Let i be a limit ordinal. We define σ_i in a canonical way along $[0,i]_T$. Then we continue to define δ_i in the next step $i+1$.

This finishes the proof of the theorem.

□

We can now prove our iterability theorem.

Proof of Theorem 4.1. Let M be a type 1 premouse such that every surviving extender is supercomplete with respect to M. Let $\mathcal{T} = \langle < M_i >, < \nu_i >, < \eta_i >, < \pi_{ij} \mid i \leq_T j >, T\rangle$ be a simple normal iteration of M of limit length θ. Assume that $\mathcal{T}$ has an ill–founded cofinal branch. Let $b = b_{\mathcal{T}}$ be the unique cofinal branch of $\mathcal{T}$. Let $\langle M_\theta, < \pi_{i\theta} \mid i \in b >\rangle$ be the direct limit of $\langle M_i, \pi_{ij} \mid i <_T j \in b\rangle$. Assume that M_θ is ill founded and that θ is minimal among all such counter examples.

Let Γ be a regular cardinal sufficiently large such that all the objects of our interests are all in H_Γ. Let $X \prec H_\Gamma$ be a countable elementary submodel such that $\{M, \mathcal{T}, b, \theta, M_\theta\} \subseteq X$. Let $\sigma : H \to H_\Gamma$ be the inverse mapping of the transitive collapsing of X. Let $\{N, \overline{\mathcal{T}}, \overline{b}, \overline{\theta}, N_{\overline{\theta}}\}$ be such that $\sigma(N) = M$, $\sigma(\overline{\mathcal{T}}) = \mathcal{T}$, $\sigma(\overline{b}) = b$, $\sigma(\overline{\theta}) = \theta$, and $\sigma(N_{\overline{\theta}}) = M_\theta$. Let $\overline{\mathcal{T}} = \langle < N_i >, < \nu_i >, < \eta_i >, < \pi_{ij} >, T\rangle$. Then $\overline{\mathcal{T}}$ is a normal iteration of N of length $\overline{\theta}$ and $\overline{b}$ is the unique cofinal branch of the iteration and $N_{\overline{\theta}}$ is the ill founded direct limit of $\langle N_i, \pi_{ij} \mid i <_T j \in \overline{\theta}\rangle$. Let $\sigma_0 = \sigma\restriction_N$. Then

$$\sigma_0 : N \to_{\Sigma^*} M.$$

Applying theorem 4.5, we have an embedding

$$\sigma_{\overline{\theta}} : N_{\overline{\theta}} \to M.$$

But M is transitive. We get a contradiction.

Therefore, M_θ must be well founded and $b_{\mathcal{T}}$ must be the unique well founded branch. Hence M is uniquely simply normally iterable.

□

§4 THE MODEL $L[\vec{E}]$.

In this section, we construct an inner model $L[\vec{E}]$ using type 1 supercompleteness as our backgrand conditions. In order to apply Jensen's theorem that normal iterability of basic premice implies full iterability [3] in our situation [every type 1 premouse is a basic premouse], we assume that there is an inaccessible cardinal in our working universe in this section when we argue that each N_ξ is iterable. Jensen's theorem reduces us to show only the normal iterability. Otherwise, there is no usage of this inaccessible cardinal. We believe that this hypothesis is redundant in our situation but not yet checked in detail.

DEFINITION 4.1 Let $M = \langle J_\alpha^A, B\rangle$ be an acceptable J–structure. Let $\gamma \in M$. We define $(\gamma)^{+M}$ to be the least $\tau \in M$ such that $\tau > \gamma$ and τ is a cardinal in M, if there is such; otherwise, we set $(\gamma)^{+M} = ht(M)$.

DEFINITION 4.2 A premouse M is a *weak mouse* if every countable Σ^* elementary submodel has a countably iterable transitive collapse, namely if $\sigma : Q \to_{\Sigma^*} M$ and Q is countable and transitive, then Q is countably iterable.

We are going to inductively define the following sequences: for an ordinal ξ,

(I) a premouse N_ξ, and $M_\xi = core(N_\xi)$ in case that N_ξ is iterable (otherwise, M_ξ is undefined and stop the inductive definition),

(II) μ_ξ, $\widetilde{\mu}_\xi$, $\widetilde{\kappa}^<_{\alpha\xi}$, $\widetilde{\kappa}^\leq_{\alpha\xi}$, $\widetilde{\mu}^<_{\alpha\xi}$, and $\widetilde{\mu}^\leq_{\alpha\xi}$ for $\alpha < \xi$.

They are related by the following specifications:

(S1) $\mu_\xi = ht(N_\xi)$ if $n(N_\xi) = 0$; $\mu_\xi = (\omega\rho^\omega_{N_\xi})^{+N_\xi}$ if $n(N_\xi) > 0$;

(S2) $\widetilde{\kappa}^\leq_{\alpha\xi} = \min\{\omega\rho^\omega_{M_\nu} \mid \alpha < \nu \leq \xi\}$.

(S3) If ξ is a limit ordinal, then $\widetilde{\kappa}^<_{\alpha\xi} = \min\{\omega\rho^\omega_{M_\nu} \mid \alpha \leq \nu < \xi\}$.

(S4) If ξ is a limit ordinal, then $\widetilde{\mu}^<_{\alpha\xi} = ht(M_\alpha)$ if $\widetilde{\kappa}^<_{\alpha\xi} = ht(M_\alpha)$; $\widetilde{\mu}^<_{\alpha\xi} = (\widetilde{\kappa}^<_{\alpha\xi})^{+M_\alpha}$ otherwise.

(S5) $\widetilde{\mu}^\leq_{\alpha\xi} = ht(M_\alpha)$ if $\widetilde{\kappa}^\leq_{\alpha\xi} = ht(M_\alpha)$; $\widetilde{\mu}^\leq_{\alpha\xi} = (\widetilde{\kappa}^\leq_{\alpha\xi})^{+M_\alpha}$ otherwise.

(S6) If ξ is a limit ordinal, then

$$\widetilde{\mu}_\xi = \sup\{\widetilde{\mu}^<_{\alpha\xi} \mid \alpha < \xi\}.$$

(S7) (*Agreement Condition*) For all $\alpha < \xi$,

$$M_\alpha||\widetilde{\mu}^\leq_{\alpha\xi} = M_\xi||\widetilde{\mu}^\leq_{\alpha\xi}.$$

Our main concerns here are the iterability of N_ξ and (S7), the Agreement Condition. As long we can maintain these two requirements, all other values are all determined by the specifications.

With all these set ups, we can now define our sequences as follows.

$N_0 = M_0 = \langle \emptyset, \emptyset \rangle$.

Successor Stage

Suppose that N_ξ and M_ξ are defined and $M_\xi = core(N_\xi)$. The following two cases set up to define $N_{\xi+1}$. If it is a premouse and normally iterable , then we define $M_{\xi+1} = core(N_{\xi+1})$; otherwise, we stop and $M_{\xi+1}$ is undefined.

Case 1. $M_\xi = \langle J^E_\nu, \emptyset \rangle$ and there exists an F such that $\langle J^E_\nu, F \rangle$ is a premouse and F is a extender of type at most 1 and F is supercomplete with respect to M_ξ.

Then we set $N_{\xi+1} = \langle J^E_\nu, F \rangle$.

Case 2. Otherwise. If $M_\xi = \langle J^E_\alpha, \emptyset \rangle$, then $N_{\xi+1} = \langle J^E_{\alpha+1}, \emptyset \rangle$; if $M_\xi = \langle J^E_\alpha, E_{\omega\alpha} \rangle$, then $N_{\xi+1} = \langle J^{E^{N_{\xi+1}}}_{\alpha+1}, \emptyset \rangle$, where $E^{N_{\xi+1}}$ extends E^{M_ξ} to length $\omega\alpha + 1$ by adding $E_{\omega\alpha}$ to its last position.

We need to check that $N_{\xi+1}$ is a premouse and if it is normally iterable, then $M_{\xi+1}$ obeys the agreement conditions.

Notice that if M_γ is defined, then it is the core of a mouse and hence it is a sound mouse.

The following Inductive Step Lemma justifies that all we need to concern at this stage is the iterability of $N_{\xi+1}$.

Lemma 4.1 (Inductive Step Lemma)

(1) Assume that $M = \langle J^E_\nu, \emptyset \rangle$ is a sound premouse. Then $P = \langle J^E_{\nu+1}, \emptyset \rangle$ is a premouse and $\omega\rho^\omega_M$ is the largest cardinal in P and $\omega\rho^\omega_P \leq \omega\rho^\omega_M$.

(2) Assume that $M = \langle J^E_\nu, \emptyset \rangle$ is a premouse. Let F be an M–extender such that $N = \langle J^E_\nu, F \rangle$ is a premouse. Then

(i) $n(M) = 0$, and hence M is sound.

(ii) $\omega\rho^\omega_N \leq$ the largest cardinal in M.

(iii) $\omega\rho^\omega_N <$ the largest cardinal in N if and only if there is some $\beta < lh(F)$ such that β generates F. (Including definition here if not yet defined.)

(iv) $core(N) = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{F} \rangle$ is a premouse.

(v) If $core(N)$ is sound, then $P = \langle J^{\overline{E}\,\overline{F}}_{\overline{\nu}+1}, \emptyset \rangle$ is a premouse and $\omega\rho^\omega_P \leq \omega\rho^\omega_N$.

(3) If $M = \langle J^E_{\nu+1}, \emptyset \rangle$ is a premouse, then $core(M) = \langle J^{\overline{E}}_{\overline{\nu}+1}, \emptyset \rangle$ is a premouse.

Lemma 4.2 (Core Mouse Agreement Lemma)

If M is a mouse and $\omega\rho^\omega_M \in M$ and $\gamma = (\omega\rho^\omega_M)^{+M}$, then $\gamma = (\omega\rho^\omega_M)^{+core(M)}$, and

$$core(M)||\gamma = M||\gamma.$$

Lemma 4.3 (Agreement at $\xi+1$) Assume that $M_{\xi+1}$ is defined. Then for $\alpha < \xi + 1$, $M_\alpha || \widetilde{\mu}^{\leq}_{\alpha\xi+1} = M_{\xi+1} || \widetilde{\mu}^{\leq}_{\alpha\xi+1}$.

Proof First, we show that $M_{\xi+1}$ and M_ξ agree upto $\widetilde{\mu}^{\leq}_{\xi\xi+1}$.

$N_{\xi+1}$ and M_ξ agree fully since M_ξ is an initial segment of $N_{\xi+1}$. Since $\omega\rho^\omega_{N_{\xi+1}} \in N_{\xi+1}$, by Jensen's lemma quoted above, $N_{\xi+1}$ and $M_{\xi+1}$ agree upto $\mu_{\xi+1}$. Since

$$\widetilde{\kappa}^{\leq}_{\xi\xi+1} = \omega\rho^\omega_{M_{\xi+1}} = \omega\rho^\omega_{N_{\xi+1}} \leq \omega\rho^\omega_{M_\xi}$$

and $\widetilde{\mu}^{\leq}_{\xi\xi+1} = \mu_{\xi+1}$, we have our desired agreement between $M_{\xi+1}$ and M_ξ.

Let $\alpha < \xi$. Then

$$\widetilde{\kappa}^{\leq}_{\alpha\xi+1} \leq \widetilde{\kappa}^{\leq}_{\alpha\xi} \ \wedge\ \widetilde{\mu}^{\leq}_{\alpha\xi+1} \leq \ \min\{\widetilde{\mu}^{\leq}_{\alpha\xi}, \mu_{\xi+1}\}.$$

Therefore,

$$M_\alpha || \widetilde{\mu}^{\leq}_{\alpha\xi+1} = M_\xi || \widetilde{\mu}^{\leq}_{\alpha\xi+1} = M_{\xi+1} || \widetilde{\mu}^{\leq}_{\alpha\xi+1}.$$

□

Limit Stage

Let λ be a limit ordinal. Assume that N_ξ and M_ξ have been defined for all $\xi < \lambda$. We would like to define N_λ and M_λ.

LEMMA 4.4 (1) If $\alpha \leq \beta < \lambda$, then $\widetilde{\mu}^{<}_{\alpha\lambda} \leq \widetilde{\mu}^{<}_{\beta\lambda}$.

(2) $\widetilde{\mu}_\lambda$ is a limit ordinal.

(3) For each $\alpha < \lambda$, there is a sequence E^α such that

$$J^{E^\alpha}_{\mu^<_{\alpha\lambda}} = J^{E^{M_\xi}}_{\mu^<_{\alpha\lambda}}$$

for all $\xi \in [\alpha, \lambda)$. More over, if $\alpha < \beta < \lambda$, then $J^{E^\alpha}_{\mu^<_{\alpha\lambda}}$ is an initial segment of $J^{E^\beta}_{\mu^<_{\beta\lambda}}$.

Proof (1) follows from the fact that if $\alpha < \beta < \lambda$ then $\widetilde{\kappa}^{<}_{\alpha\lambda} \leq \widetilde{\kappa}^{<}_{\beta\lambda}$.

We show (2) and (3) at the same time.

For $\alpha < \lambda$, let ξ_α be the least $\xi > \alpha$ such that $\widetilde{\kappa}^{<}_{\alpha\lambda} = \omega\rho^\omega_{M_\xi}$.

We claim that $\widetilde{\mu}^{<}_{\alpha\lambda} < \widetilde{\mu}^{<}_{(\xi_\alpha+1)\lambda}$ and M_{ξ_α} is an initial segment of M_η for $\eta \in [\xi_\alpha, \lambda)$ and $\widetilde{\mu}^{<}_{\alpha\lambda} = \widetilde{\mu}^{\leq}_{\alpha\xi_\alpha}$.

Then (2) and (3) follow from this claim and the Agreement Condition.

To see the claim, observe that

$$\omega\rho^\omega_{M_{\xi_\alpha}} = \omega\rho^\omega_{N_{\xi_\alpha+1}} = \omega\rho^1_{N_{\xi_\alpha+1}}$$

and it is the largest cardinal in $N_{\xi_\alpha+1}$. Hence, by Lemma 1.2, $N_{\xi_\alpha+1}$ is sound. Therefore, $M_{\xi_\alpha+1} = N_{\xi_\alpha+1}$. It follows that

$$\widetilde{\mu}^{<}_{(\xi_\alpha+1)\lambda} = ht(N_{\xi_\alpha+1}) = ht(M_{\xi_\alpha}) + 1 > ht(M_{\xi_\alpha}) \geq \widetilde{\mu}^{<}_{\alpha\lambda}.$$

Now let $\xi \in (\xi_\alpha+1, \lambda)$. Then $\omega\rho^\omega_{M_{\xi_\alpha}} \leq \widetilde{\kappa}^{\leq}_{(\xi_\alpha+1)\xi}$. Since $\omega\rho^\omega_{M_{\xi_\alpha}}$ is the largest cardinal in $N_{\xi_\alpha+1} = M_{\xi_\alpha+1}$, we have $\widetilde{\mu}^{\leq}_{(\xi_\alpha+1)\xi} = ht(M_{\xi_\alpha+1})$. Therefore, by our Agreement Condition, $M_{\xi_\alpha+1}$ is an initial segment of M_ξ. Hence, for $\xi \in [\xi_\alpha, \lambda)$, M_{ξ_α} is an initial segment of M_ξ.

□

Equipped with this lemma, we let

$$J^{E}_{\widetilde{\mu}_\lambda} = \bigcup_{\alpha<\lambda} J^{E^\alpha}_{\widetilde{\mu}^{<}_{\alpha\lambda}}.$$

Define $N_\lambda = \langle J^{E}_{\widetilde{\mu}_\lambda}, \emptyset\rangle$. If N_λ is iterable, then let $M_\lambda = core(N_\lambda)$. Otherwise, we stop and M_λ is undefined.

LEMMA 4.5 (Limit Step Lemma)
(1) N_λ is a premouse.
(2) Assume that N_λ is iterable. Then for all $\alpha < \lambda$,

$$M_\alpha||\widetilde{\mu}^{\leq}_{\alpha\lambda} = M_\lambda||\widetilde{\mu}^{\leq}_{\alpha\lambda}.$$

Proof (1) follows from the fact that $\widetilde{\mu}_\lambda$ is a limit ordinal and every initial segment of a mouse is sound and $J^{E}_{\widetilde{\mu}_\lambda}$ is a stack of a sequence of initial segments of mice.

For (2), notice that for all $\alpha < \lambda$,

$$\widetilde{\kappa}^{\leq}_{\alpha\lambda} \leq \widetilde{\kappa}^{<}_{\alpha\lambda} \wedge \widetilde{\mu}^{\leq}_{\alpha\lambda} \leq \widetilde{\mu}^{<}_{\alpha\lambda}.$$

Therefore,

$$M_\alpha||\widetilde{\mu}^{\leq}_{\alpha\lambda} = N_\lambda||\widetilde{\mu}^{\leq}_{\alpha\lambda} \ \wedge \ \widetilde{\mu}^{\leq}_{\alpha\lambda} \leq \mu_\lambda.$$

If $n(N_\lambda) = 0$, then N_λ is sound and $M_\lambda = N_\lambda$. Otherwise, $\omega\rho^{\omega}_{N_\lambda} \in N_\lambda$. By Jensen's lemma above, N_λ and M_λ agree upto μ_λ. Hence,

$$M_\alpha||\widetilde{\mu}^{\leq}_{\alpha\lambda} = M_\lambda||\widetilde{\mu}^{\leq}_{\alpha\lambda}.$$

□

Before we go on to show that each M_ξ is defined, we introduce some technical devices that shall be used in the proof.

DEFINITION 4.3 (1) Let M be a premouse. Let $\nu \in M$ be such that $E^M_\nu \neq \emptyset$. Define $\beta(M,\nu) =$ the least $\beta < ht(M)$ such that $\nu \leq \beta$ and $\omega\rho^{\omega}_{M||\beta} \leq \omega\rho^{\omega}_{M||\xi}$ for $\nu \leq \xi < ht(M)$.

(2) Define $\beta^+(M,\nu) =$ the least $\beta \leq ht(M)$ such that $\nu \leq \beta$ and $\omega\rho^{\omega}_{M||\beta} \leq \omega\rho^{\omega}_{M||\xi}$ for $\nu \leq \xi \leq ht(M)$.

Hence, $\beta(M,\nu)$ is the first realization of the minimum of all the ω–th projectums of proper initial segments $M||\xi$ for $\xi \in [\nu, ht(M))$.

We are going to use the β–operator iteratively to search the origin of $E^{N_\xi}_\nu$ for $\nu < ht(N_\xi)$. We define two Minimal Walks below to achieve such searching.

LEMMA 4.6 (Minimal Walk from $ht(M)$ to ν)

Let M be a premouse and $\nu \in ht(M)$. Assume that $E^M_\nu \neq \emptyset$. Then there is uniquely a sequence $\langle \overline{\beta}_i \mid i \leq k(M,\nu)\rangle$, *the Minimal Walk from* $ht(M)$ *to* ν, with the following three properties:

(MW1) $\overline{\beta}_0 = ht(M)$, and $\nu = \overline{\beta}_{k(M,\nu)}$.

(MW2) $\overline{\beta}_{k(M,\nu)} < \overline{\beta}_{k(M,\nu)-1} < \cdots < \overline{\beta}_1 < \overline{\beta}_0$.

(MW3) $\overline{\beta}_{i+1} = \beta(M||\overline{\beta}_i, \nu)$ for all $i < k(M,\nu)$.

[Sometimes, we write $\overline{\beta}_i[M,\nu]$ for $\overline{\beta}_i$.]

Proof Iterating the β operator, we define the sequence by induction as follows.

$\overline{\beta}_0 = ht(M)$.

If $\overline{\beta}_m$ is defined and $\overline{\beta}_m > \nu$, then define $\overline{\beta}_{m+1} = \beta(M||\overline{\beta}_m, \nu)$. Otherwise, $\overline{\beta}_{m+1}$ is undefined and we stop. When we stop, we set $k = k(M,\nu)$ to be the maximum m such that $\overline{\beta}_m$ is defined. [Hence, $\overline{\beta}_k = \nu$.]

□

Notice that if $\nu < \beta(M,\nu)$, then for all $\xi \in [\nu, \beta(M,\nu))$, we have $\omega\rho^\omega_{M||\beta(M,\nu)} < \omega\rho^\omega_{M||\xi}$. It turns out that this property alone can be used to compute the β–operator. In our applications, we shall use this property to compute.

Definition 4.4 Let M be a premouse. Let $\nu \in M$ be such that $E^M_\nu \neq \emptyset$. Define $B(M,\nu)$ by the following:

$$\eta \in B(M,\nu) \iff \nu \leq \eta \ \wedge \ \eta \in ht(M) \ \wedge \ (\nu < \eta \to \forall \xi \in [\nu, \eta)\ (\omega\rho^\omega_{M||\eta} < \omega\rho^\omega_{M||\xi})).$$

$B(M,\nu) \neq \emptyset$ since $\nu \in B(M,\nu)$. $B(M,\nu)$ is finite since if $\eta < \beta$ are in $B(M,\nu)$ then $\omega\rho^\omega_{M||\beta} < \omega\rho^\omega_{M||\eta}$. The precise relationship between the Minimal Walk to ν sequence and the set $B(M,\nu)$ is given by the following lemma.

Lemma 4.7 Let M be a premouse and let $\nu < ht(M)$ be such that $E^M_\nu \neq \emptyset$. Then the Minimal Walk to ν sequence $\langle \overline{\beta}_0[M,\nu], \overline{\beta}_1[M,\nu], \cdots, \overline{\beta}_{k(M,\nu)}[M,\nu]\rangle$ is a strict decreasing enumeration of $B(M,\nu) \cup \{ht(M)\}$. In fact, $\overline{\beta}_{i+1} = \max(B(M,\nu) \cap \overline{\beta}_i)$ for $i < k(M,\nu)$.

Proof By induction on $i < k(M,\nu)$, we show that $\overline{\beta}_{i+1} = \max(B(M,\nu) \cap \overline{\beta}_i)$.

Let $\overline{\beta}_1 = \beta(M,\nu)$. If $\overline{\beta}_1 = \nu$, then $\overline{\beta}_1 \in B(M,\nu)$. If $\nu < \overline{\beta}_1$, then for all $\xi \in [\nu, \overline{\beta}_1)$, we have $\omega\rho^\omega_{M||\overline{\beta}_1} < \omega\rho^\omega_{M||\xi}$ by definition of the β–operator. Hence $\overline{\beta}_1 \in B(M,\nu)$.

Let $\eta = \max(B(M,\nu))$. If $\overline{\beta}_1 < \eta$, then $\omega\rho^\omega_{M||\eta} < \omega\rho^\omega_{M||\overline{\beta}_1}$ and $\eta < ht(M)$. Hence $\overline{\beta}_1 = \eta$.

Assume that $\overline{\beta}_m = \max(B(M,\nu) \cap \overline{\beta}_{m-1})$ for $m > 0$.

If $m = k(M,\nu)$, then $\overline{\beta}_m = \nu$. Then we have done. So assume that $m < k(M,\nu)$. Hence $\overline{\beta}_m > \nu$.

Let $\overline{\beta}_{m+1} = \beta(M||\overline{\beta}_m, \nu)$. If $\overline{\beta}_{m+1} = \nu$, then $\overline{\beta}_{m+1} \in B(M,\nu)$. If $\overline{\beta}_{m+1} > \nu$, then for all $\xi \in [\nu, \overline{\beta}_{m+1})$, $\omega\rho^\omega_{M||\overline{\beta}_{m+1}} < \omega\rho^\omega_{M||\xi}$ by definition of the β–operator. Hence $\overline{\beta}_{m+1} \in B(M,\nu) \cap \overline{\beta}_m$.

Let $\eta = \max(B(M,\nu) \cap \overline{\beta}_m)$.
If $\eta > \overline{\beta}_{m+1}$, then $\omega\rho^\omega_{M||\eta} < \omega\rho^\omega_{M||\overline{\beta}_{m+1}}$. This contradicts to the definition of $\overline{\beta}_{m+1}$.
Hence $\overline{\beta}_{m+1} = \max(B(M,\nu) \cap \overline{\beta}_m)$.

□

The following lemma explains that the β–operator has certain invariant character under Σ_1–embeddings.

Lemma 4.8 Let M and N be two premice and let $\sigma : M \rightarrow_{\Sigma_1} N$. Let $\nu < ht(M)$ be such that $E^M_\nu \neq \emptyset$. Then $\sigma(\nu) < ht(M)$ and $E^N_{\sigma(\nu)} \neq \emptyset$ and $k(M,\nu) = k(N,\sigma(\nu))$ and for all $1 \leq i \leq k(M,\nu)$, $\sigma(\overline{\beta}_i[M,\nu]) = \overline{\beta}_i[N,\sigma(\nu)]$.

Proof The first part of the conclusion is trivial but included in order for the second part to make sense.

We just have to prove that $\sigma(\overline{\beta}_i[M,\nu]) = \overline{\beta}_i[N,\sigma(\nu)]$ for all $1 \leq i \leq k(M,\nu)$.

We prove that $\sigma(\beta(M||\overline{\beta}_i,\nu)) = \beta(N||\sigma(\overline{\beta}_i),\sigma(\nu))$ for all $i < k(M,\nu)$, where $\sigma(\overline{\beta}_0) = ht(N)$.

Let us first observe the following general fact.

Fact If $\lambda < \xi < ht(M)$ is a cardinal in $M||\xi$, then $\sigma(\lambda)$ is a cardinal in $N||\sigma(\xi)$.

To see this, assume that $\sigma(\lambda)$ is not a cardinal in $N||\sigma(\xi)$. Then

$$N \models \exists\mu < \sigma(\lambda)\ \exists\, f \in J^{E^N}_{\sigma(\xi)}\ f : \mu \rightarrow_{onto} \sigma(\lambda).$$

By Σ_1–elementarity,

$$M \models \exists\mu < \lambda\ \exists\, f \in J^{E^M}_{\xi}\ f : \mu \rightarrow_{onto} \lambda.$$

But $M||\xi \models \lambda$ is a cardinal. This is a contradiction.

Let $\overline{\beta}_{m+1} = \beta(M||\overline{\beta}_m,\nu) < \overline{\beta}_m$. Then $\sigma(\overline{\beta}_{m+1}) < \sigma(\overline{\beta}_m)$.
By definition, for all $\eta \in [\nu,\overline{\beta}_m)$, we have

$$\omega\rho^\omega_{M||\overline{\beta}_{m+1}} \leq \omega\rho^\omega_{M||\eta}.$$

Hence $\omega\rho^\omega_{M||\overline{\beta}_{m+1}}$ is a cardinal in $M||\overline{\beta}_m$ since $\nu \leq \overline{\beta}_{m+1} < \overline{\beta}_m$ and $\omega\rho^\omega_{M||\overline{\beta}_{m+1}} < \nu$. It follows that $\sigma(\omega\rho^\omega_{M||\overline{\beta}_{m+1}})$ is a cardinal in $N||\sigma(\overline{\beta}_m)$ by Σ_1–elementarity of σ.

Hence $\sigma(\omega\rho^\omega_{M||\overline{\beta}_{m+1}}) \leq \omega\rho^\omega_{N||\sigma(\overline{\beta}_{m+1})}$.

Let $\kappa = \omega\rho^\omega_{M||\overline{\beta}_{m+1}}$.

Let $f : \kappa \rightarrow_{onto} \omega\overline{\beta}_{m+1}$ be in $J^{E^M}_{\overline{\beta}_{m+1}+1}$ by acceptability since $\kappa = \omega\rho^\omega_{M||\overline{\beta}_{m+1}} < \overline{\beta}_{m+1}$ and hence $P(\kappa) \cap (J^{E^M}_{\overline{\beta}_{m+1}+1} - J^{E^M}_{\overline{\beta}_{m+1}}) \neq \emptyset$.

Then $\sigma(f) : \sigma(\kappa) \rightarrow_{onto} \sigma(\omega\overline{\beta}_{m+1})$ and $\sigma(f) \in (J^{E^N}_{\sigma(\overline{\beta}_{m+1})+1} - J^{E^N}_{\sigma(\overline{\beta}_{m+1})})$.

Hence $\sigma(\omega\rho^{\omega}_{M||\overline{\beta}_{m+1}}) \geq \omega\rho^{\omega}_{N||\sigma(\overline{\beta}_{m+1})}$.

Therefore, $\sigma(\omega\rho^{\omega}_{M||\overline{\beta}_{m+1}}) = \omega\rho^{\omega}_{N||\sigma(\overline{\beta}_{m+1})}$.

This gives us that $\omega\rho^{\omega}_{N||\sigma(\overline{\beta}_{m+1})}$ is a cardinal in $N||\sigma(\overline{\beta}_m)$.

It follows that for all $\eta \in [\sigma(\overline{\beta}_{m+1}), \sigma(\overline{\beta}_m))$, $\omega\rho^{\omega}_{N||\sigma(\overline{\beta}_{m+1})} \leq \omega\rho^{\omega}_{N||\eta}$.

Hence $\sigma(\overline{\beta}_{m+1}) \geq \beta(N||\sigma(\overline{\beta}_m), \sigma(\nu))$.

If $\overline{\beta}_{m+1} = \nu$, then $\sigma(\overline{\beta}_{m+1}) = \sigma(\nu)$ and $\sigma(\nu) \leq \beta(N||\sigma(\overline{\beta}_m), \sigma(\nu))$.

Hence if $\overline{\beta}_{m+1} = \nu$ then $\sigma(\overline{\beta}_{m+1}) = \beta(N||\sigma(\overline{\beta}_m), \sigma(\nu))$.

Now we assume that $\overline{\beta}_{m+1} > \nu$. We show that $\sigma(\overline{\beta}_{m+1}) = \beta(N||\sigma(\overline{\beta}_m), \sigma(\nu))$.

Let $\tau = (\omega\rho^{\omega}_{M||\overline{\beta}_{m+1}})^{+M||\nu}$. Then $\tau \leq \omega\rho^{\omega}_{M||\nu}$ is a cardinal in $M||\nu$. Since for $\eta \in [\nu, \overline{\beta}_{m+1})$, $\omega\rho^{\omega}_{M||\overline{\beta}_{m+1}} < \omega\rho^{\omega}_{M||\eta}$, τ is a cardinal in $M||\overline{\beta}_{m+1}$ and $\tau < \nu$.

It follows that $\sigma(\tau)$ is a cardinal in $N||\sigma(\overline{\beta}_{m+1})$ and $\sigma(\tau) < \sigma(\nu)$.

Let $\eta \in [\sigma(\nu), \sigma(\overline{\beta}_{m+1}))$. Then

$$\omega\rho^{\omega}_{N||\sigma(\overline{\beta}_{m+1})} = \sigma(\omega\rho^{\omega}_{M||\overline{\beta}_{m+1}}) < \sigma(\tau) \leq \omega\rho^{\omega}_{N||\eta}.$$

Therefore, $\sigma(\overline{\beta}_{m+1}) = \beta(N||\overline{\beta}_m, \sigma(\nu))$.

□

The following lemma explains one of their usages of these two β–operators.

LEMMA 4.9 Let us assume that N_ξ is defined and the induction hypothesis are maintained below ξ. Then

(i) Let $\nu < ht(N_\xi)$ be such that $E^{N_\xi}_\nu \neq \emptyset$. Let $\beta = \beta(N_\xi, \nu)$. There is exactly one $\eta < \xi$ such that $N_\xi||\beta = M_\eta$.

(ii) Assume that N_ξ is iterable. Let $\nu \leq ht(M_\xi)$ be such that $E^{M_\xi}_\nu \neq \emptyset$. Let $\beta = \beta^+(M_\xi, \nu)$. There is exactly one $\eta \leq \xi$ such that $M_\xi||\beta = M_\eta$.

Proof By induction on ξ, we prove (i) and (ii).

We prove (i) first. For $\xi = 0$, nothing needs a proof.

Case 1 $\xi = \gamma + 1$.

If $M_\gamma = \langle J^E_\alpha, \emptyset\rangle$ and $N_\xi = \langle J^E_\alpha, F\rangle$, then $\omega\rho^{\omega}_{M_\gamma} = Ord^{M_\gamma}$ and hence

$$\beta(N_\xi, \nu) = \beta^+(M_\gamma, \nu).$$

The conclusion follows from our induction hypothesis on (ii).

If $M_\gamma = \langle J^E_\alpha, \emptyset\rangle$ and $N_\xi = \langle J^E_{\alpha+1}, \emptyset\rangle$, then

$$\beta(N_\xi, \nu) = \beta^+(M_\gamma, \nu).$$

The conclusion follows from our induction hypothesis on (ii).

Assume now that $M_\gamma = \langle J^E_\alpha, E_\alpha\rangle$ with $E_\alpha \neq \emptyset$ and $N_\xi = \langle J^E_{\alpha+1}, \emptyset\rangle$.

If $\nu < \alpha$, then

$$\beta(N_\xi, \nu) = \beta^+(M_\gamma, \nu).$$

The conclusion follows from our induction hypothesis on (ii).

If $\nu = \alpha$, then $\beta(N_\xi, \nu) = \alpha$ and $N_\xi || \alpha = M_\gamma$. Hence in this case, the unique $\eta = \alpha$.

Case 2. ξ is a limit ordinal.

First let us observe two simple facts about the $\widetilde{\kappa}^<_{\alpha\xi}$–sequence for $\alpha < \xi$.

Fact 1. For $\alpha < \xi$, $\widetilde{\kappa}^<_{\alpha\xi}$ is a cardinal in M_α.

Let $\alpha < \xi_\alpha < \xi$ be the least realization of $\widetilde{\kappa}^<_{\alpha\xi}$, i.e., the first place that $\omega\rho^\omega_{M_{\xi_\alpha}} = \widetilde{\kappa}^<_{\alpha\xi}$. Then

$$\widetilde{\kappa}^<_{\alpha\xi} = \widetilde{\kappa}^\leq_{\alpha\xi_\alpha} = \widetilde{\kappa}^\leq_{\alpha\xi_\alpha+1}$$

and

$$\widetilde{\mu}^<_{\alpha\xi} = \widetilde{\mu}^\leq_{\alpha\xi_\alpha+1}.$$

Since $M_\alpha || \widetilde{\mu}^<_{\alpha\xi} = M_{\xi_\alpha+1} || \widetilde{\mu}^<_{\alpha\xi}$ and either $\widetilde{\mu}^<_{\alpha\xi} = ht(M_\alpha)$ or $\widetilde{\mu}^<_{\alpha\xi}$ is the cardinal successor of $\widetilde{\kappa}^<_{\alpha\xi}$ in M_α, and $\widetilde{\kappa}^<_{\alpha\xi}$ is a cardinal in $M_{\xi_\alpha+1}$, we have that $\widetilde{\kappa}^<_{\alpha\xi}$ is a cardinal in M_α.

Fact 2. If the sequence $\langle \widetilde{\kappa}^<_{\alpha\xi} \mid \alpha < \xi \rangle$ is not eventually constant, then

$$\sup\{\widetilde{\kappa}^<_{\alpha\xi} \mid \alpha < \xi\} = \sup\{\widetilde{\mu}^<_{\alpha\xi} \mid \alpha < \xi\}.$$

Let $\alpha < \xi$. Let $\alpha < \xi_\alpha < \xi$ be the first realization place of $\widetilde{\kappa}^<_{\alpha\xi}$. Let $\gamma < \xi$ be the least such that $\xi_\alpha < \gamma$ and $\widetilde{\kappa}^<_{\gamma\xi} > \widetilde{\kappa}^<_{\alpha\xi}$. Let $\xi_\gamma > \gamma$ be the first realization place of $\widetilde{\kappa}^<_{\gamma\xi}$. Then $\widetilde{\kappa}^<_{\gamma\xi}$ is the largest cardinal in $M_{\xi_\gamma+1}$. Hence $(\widetilde{\kappa}^<_{\alpha\xi})^{+M_{\xi_\gamma+1}}$ is a cardinal in $M_{\xi_\gamma+1}$ and is $\leq \widetilde{\kappa}^<_{\gamma\xi}$. Therefore, $\widetilde{\mu}^<_{\alpha\xi} \leq \widetilde{\kappa}^<_{\gamma\xi}$.

Now let us go back to our inductive proof in case that ξ is a limit ordinal.

Case 2.1 $\widetilde{\kappa}^<_{\alpha\xi} \geq \nu$ for some $\alpha < \xi$.

Let $\alpha_0 < \xi$ be such that $\widetilde{\kappa}^<_{\alpha_0\xi} \geq \nu$. Then for all $\alpha \in [\alpha_0, \xi)$, $\widetilde{\kappa}^<_{\alpha\xi} \geq \nu$. Pick α large enough such that $\widetilde{\mu}^<_{\alpha\xi} > \beta = \beta(N_\xi, \nu)$.

It follows that

$$\beta = \beta(N_\xi, \nu) = \beta(N_\xi || \widetilde{\mu}^<_{\alpha\xi}, \nu).$$

By our Agreement Condition, $M_\alpha || \widetilde{\mu}^<_{\alpha\xi} = N_\xi || \widetilde{\mu}^<_{\alpha\xi}$. Hence

$$\beta = \beta(N_\xi, \nu) = \beta(M_\alpha || \widetilde{\mu}^<_{\alpha\xi}, \nu).$$

We observe that if $\eta \in [\widetilde{\mu}^<_{\alpha\xi}, ht(M)]$, then

$$\omega\rho^\omega_{M_\alpha||\eta} \geq \widetilde{\kappa}^<_{\alpha\xi}.$$

This is because $\widetilde{\kappa}^<_{\alpha\xi}$ is a cardinal in M_α and $\omega\rho^\omega_{M_\alpha} \geq \widetilde{\kappa}^<_{\alpha\xi}$ by definition.

This gives us that

$$\min\{\omega\rho^\omega_{M_\alpha||\eta} \mid \nu \leq \eta \leq ht(M_\alpha)\} = \min\{\omega\rho^\omega_{M_\alpha||\eta} \mid \nu \leq \eta < \widetilde{\mu}^<_{\alpha\xi}\}$$

since
$$\min\{\omega\rho^{\omega}_{M_\alpha||\eta} \mid \nu \leq \eta \leq ht(M_\alpha)\} < \nu \leq \widetilde{\kappa}^{<}_{\alpha\xi}.$$
Hence, we have $\beta^{+}(M_\alpha, \nu) = \beta(M_\alpha||\widetilde{\mu}^{<}_{\alpha\xi})$.

Therefore, $\beta(N_\xi, \nu) = \beta^{+}(M_\alpha, \nu)$. By induction hypothesis on (ii), we are done.

Case 2.2 For all $\alpha < \xi$, $\widetilde{\kappa}^{<}_{\alpha\xi} < \nu$.

Since this sequence is bounded by $\nu < ht(N_\xi) = \widetilde{\mu}_\xi$, the sequence must be eventually constant by fact 2 above. Let κ be such eventual constant of this sequence. Then κ is the largest cardinal in N_ξ.

Pick $\alpha < \xi$ large enough such that $\widetilde{\kappa}^{<}_{\alpha\xi} = \kappa$ and $\widetilde{\mu}^{<}_{\alpha\xi} > \beta = \beta(N_\xi, \nu)$.

It follows that $M_\alpha||\widetilde{\mu}^{<}_{\alpha\xi} = N_\xi||\widetilde{\mu}^{<}_{\alpha\xi}$.

We observe that $\omega\rho^{\omega}_{N_\xi||\beta} = \omega\rho^{\omega}_{M_\alpha||\beta} = \kappa$.

First, $\omega\rho^{\omega}_{N_\xi||\beta} \geq \kappa$ since κ is a cardinal in N_ξ and $\beta < ht(N_\xi)$.

Secondly, $\omega\rho^{\omega}_{N_\xi||\beta} \leq \kappa$. This is because κ is the largest cardinal in N_ξ and $\omega\rho^{\omega}_{N_\xi||\beta}$ is a cardinal in N_ξ. [By definition, we have that for all $\eta \in [\nu, ht(N_\xi))$, $\omega\rho^{\omega}_{N_\xi||\beta} \leq \omega\rho^{\omega}_{N_\xi||\eta}$. $\omega\rho^{\omega}_{N_\xi||\beta}$ is a cardinal in $N_\xi||\beta$. Hence remains to be a cardinal in N_ξ.]

Hence $\omega\rho^{\omega}_{N_\xi||\beta} = \kappa$.

By definition, we have for $\eta \in [\nu, \beta)$,
$$\kappa < \omega\rho^{\omega}_{M_\alpha||\eta}$$
and for $\eta \in [\beta, \widetilde{\mu}^{<}_{\alpha\xi})$,
$$\kappa \leq \omega\rho^{\omega}_{M_\alpha||\eta}.$$
Since $\kappa = \widetilde{\kappa}^{<}_{\alpha\xi}$ is a cardinal in M_α and $\omega\rho^{\omega}_{M_\alpha} \geq \widetilde{\kappa}^{<}_{\alpha\xi} = \kappa$, we have that for all $\eta \in [\widetilde{\mu}^{<}_{\alpha\xi}, ht(M_\alpha)]$, $\kappa \leq \omega\rho^{\omega}_{M_\alpha||\eta}$.

Therefore, $\beta = \beta(N_\xi, \nu) = \beta^{+}(M_\alpha, \nu)$. By induction hypothesis on (ii), we are done.

This proves (i).

To prove (ii), let $\beta = \beta^{+}(M_\xi, \nu)$. If $\beta = ht(M_\xi)$, then the unique $\eta = \xi$. So we assume that $\beta < ht(M_\xi)$. Hence $\nu < ht(M_\xi)$ and $\beta^{+}(M_\xi, \nu) = \beta(M_\xi, \nu)$.

Then
$$\omega\rho^{\omega}_{N_\xi} = \omega\rho^{\omega}_{M_\xi} \geq \omega\rho^{\omega}_{M_\xi||\beta}.$$
If $\rho^{\omega}_{N_\xi} = ht(N_\xi)$, then $N_\xi = M_\xi$, and $\beta(M_\xi, \nu) = \beta(N_\xi, \nu)$. By (i), we are done.

Let us assume that $\rho^{\omega}_{N_\xi} < ht(N_\xi)$. Since N_ξ is iterable, letting $\tau = (\omega\rho^{\omega}_{N_\xi})^{+N_\xi}$, we have that $\tau = (\omega\rho^{\omega}_{M_\xi})^{+M_\xi}$ and
$$N_\xi||\tau = M_\xi||\tau.$$
If $\tau = ht(N_\xi)$, then $M_\xi = N_\xi$ and $\beta(N_\xi, \nu) = \beta(M_\xi, \nu)$. By (i), we are done.

So assume that τ is a cardinal in N_ξ. Let $\eta \in [\tau, ht(N_\xi))$. Then
$$\omega\rho^{\omega}_{N_\xi||\eta} \geq \tau > \omega\rho^{\omega}_{N_\xi} \geq \omega\rho^{\omega}_{M_\xi||\beta}.$$

If $\tau = ht(M_\xi)$, then $\beta < \tau$ by our assumption.

So assume that τ is a cardinal in M_ξ. Let $\eta \in [\tau, ht(M_\xi))$. Then

$$\omega\rho^\omega_{M_\xi||\eta} \geq \tau > \omega\rho^\omega_{M_\xi} \geq \omega\rho^\omega_{M_\xi||\beta}.$$

Therefore, $\beta < \tau$ in any case.

Hence,

$$\beta(M_\xi, \nu) = \beta(M_\xi||\tau, \nu) = \beta(N_\xi||\tau, \nu) = \beta(N_\xi, \nu).$$

By (i), we are done.

□

We now define our second Minimal Walk to search for the origin of an extender $E^{N_\xi}_\nu$ for $\nu < ht(N_\xi)$.

Lemma 4.10 (Minimal Walk to the Origin of $E^{N_\xi}_\nu$)

Assume that N_ξ is defined and our induction hypothesis of the construction are maintained below ξ. Let $\nu < ht(N_\xi)$ be such that $E^{N_\xi}_\nu \neq \emptyset$. Then there is uniquely a sequence, *Minimal Walk to the Origin of* $E^{N_\xi}_\nu$,

$$\langle(\beta_0, \gamma_0, c_0), (\beta_1, \gamma_1, c_1), \cdots, (\beta_{e(\xi,\nu)}, \gamma_{e(\xi,\nu)}, c_{e(\xi,\nu)})\rangle$$

satisfying the following specifications:

(MWO1) $\beta_0 = ht(N_\xi), \gamma_0 = \xi, c_0 = id\restriction_{N_\xi}$.

(MWO2) $0 < e(\xi, \nu) < \omega$ and $\gamma_0 > \gamma_1 > \cdots > \gamma_{e(\xi,\nu)}$.

(MWO3) $c_i \circ \cdots \circ c_0(\nu) < ht(N_{\gamma_i})$ and $E^{N_{\gamma_i}}_{c_i\circ\cdots\circ c_0(\nu)} \neq \emptyset$ for $i < e(\xi, \nu)$.

(MWO4) $\beta_{i+1} = \beta(N_{\gamma_i}, c_i \circ \cdots \circ c_0(\nu))$ for $i < e(\xi, \nu)$.

(MWO5) γ_{i+1} is the unique $\eta < \gamma_i$ such that $M_\eta = N_{\gamma_i}||\beta_{i+1}$.

(MWO6) c_{i+1} is the core map $\sigma : M_{\gamma_{i+1}} \to N_{\gamma_{i+1}}$ for $i < e(\xi, \nu)$.

(MWO7) $\beta_{e(\xi,\nu)} = ht(M_{\gamma_{e(\xi,\nu)}}) = c_{e(\xi,\nu)-1} \circ \cdots \circ c_0(\nu)$.

Proof Assume that $\nu < ht(N_\xi)$ and that $E^{N_\xi}_\nu \neq \emptyset$.

Set $\gamma_0 = \xi, \beta_0 = ht(N_\xi), c_0 = id\restriction_{N_\xi}$.

Let $\beta_1 = \beta(N_\xi, \nu)$, γ_1 be the unique $\eta < \xi$ such that $N_\xi||\beta_1 = M_\eta$, and

$$c_1 = \text{ the core map } \sigma : M_{\gamma_1} \to N_{\gamma_1}.$$

We have (β_1, γ_1, c_1) defined.

If $\beta_1 = \nu$, then set $e(\xi, \nu) = 1$ and we stop.

If $\beta_1 > \nu$, then we continue as follows:

Since $\nu < ht(M_{\gamma_1})$ and $E^{M_{\gamma_1}}_\nu \neq \emptyset$, we have $c_1(\nu) < ht(N_{\gamma_1})$ and $E^{N_{\gamma_1}}_{c_1(\nu)} \neq \emptyset$.

Let $\beta_2 = \beta(N_{\gamma_1}, c_1(\nu))$.

Let γ_2 be the unique $\eta < \gamma_1$ such that $M_\eta = N_{\gamma_1}||c_1(\nu)$.

Let c_2 be the core map $\sigma : M_{\gamma_2} \to N_{\gamma_2}$.

We have (β_2, γ_2, c_2) defined.

If $\beta_2 = c_1(\nu)$, then we set $e(\xi,\nu) = 2$ and stop.

Otherwise, we continue.

Inductively, assume that (β_m, γ_m, c_m) is defined, $\beta_m > c_{m-1} \circ \cdots \circ c_0(\nu)$, $\beta_m = ht(M_{\gamma_m})$, $E^{M_{\gamma_m}}_{c_{m-1}\circ\cdots\circ c_0(\nu)} \neq \emptyset$ and c_m is the core map $\sigma : M_{\gamma_m} \to N_{\gamma_m}$.

Then $c_m \circ \cdots \circ c_0(\nu) < ht(N_{\gamma_m})$ and $E^{N_{\gamma_m}}_{c_m\circ\cdots\circ c_0(\nu)} \neq \emptyset$.

Let $\beta_{m+1} = \beta(N_{\gamma_m}, c_m \circ \cdots \circ c_0(\nu))$.

Let γ_{m+1} be the unique $\eta < \gamma_m$ such that $M_\eta = N_{\gamma_m} || c_m \circ \cdots \circ c_0(\nu)$.

Let c_{m+1} be the core map $\sigma : M_{\gamma_{m+1}} \to N_{\gamma_{m+1}}$.

We have $(\beta_{m+1}, \gamma_{m+1}, c_{m+1})$ defined.

If $\beta_{m+1} = c_m \circ \cdots \circ c_0(\nu)$, then we set $e(\xi,\nu) = m+1$ and stop.

Otherwise, we continue.

Since $\gamma_0 > \gamma_1 > \cdots > \gamma_m > \gamma_{m+1} \cdots$, we must stop at some point. This proves the lemma.

□

LEMMA 4.11 Let N_ξ be defined and our induction hypothesis are maintained below ξ. Let $\nu < ht(N_\xi)$ be such that $E^{N_\xi}_\nu \neq \emptyset$. Then $k(N_\xi, \nu) = e(\xi,\nu)$. Namely the lengths of the two Minimal Walk sequences are the same. Further more, $\beta_{i+1} = c^{(i)}(\overline{\beta}_{i+1})$ for all $i < e(\xi,\nu)$, where $c^{(i)} = c_i \circ \cdots \circ c_0$.

Proof Let $\langle(\beta_0,\gamma_0,c_0),(\beta_1,\gamma_1,c_1),\cdots,(\beta_{e(\xi,\nu)},\gamma_{e(\xi,\nu)},c_{e(\xi,\nu)})\rangle$ be the Minimal Walk to the Origin of $E^{N_\xi}_\nu$ sequence. Then we have that for all $0 < i < e(\xi,\nu)$, $k(M_{\gamma_i}, c_{i-1} \circ \cdots \circ c_0(\nu)) = k(N_{\gamma_i}, c_i \circ \cdots \circ c_0(\nu))$.

Then by induction, we have $k(N_{\gamma_i}, c_i \circ \cdots \circ c_0(\nu)) = k(N_\xi,\nu) - i$ for all $0 < i < e(\xi,\nu)$.

Hence $k(N_\xi,\nu) = e(\xi,\nu)$.

By induction on $k(N_\xi,\nu)$, we show that $\beta_{i+1}[N_\xi,\nu] = c^{(i)}(\overline{\beta}_{i+1}[N_\xi,\nu])$ for all $0 \leq i < k(N_\xi,\nu)$.

$\beta_1 = \overline{\beta}_1 = c_0(\overline{\beta}_1)$.

By definition, we have $\overline{\beta}_i[M_{\gamma_1},\nu] = \overline{\beta}_{i+1}[N_{\gamma_0},\nu]$. Hence

$$c_1(\overline{\beta}_{i+1}[N_{\gamma_0},\nu]) = \overline{\beta}_i[N_{\gamma_1}, c_1(\nu)]$$

for $1 \leq i < k(N_{\gamma_0},\nu)$, and $k(N_{\gamma_1}, c_1(\nu)] = k(N_{\gamma_0},\nu) - 1$.

By induction hypothesis, we have for all $0 \leq i < k(N_{\gamma_1}, c_1(\nu))$,

$$\beta_{i+1}[N_{\gamma_1}, c_1(\nu)] = c^{(i)}[N_{\gamma_1}, c_1(\nu)](\overline{\beta}_{i+1}[N_{\gamma_1}, c_1(\nu)]).$$

Hence for $1 \leq i \leq k(N_{\gamma_1}, c_1(\nu))$,

$$\beta_{i+1}[N_{\gamma_0},\nu] = \beta_i[N_{\gamma_1}, c_1(\nu)] = c^{(i-1)}[N_{\gamma_1}, c_1(\nu)](c_1(\overline{\beta}_{i+1}[N_{\gamma_0},\nu]))$$

and

$$c^{(i-1)}[N_{\gamma_1}, c_1(\nu)](c_1(\overline{\beta}_{i+1}[N_{\gamma_0},\nu])) = c(i)[N_{\gamma_0},\nu](\overline{\beta}_{i+1}[N_{\gamma_0},\nu]).$$

Therefore, $\beta_{i+1} = c^{(i)}(\overline{\beta}_{i+1})$ for all $0 \leq i < k(N_{\gamma_0}, \nu)$.

□

For each $m \leq e(\xi, \nu)$, let $c^{(m)} = c_m \circ \cdots \circ c_0$. Let $c^* = c^{(e(\xi,\nu))}$.

LEMMA 4.12 Let $0 < m \leq e(\xi, \nu)$. Then
(1) $c^{(m)} : N_\xi || \overline{\beta}_m \to_{\Sigma^*} N_{\gamma_m}$,
(2) If $\kappa < \nu$ is a cardinal in N_ξ, then $c^{(m)}$ is identity on κ.
(3) If $\kappa < \nu$ is a successor cardinal in N_ξ, then $c^{(m)}$ is identity on $\kappa + 1$.

Proof (1) is clear form the definition of the Minimal Walk sequence.
To see (2) and (3), we do induction on m. Let $m = i + 1$. Then

$$c_m : N_{\gamma_i} || \beta_m \to N_{\gamma_m}$$

is the core map. Since $c^{(i)}(\kappa)$ is a cardinal in N_{γ_i}, and $c^{(i)}(\kappa) < c^{(i)}(\nu)$, and $\beta_m \in N_{\gamma_i}$, we must have $\omega\rho^\omega_{N_{\gamma_i}||\beta_m} \geq c^{(i)}(\kappa)$. Hence c_m is identity on $c^{(i)}(\kappa)$.

If κ is a successor cardinal, so is $c^{(i)}(\kappa)$ and we have that $c_m(c^{(i)}(\kappa)) = c^{(i)}(\kappa)$.
Since $c^{(m)} = c_m \circ c^{(i)}$, we are done by induction hypothesis.

□

LEMMA 4.13 Every surviving extender of N_ξ is supercomplete with respect to N_ξ.

Proof Let $F = E^{N_\xi}_\nu$ be a total extender on N_ξ. Let $\kappa = crit(F)$ and $\tau = (\kappa)^{+N_\xi||\nu}$. Then τ is a cardinal in N_ξ. Let c^* and ξ^* be such that $c^* : N_\xi||\nu \to_{\Sigma^*} N_{\xi^*}$ is the core map. Let $\alpha(F, N_\xi)$ be the largest cardinal of N_ξ below ν. Let $\lambda = lh(F)$. Then $\tau \leq \alpha(F, N_\xi) \leq \lambda$.

By the above Lemma, c^* is identity on $\alpha(F, N_\xi)$ and $c^*(\alpha(F, N_\xi))$ is a cardinal in N_{ξ^*} and $c^*(\alpha(F, N_\xi)) \leq c^*(\lambda)$, which is the largest cardinal in N_{ξ^*}.

Let $X \subseteq \lambda$ be countable and let $W \subseteq P(\kappa) \cap N_\xi$ be countable. Let F^* be the top extender of N_{ξ^*} and let $\kappa^* = crit(F^*)$, $\lambda^* = c^*(\lambda)$. Then $\kappa^* = \kappa$ and $c^*(a) = a$ for all $a \in P(\kappa) \cap N_\xi$. Let $W^* = \{c^*(a) \mid a \in W\}$ and let $X^* = \{c^*(\gamma) \mid \gamma \in X\}$. Then $W^* = W$. Since F^* is supercomplete on N_{ξ^*}, there is a strong connection $\overline{\delta} : X^* \to \kappa^*$ with respect to (X^*, W^*). We then define that $\delta(\gamma) = \overline{\delta}(c^*(\gamma))$ for each $\gamma \in X$. It follows that δ is a weak connection with respect to (X, W) since

$$\prec \vec{\gamma} \succ \in F(a) \iff \prec c^*(\vec{\gamma}) \succ \in F^*(c^*(a)) \iff \prec \overline{\delta}(c^*(\vec{\gamma})) \succ \in c^*(a) \iff \prec \delta(\vec{\gamma}) \succ \in a.$$

Let $Y \subseteq X \cap \alpha(F, N_\xi)$ be such that $\bigcup_{\gamma \in Y} t_\gamma$ is a well founded relation. Since c^* is identity on $\alpha(F, N_\xi)$, it is the same as $\bigcup_{\gamma \in Y} t_{c^*(\gamma)}$, hence well founded in N_{ξ^*}. Therefore, $\bigcup_{\gamma \in Y} t_{\overline{\delta}(c^*(\gamma))} = \bigcup_{\gamma \in Y} t_{\delta(\gamma)}$ is well founded.

□

Now we are finally ready to show that our construction never breaks down.

LEMMA 4.14 M_ξ is defined for all ξ.

Proof Let ξ be the least such that M_ξ is undefined. Hence, N_ξ is not a weak mouse, i.e., there is a countable Q such that $\sigma : Q \to_{\Sigma^*} N_\xi$ and Q is not countably iterable.

Let $\sigma : Q \to_{\Sigma^*} N_\xi$ be such a witness. We derive a contradiction.

Let $\mathcal{T} = \langle < Q_i >, < \nu_i >, < \eta_i >, < \pi_{ij} >, T \rangle$ be a direct normal iteration.

We are going to show that $\mathcal{T}$ can be uniquely continued. This shall give us the desired contradiction.

Let $\langle \lambda_{ij} \mid i < j < lh(\mathcal{T}) \rangle$, $\langle U_j \mid j < lh(\mathcal{T}) \rangle$, and $\langle \kappa_{ij} \mid i \in U_j, j < lh(\mathcal{T}) \rangle$ be the objects associated with the iteration as defined in Definition 4.1.

We need to define two integral valued functions associated with the iteration in the following.

For each $i < lh(\mathcal{T})$, we define $t(i) \in \omega$ as follows:

Recall that $\eta_i =$ the maximum $\eta \leq ht(Q_{T(i+1)})$ such that

$$(\kappa_i^+)^{Q_{T(i+1)}||\eta} = (\kappa_i^+)^{Q_i||\nu_i}.$$

Let $\tau_i = (\kappa_i^+)^{Q_i||\nu_i}$. Since τ_i is a cardinal in $Q_{T(i+1)}||\nu_{T(i+1)}$, we have $\nu_{T(i+1)} \leq \eta_i \leq ht(Q_{T(i+1)})$. If $\nu_{T(i+1)} < \eta_i < ht(Q_{T(i+1)})$, then $\eta_i \in B(Q_{T(i+1)}, \nu_{T(i+1)})$ since

$$\omega\rho^\omega_{Q_{T(i+1)}||\eta_i} < \tau_i \leq \omega\rho^\omega_{Q_{T(i+1)}||\eta}$$

for all $\eta \in [\nu_{T(i+1)}, \eta_i)$.

Let $\langle \overline{\beta}_0, \overline{\beta}_1, \cdots, \overline{\beta}_{k(Q_{T(i+1)}, \nu_{T(i+1)})} \rangle$ be the Minimal Walk to $\nu_{T(i+1)}$ sequence. Then there is a unique m such that $0 \leq m \leq k(Q_{T(i+1)}, \nu_{T(i+1})$ and $\eta_i = \overline{\beta}_m$.

We then define $t(i) = m \iff \eta_i = \overline{\beta}_m[Q_{T(i+1)}, \nu_{T(i+1)}]$.

Next, we define $s(i,j) \in \omega$ for $i \in U_j$.

Let $i \in U_j$. Recall that $T_\wedge(i,j)$ is the joint point of i and j on the tree and $T_\wedge(i,j) \in U_j$ and $\kappa_{ij} = \min\{\kappa_{T_\wedge(i,j),i}, \kappa_{T_\wedge(i,j),j}\}$. Hence $\kappa_{ij} < \lambda_{T_\wedge(i,j)} \leq \lambda_i$.

Let $\tau_{ij} = (\kappa_{ij}^+)^{Q_i||\nu_i}$.

Let $\eta_{ij} =$ the maximum $\eta \leq ht(Q_i)$ such that τ_{ij} is a cardinal in $Q_i||\eta$. We have $\nu_i \leq \eta_{ij} \leq ht(Q_i)$.

By the maximality, $\eta_{ij} \in B(Q_i, \nu_i) \cup \{ht(Q_i)\}$, since if $\eta_{ij} < ht(Q_i)$, then

$$\omega\rho^\omega_{Q_i||\eta_{ij}} < \tau_{ij} \leq \omega\rho^\omega_{Q_i||\eta}$$

for all $\eta \in [\nu_i, \eta_{ij})$.

Let $\langle \overline{\beta}_0, \overline{\beta}_1, \cdots, \overline{\beta}_{k(Q_i,\nu_i)} \rangle$ be the Minimal Walk to ν_i sequence. Then there is a unique m such that $0 \leq m \leq k(Q_i, \nu_i)$ and $\eta_{ij} = \overline{\beta}_m$.

We then define $s(i,j) = m \iff \eta_{ij} = \overline{\beta}_m[Q_i, \nu_i]$.

We are going to construct $\sigma_i : Q_i \to N_{\xi_i}$ so that

(a) $\xi_0 = \xi$, $\sigma_0 = \sigma$,

(b) If $i <_T j$ and j is simple above i, then $\xi_i = \xi_j$ and $\sigma_j \pi_{ij} = \sigma_i$, and

(c) Let $j = i+1$ and $h = T(j)$ and let $n = t(i)$. (Hence $\eta_i = \overline{\beta}_n[Q_h, \nu_h]$.) Let $c^{(n)} = c^{(n)}[\xi_h, \sigma_h(\nu_h)]$, derived from the Minimal Walk sequence associated with $(N_{\xi_h}, \sigma_h(\nu_h))$. Then

$$\sigma_j \pi_{hj} = c^{(n)} \sigma_h \restriction_{Q_h || \overline{\beta}_n}.$$

(Here, $c^{(0)} = id$, $\overline{\beta}_0 = ht(Q_h)$.)

(d) Let $i \in U_j$. Let $n = s(i,j)$. (Hence, $\eta_{ij} = \overline{\beta}_n[Q_i, \nu_i]$.) Let $c^{(n)} = c^{(n)}[\xi_i, \sigma_i(\nu_i)]$, $c^* = c^*[\xi_i, \sigma_i(\nu_i)]$ and $\xi_i^* = \gamma_{e(\xi_i, \sigma_i(\nu_i))}$ be derived from the Minimal Walk sequence associated with $(N_{\xi_i}, \sigma_i(\nu_i))$. Then $\sigma_j \restriction_{\kappa_{ij}} = c^{(n)} \sigma_i \restriction_{\kappa_{ij}}$, and $\sigma_j[\lambda_{ij}] \subseteq c^{(n)} \sigma_i[\kappa_{ij}]$, and whenever $X \subseteq \lambda_{ij}$ such that

$$\bigcup \{t_\alpha \in N_{\xi_i^*} \mid \alpha \in c^* \sigma_i[X]\}$$

is well founded, then

$$\bigcup \{t_\alpha \in N_{\xi_j} \mid \alpha \in \sigma_j[X]\}$$

is well founded.

We construct these σ_j by induction on j, verifying the four requirements at each stage.

Case1. $j = 0$. This is already given.

Case2. $j = i+1$. Let $h = T(j)$. Then if $h < i$ then $h \in U_i$ and if $h = i$ then κ_{hi} is λ_h by definition.

Case2.1 $h = i$. Hence $\kappa_{hi} = \lambda_{hi} = \lambda_i$.

Let $\langle \overline{\beta}_m \mid m \leq k(Q_i, \nu_i) \rangle$ be the Minimal Walk to ν_i sequence.

Let $\eta = \eta_i$, which is the maximum $\gamma \leq ht(Q_i)$ such that τ_i is a cardinal in $Q_i || \gamma$.

Let $n = t(i)$. Hence $\eta_i = \overline{\beta}_n[Q_i, \nu_i]$.

Let

$$\langle (\beta_0, \gamma_0, c_0), (\beta_1, \gamma_1, c_1), \cdots, (\beta_{e(\xi_i, \sigma_i(\nu_i))}, \gamma_{e(\xi_i, \sigma_i(\nu_i))}, c_{e(\xi_i, \sigma_i(\nu_i))}) \rangle$$

be the Minimal Walk Sequence of $\sigma_i(E_{\nu_i}^{Q_i})$.

Let $p = k(Q_i, \nu_i) = e(\xi_i, \sigma_i(\nu_i))$.

Let $c^* = c_p \circ \cdots \circ c_0 = c^{(p)}[\xi_i, \sigma_i(\nu_i)]$, $\xi^* = \gamma_p[\xi_i, \sigma_i(\nu_i)]$ and $c^{(n)} = c^{(n)}[\xi_i, \sigma_i(\nu_i)]$, where n is given above, i.e., determined by $\eta = \overline{\beta}_n [Q_i, \nu_i]$.

Set $\xi_i^n = \gamma_n[\xi_i, \sigma_i(\nu_i)]$. Then

$$c^{(n)} : N_{\xi_i} || \sigma_i(\eta) \rightarrow_{\Sigma^*} N_{\xi_i^n}$$

and

$$c^* = \widetilde{\sigma} \circ c^{(n)} : N_{\xi_i} || \sigma_i(\nu_i) \rightarrow_{\Sigma^*} N_{\xi^*}$$

such that $\widetilde{\sigma} \restriction_{c^{(n)}(\sigma_i(\tau_i+1))} = id$. In particular, for $X \in P(\kappa_i) \cap J_{\nu_i}^{E^{Q_i}}$, we have

$$c^* \circ \sigma_i(X) = c^{(n)} \circ \sigma_i(X).$$

Let $N = N_{\xi^*}$. Then $N = \langle J_\nu^{E^N}, E_\nu \rangle$ with $E_\nu \neq \emptyset$ and E_ν is supercomplete with respect to N. $c^* \circ \sigma_i[J_{\lambda_i}^{E^{Q_i}}]$ is a countable subset of $J_\nu^{E^N}$.

Choose

$$\widetilde{\delta} : c^* \circ \sigma_i[J^{E^{Q_i}}_{\lambda_i}] \rightarrow c^* \circ \sigma_i(J^{E^{Q_i}}_{\kappa_i})$$

such that, setting $\delta(\alpha) = \widetilde{\delta} \circ c^* \circ \sigma_i(\alpha)$ for $\alpha < \lambda_i$,

(i) $\prec \delta(\vec{\alpha}) \succ \in c^*\sigma_i(X) \iff \prec c^*\sigma_i(\vec{\alpha}) \succ \in E_\nu(c^*\sigma_i(X)) \iff \prec \vec{\alpha} \succ \in E^{Q_i}_{\nu_i}(X)$ for all $X \in P(\kappa_i) \cap J^{E^{Q_i}}_{\nu_i}$ and $\vec{\alpha} \in [\lambda_i]^{<\omega}$.

(ii) If $X \subseteq J^{E^{Q_i}}_{\lambda_i}$ is such that $\bigcup c^*\sigma_i[X]$ is a well founded relation, then $\bigcup \widetilde{\delta}[c^*\sigma_i[X]]$ is a well founded relation.

Set $\sigma_j(\pi_{ij}(f)(\alpha)) = c^{(n)}\sigma_i(f)(\delta(\alpha))$ and set $\xi_j = \xi^n_i$.

Then $\sigma_j : Q_j \rightarrow N_{\xi_j}$,

$$\sigma_j[\lambda_{ij}] \subseteq c^{(n)} \circ \sigma_i[\kappa_{ij}] = c^{(n)} \circ \sigma_i[\kappa_i] = c^* \circ \sigma_i[\kappa_i],$$

and $\sigma_j\pi_{ij} = \sigma^{(n)} \circ \sigma_i$ since $\kappa_{ij} = \min(\kappa_i, \kappa_{ii}) = \kappa_i$, and λ_{ij} is λ_i if κ_i is of type 1 and λ_{ij} is κ_i if κ_i is of type 0.

Thus (b) and (c) holds at j.

We verify that (d) holds at (i,j). Let $X \subseteq J^{E^{Q_i}}_{\lambda_i}$ be such that $\bigcup c^* \circ \sigma_i[X]$ is well founded. Note that in our case, $i \in U_j$ if and only if $\lambda_{ij} > \sup_{h<i}\lambda_h$ if and only if either κ_i is of type 1 or $\kappa_i > \sup_{h<i}\lambda_h$. Let $h < i$. Then $\lambda_{hi} \leq \lambda_h \leq \kappa_i$ and hence $\lambda_{hj} = \lambda_{hi}$, and $\kappa_{hj} = \kappa_{hi}$. Hence $h \in U_j$ if and only if $i \in U_j$. Let $h \in U_j$. Then

$$\sigma_j[\lambda_{hj}] = \sigma_i[\lambda_{hj}] \subseteq c^{(t(h))} \circ \sigma_h[\kappa_{hj}].$$

(Recall that $c^{(t(h))} = c^{(t(h))}[\xi_h, \sigma_h(\nu_h)]$.) Moreover, if $Y \subseteq J^{E^{Q_h}}_{\lambda_{hj}}$ is such that $\bigcup c^*_h \circ \sigma_h[Y]$ is well founded, then $\bigcup \sigma_j[Y] = \bigcup \sigma_i[Y]$ is well founded. (Recall that $c^*_h = c^*[\xi_h, \sigma_h(\nu_h)] = c^{(e(\xi_h, \sigma_h(\nu_h))}[\xi_h, \sigma_h(\nu_h)]$.)

This verifies (d).

Case2.2 $h < i$. (Hence $h \in U_i$ since $h = T(i+1)$.)

Case2.2.1 $\kappa_i < \kappa_{hi}$.

Let $n = t(i)$. Set $c^{(n)}_i = c^{(n)} \circ \sigma_i$ and $c^*_i = c^*\sigma_i$, where $c^{(n)} = c^{(n)}[\xi_i, \sigma_i(\nu_i)]$ and $c^* = c^*[\xi_i, \sigma_i(\nu_i)]$.

Let $m = s(h,i)$. Then $\eta_{hi} = \overline{\beta}_m[Q_h, \nu_h]$ and $\eta_i = \overline{\beta}_n[Q_h, \nu_h]$. Then $\eta_{hi} \leq \eta_i$ since $\kappa_i < \kappa_{hi}$. Hence $m \geq n$ and $n = m$ iff $\eta_{hi} = \eta_i$. Then $c^{(m)}_h = \widetilde{\sigma}_1 \circ c^{(n)}_h$, where $\widetilde{\sigma}_1\restriction_{c^{(n)}_h(\kappa_i)+1} = id$. Hence

$$c^{(n)}_h\restriction_{\kappa_i+1} = c^{(m)}_h\restriction_{\kappa_i+1}.$$

But $c^*_h = \widetilde{\sigma} c^{(m)}_h$, where $\widetilde{\sigma}\restriction_{c^{(m)}_h(\kappa_{hi}+1)} = id$. Hence

$$c^*_h\restriction_{(\kappa_{hi}+1)} = c^{(m)}_h\restriction_{(\kappa_{hi}+1)}$$

and

$$c^*_h\restriction_{\kappa_i+1} = c^{(n)}_h\restriction_{\kappa_i+1}.$$

We know that $c_h^{(n)}\restriction_{\kappa_{hi}} = \sigma_i\restriction_{\kappa_{hi}}$. Since $\lambda_h < \lambda_i$ is a cardinal in Q_i, we have $c_i^*\restriction_{\lambda_h} = \sigma_i\restriction_{\lambda_h}$. Since F^* = the top extender of $N_{\xi_i^*}$ is supercomplete, (where $\xi_i^* = \gamma_{e(\xi_i,\sigma_i(\nu_i))}$), there is a strong connection

$$\widetilde{\delta} : c_i^*[J_{\lambda_i}^{E^{Q_i}}] \to c_i^*(J_{\kappa_i}^{E^{Q_i}})$$

such that, letting $\delta = \widetilde{\delta}\, c_i^*$, we have

(a) $\prec\delta(\vec{\alpha})\succ \in c_i^*(X) \iff \prec\vec{\alpha}\succ \in F(X)$ for all $X \in P(\kappa_i) \cap J_{\nu_i}^{E^{Q_i}}$ and $\vec{\alpha} < \lambda_i$, where $F = E_{\nu_i}^{Q_i}$, and

(b) if $X \subseteq J_{\lambda_i}^{E^{Q_i}}$ is such that $\bigcup c_i^*[X]$ is well founded, then $\bigcup \delta[X]$ is well founded.

We then define that

$$\sigma_j(\pi_{hj}(f)(\alpha)) = c_h^{(n)}(f)(\delta(\alpha)).$$

Hence (i) $\sigma_j\pi_{hj} = c_h^{(n)}$ and $\sigma_j\restriction_{\lambda_i} = \delta$.

(ii) $\sigma_j\restriction_{\kappa_i} = c_h^{(n)}\restriction_{\kappa_i}$.

(iii) $\sigma_j[\lambda_i] \subseteq c_h^{(n)}(\kappa_i)$.

(iv) Let $X \subseteq J_{\lambda_i}^{E^{Q_i}}$ be such that $\cup\sigma_i^*[X]$ is well founded. Then $\cup\sigma_j[X]$ is well founded.

In particular, (b) and (c) hold. We prove that (d) holds as well.

If κ_i is of type 1, then $\lambda_{ij} = \lambda_i$ and $\lambda_{hj} = \lambda_{hi}$ for $h < i$.

If κ_i is of type 0, then $\lambda_{lj} = \min(\lambda_{li}, \kappa_i)$. Hence, if κ_i is of type 0, then $\lambda_{lj} = \kappa_i < \lambda_h$ for $h \leq l < j$. It follows that $l \notin U_j$ for $h < l < j$.

For $l < h$, (d) follows (with l in place of i) exactly as before, since $\kappa_{li} = \kappa_{lj}$, $\lambda_{li} = \lambda_{lj}$ and $\sigma_j\restriction_{\lambda_l} = \sigma_i\restriction_{\lambda_l}$.

For $l = h$, (d) follows as before.

Now let $l > h$. Then κ_i is of type 1 since otherwise $l \notin U_j$. Hence $\kappa_i < \kappa_{li}$ since $\kappa_{li} \leq \kappa_i < \lambda_{li}$ is impossible for κ_i of type 1. Then we have $\kappa_{lj} = \kappa_i$ and $\lambda_{lj} = \lambda_{li}$.

Let $X \subseteq J_{\lambda_{li}}^{E^{Q_j}} = J_{\lambda_{li}}^{E^{Q_i}} = J_{\lambda_{li}}^{E^{Q_l}}$ be such that $\cup c_l^*[X]$ is well founded. Then $\cup\sigma_i[X] \subseteq J_{\kappa_{li}}^{E^{Q_i}}$ is well founded. λ_l is a cardinal in Q_i. It follows that $c_i^*\restriction_{\kappa_{li}} = \sigma_i\restriction_{\kappa_{li}}$ and $\cup c_i^*[X]$ is well founded. Hence $\cup\sigma_j[X]$ is well founded.

Case2.2.2 $\kappa_{hi} \leq \kappa_i$.

Then κ_i is of type 0. It then follows that $\kappa_i < \lambda_{hi}$.

We also have that $\eta_{hi} \geq \eta_i$. Let m and n be such that $\eta_{hi} = \overline{\beta}_m[Q_h, \nu_h]$ and $\eta_i = \overline{\beta}_n[Q_h, \nu_h]$. Then $m \leq n$.

Let $e(h) = e(\xi_h, \sigma_h(\nu_h))$ and let

$$\langle(\beta_0, \gamma_0, c_0), (\beta_1, \gamma_1, c_1), \cdots, (\beta_{e(h)}, \gamma_{e(h)}, c_{e(h)})\rangle$$

be the Minimal Walk Sequence associated with $(N_{\xi_h}, \sigma_h(\nu_h))$. Let $c_h^{(n)} = c^{(n)}\sigma_h$ and $c_h^* = c^{(e(h))}\sigma_h$. Let $\xi_h^n = \gamma_n[\xi_h, \sigma_h(\nu_h)]$ and $\xi_h^* = \gamma_{e(h)}[\xi_h, \sigma_h(\nu_h)]$.

We want to define $\delta : \lambda_h \to c_h^{(n)}(\kappa_i)$ so that whenever $\vec{\alpha} < \lambda_h$ and $X \in P(\kappa_i) \cap J_{\lambda_h}^{E^{Q_h}}$, we have that

$$\prec\delta(\vec{\alpha})\succ \in c_h^{(n)}(X) \iff \prec\vec{\alpha}\succ \in E_{\nu_i}^{Q_i}(X).$$

We can then set

$$\sigma_j(\pi_{hj}(f)(\alpha)) = c_h^{(n)}(f)(\delta(\alpha))$$

for $\alpha < \lambda_h$ and $f : \kappa_i \to Q_i^* = Q_h || \eta_i$ within the domains.

Let $\langle \alpha_m \mid m < \omega \rangle$ be an enumeration of λ_h and let $\langle X_m \mid m < \omega \rangle$ be an enumeration of $P(\kappa_i) \cap J_{\lambda_i}^{E^{Q_i}}$. The existence of δ says that a certain relation is not well founded. Let $\overline{v}_p$ $(p < \omega)$ be the set of all $f : p \to \kappa_i$ such that for all $\vec{k} < p$ and all $l < p$,

$$\prec f(\vec{k}) \succ \in X_l \iff \prec \alpha_{k_1}, \cdots, \alpha_{k_{|\vec{k}|}} \succ \in E_{\nu_i}^{Q_i}(X_l).$$

Then $\overline{v}_p \in Q_i^*$ since $\overline{v}_p \in Q_i || \lambda_h = Q_h || \lambda_h$.

Let $v_p = c_h^{(n)}(\overline{v}_p) \in N_{\xi_h^n}$. Note that $c_h^* = \widetilde{\sigma} c_h^{(n)}$ with

$$\widetilde{\sigma} \restriction_{c_h^{(n)}(\kappa_i+1)} = id.$$

(In fact, $\widetilde{\sigma}$ is identity on $c_h^{(n)}(\tau_i) + 1$ since $c_h^{(n)}(\tau_i) < c_h^{(n)}(\nu_h)$ is a successor cardinal in N_{γ_n}, where $\tau_i = (\kappa_i^+)^{Q_h || \nu_h}$.)

Hence $\widetilde{\sigma}(v_p) = v_p$ and $v_p = c_h^*(\overline{v}_p)$.

Set $\overline{r}_p =$ the set of $\langle a, b \rangle$ and $\langle c, d \rangle$ such that $p > a > c$ and $b \in v_a$, $d \in v_c$ and $d \subseteq b$.

Let $r_p = c_h^{(n)}(\overline{r}_p)$. Then $r_p = c^*(\overline{r}_p)$ for the same reason as above.

Set $R = \bigcup \{r_p \mid p < \omega\}$. The existence of δ reduces to show that R is ill–founded.

To see this, let $R' = \bigcup \{\sigma_i(\overline{r}_p) \mid p < \omega\}$. Note that $\overline{r}_p \in J_{\lambda_{hi}}^{E^{Q_i}}$. By the ω–completeness of F^*, the top extender of $N_{\xi_i^*}$, there is a $\delta' : \omega \to \sigma_i(\kappa_i)$ such that for all $\vec{k} < \omega$ and all $m < \omega$, we have that

$$\prec \delta'(\vec{k}) \succ \in \sigma_i(X_m) \iff \prec \vec{\alpha_k} \succ \in F^*(c_i^*(X_m)).$$

Recall that $\sigma_i \restriction_{\lambda_h} = c_i^* \restriction_{\lambda_h}$ since $\lambda_h > \kappa_i$ is a cardinal in Q_i and $\lambda_h < \lambda_i < \nu_i$. It follows then for all $m < \omega$,

$$(\delta' \restriction_{(m+1)}, \delta' \restriction_m) \in R'.$$

Hence R' is ill–founded. So R is ill–founded since (d) holds at i.

This gives us a desired δ and hence this completes the construction of σ_j in the case that $j = i + 1$ and $\kappa_{hi} \leq \kappa_i$. We now verify the required properties.

First we observe the following:

(1) $\sigma_j \restriction_{\kappa_i} = c_h^{(n)} \restriction_{\kappa_i}$. For $\alpha < \kappa_i$, let $X = \{\alpha\}$. Then

$$\delta(\beta) \in c_h^{(n)}(X) = \{c_h^{(n)}(\alpha)\} \iff \beta \in E_{\nu_i}^{Q_i}(\{\alpha\}) = \{\alpha\}.$$

Also, $\delta(\alpha) = \sigma_j(\alpha)$.

(2) $\sigma_j \pi_{hj} = c_h^{(n)}$. Hence (b) and (c) holds.

To see that (d) holds, notice that since κ_i is of type 0, we have that $\lambda_{hj} = \kappa_{hj} = \kappa_i$. Thus $\lambda_{lj} = \kappa_i$ for $l > h$. Hence $l \notin U_j$ for $l > h$.

Let $l \in U_j$ (hence $l \leq h$). If $l = h$, then (d) is trivial since $\kappa_{hj} = \lambda_{hj} = \kappa_i$ and hence $\sigma_j \restriction_{\lambda_{hj}} = c_h^{(n)} \restriction_{\lambda_{hj}}$.

For $l < h$, we have that $\lambda_l < \kappa_i$ and $\kappa_{lj} = \kappa_{lh}$ and $\lambda_{lj} = \lambda_{lh}$. Then we have

$$c_h^{(n)}\restriction_{\lambda_l} = \sigma_h\restriction_{\lambda_l}$$

since λ_l is a cardinal in Q_h and $\lambda_l < \nu_h$. Hence $\sigma_j\restriction_{\lambda_{lj}} = \sigma_h\restriction_{\lambda_{lj}}$ and (d) holds for l at j since it holds for l at i.

This finishes the case 2.2.2.

Case3. j is a limit ordinal.

Let $i <_T j$ be such that j is simple above i. Then we simply set $\xi_j = \xi_i$ and σ_j is defined canonically so that $\sigma_j \pi_{hj} = \sigma_h$ for all $h <_T j$. The requirements are satisfied by induction hypothesis and the properties of the sequences κ_{mn} and λ_{mn}.

This finishes the definition of σ_j. We now show that $\mathcal{T}$ can be uniquely continued.

Case 1 $lh(\mathcal{T}) = i + 1$.

Let $\nu > \nu_i$ be the least such that $E_\nu^{Q_i} \neq \emptyset$. Notice that if there is no such extender available from Q_i, then we are done as far as normal iterability concerned.

Let $\nu_{i+1} = \nu$ and $\kappa_{i+1} = crit(E_\nu^{Q_i})$ and $\lambda_{i+1} = E_\nu^{Q_i}(\kappa_{i+1})$.

Let $h \leq i$ be the least such that $(h = i$ or $\kappa_{i+1} < \lambda_h)$.

Let $\eta_{i+1} =$ the maximal $\eta \leq ht(Q_h)$ such that

$$\tau_{i+1} = (\kappa_{i+1}^+)^{Q_i||\nu_{i+1}} = (\kappa_{i+1}^+)^{Q_h||\eta}.$$

Then $\eta_{i+1} \in B(Q_h, \nu_h) \cup \{ht(Q_h)\}$.

Let $\sigma_h : Q_h \rightarrow_{\Sigma^*} N_{\xi_h}$ and let

$$Q_{i+1} = ult^*(Q_h||\eta_{i+1}, E_{\nu_{i+1}}^{Q_i}).$$

We check that Q_{i+1} is well founded. To see this, let n be such that $\eta_{i+1} = \overline{\beta}_n[Q_h, \nu_h]$.

Let $c^{(n)} = c^{(n)}[N_{\xi_h}, \sigma_h(\nu_h)]$, $c^* = c^{(e(\xi_h, \sigma_h(\nu_h))}[N_{\xi_h}, \sigma_h(\nu_h)]$, and $\xi_h^* = \gamma^*[N_{\xi_h}, \sigma_h(\nu_h)]$.

Then we define $\pi(\langle \alpha, f\rangle) = c^{(n)}\sigma_h(f)(c^*\sigma_h(\alpha))$. It then follows that

$$\pi : Q_{i+1} \rightarrow_{\Sigma_0} N_{\xi_h^*}.$$

Therefore, Q_{i+1} is well founded.

Case 2 $lh(\mathcal{T}) = \theta$ is a limit ordinal.

Let b be the unique cofinal branch of $\mathcal{T}$. Let $i \in b$ be large such that $[i, \theta)_T$ is simple. Let $\xi = \xi_i$. Then for all $j \in [i, \theta)_T$, $\xi = \xi_j$ and $\sigma_j : Q_j \rightarrow_{\Sigma^*} N_\xi$. Therefore, the direct limit along the branch b is well founded since it can be embedded into N_ξ.

This finishes the proof that M_ξ is defined for every ξ.

□

We now show that at Successor Stage, in Case 1, there is a unique supercomplete extender F of type at most 1 such that $\langle J_\nu^E, F\rangle$ is a premouse.

LEMMA 4.15 Let $M_\xi = \langle J_\nu^E, \emptyset\rangle$ be defined at stage ξ. Assume that both F and G are supercomplete extenders of type at most 1 and that both $\langle J_\nu^E, F\rangle$ and $\langle J_\nu^E, G\rangle$ are premise. Then $F = G$.

Proof Let $M_\xi = \langle J_\nu^E, \emptyset\rangle$ be defined at stage ξ and assume that both F and G are supercomplete extender of type at most 1 and both $\langle J_\nu^E, F\rangle$ and $\langle J_\nu^E, G\rangle$ are premice. Consider the structure $N = \langle J_\nu^E, F, G\rangle$. N is presolid, namely, for $\beta < \nu$, $N||\beta$ is solid. So N is a presolid prebicephalus.

DEFINITION 4.5 A prebicephalus is a structure $\langle J_\alpha^E, F, G\rangle$ such that both $\langle J_\alpha^E, F\rangle$ and $\langle J_\alpha^E, G\rangle$ are premice and both F and G are not empty.

The proof now is to show that N is a bicephalus.

DEFINITION 4.6 A bicephalus is a presolid prebicephalus M such that for all Q, if Q is countable and there is $\sigma : Q \to_{\Sigma_1} M$, then Q is countably normally Σ_0–iterable.

We are going to explain what the Σ_0–iterability of a prebicephalus means in just a moment. The basic point here is that if N is a bicephalus, then $F = G$. Therefore, the uniqueness of next extender is reduced to define Σ_0–iteration of prebicephalus and to show that every bicephalus trivializes and to show that N is a bicephalus.

We first recall the basic theory of Σ_0–iteration.

DEFINITION 4.7 Let $M = \langle J_\alpha^E, F, G\rangle$ be a prebicephalus. For $\nu < \alpha$ and $h < 2$, set $E_{\nu,h}^M = E_\nu^M$. Set $E_{\alpha,0}^M = F$ and $E_{\alpha,1}^M = G$.

For a premouse M, we also set $E_{\nu,h}^M = E_\nu^M$ for $\nu \le ht(M)$ and $h < 2$.

With these notation convention, we can now define that a generalized Σ_0–iteration of a prebicephalus or a premouse M,

$$\mathcal{T} = \langle\langle M_i \mid i < \theta\rangle, \langle < \nu_i, h_i > \mid i \in D\rangle, \langle \eta_i \mid i+1 < \theta\rangle, \langle \pi_{ij}\ i \le_T j\rangle, T\rangle$$

is as that of a generalized iteration with one exception, the requirement (f), where all the occurrences of the index ν_i are replaced by the double index ν_i, h_i and the clause that

$$\pi_{\xi,i+1} : M_\xi||\eta_i \to^*_{E_{\nu_i}^{M_i}} M_{i+1}$$

is replaced by the following: if $i+1$ is simple, then

$$\pi_{\xi,i+1} : M_\xi||\eta_i \to_{E_{\nu_i,h_i}^{M_i},\Sigma_0} M_{i+1}$$

(namely, $M_{i+1} = ult(M_\xi||\eta_i, E_{\nu_i,h_i}^{M_i})$ using only functions which are elements of the model $M_\xi||\eta_i$) and if $i+1$ is not simple, then

$$\pi_{\xi,i+1} : M_\xi||\eta_i \to^*_{E_{\nu_i,h_i}^{M_i}} M_{i+1}.$$

Then all relevant concepts of Σ_0–iteration and Σ_0–iterability are defined in the same way.

DEFINITION 4.8 Let M^h be a premouse or prebicephalus ($h = 0, 1$). The coiteration $\langle \mathcal{T}^0, \mathcal{T}^1 \rangle$ of M^0 and M^1 is the pair of normal Σ_0–iterations:

$$\mathcal{T}^h = \langle < M_i^h >, \langle < \nu_i, l_i^h > \rangle, < \eta_i^h >, < \pi_{ij}^h >, T^h \rangle$$

defined by setting

$M_0^h = M^h$;

ν_i is the least ν such that there are $l, k < 2$ with $E_{\nu,l}^{M_i^0} \neq E_{\nu,k}^{M_i^1}$;

$i \in D^h \iff E_{\nu_i,0}^h \neq \emptyset$;

if $i \notin D^{1-h}$, then set $l_i^h = 0$;

if $i \in D^h \cap D^{1-h}$, let $\langle l_i^0, l_i^1 \rangle$ be the lexicographically least (l, k) such that $E_{\nu_i,l}^{M_i^0} \neq E_{\nu_i,k}^{M_i^1}$.

LEMMA 4.16 (Comparison Lemma) If M^0 and M^1 are normally Σ_0–iterable, then the coiteration terminates. If both are presolid and N^0 and N^1 are the two last iterates, then one side of the coiteration is simple on the main branch and if M^h to N^h is nonsimple then N^{1-h} is a segment of N^h.

Applying this comparison lemma, we can now that bicephali trivialize.

LEMMA 4.17 Let $M = \langle J_\alpha^E, F, G \rangle$ be a bicephalus. Then $F = G$.

Proof By Löwenheim–Skolem, taking a countable elementary submodel if necessary, it suffices to prove the lemma for countable M. Coiterate M against itself and the coiteration terminates in less than ω_1 steps, resulting two structures N and N'. Without loss of generality, we may assume that N is an initial segment of N' and N is a simple iterate of M. Let $N = \langle J_{\overline{\alpha}}^{\overline{E}}, \overline{F}, \overline{G} \rangle$. Then $\overline{F} = \overline{G} = E_{\overline{\alpha},l}^{N'}$ ($l = 0, 1$). Hence $F = G$ since $\pi_{0,\overline{\alpha}} : M \to_{\Sigma_1} N$.

LEMMA 4.18 N is a bicephalus.

Proof Let $\sigma : Q \to_{\Sigma_1} N$. Assume that Q is countable. We show that Q is countably normally Σ_0–iterable.

Let $\mathcal{T}$ be a countable normal Σ_0–iteration of Q of length θ, with

$$\mathcal{T} = \langle < Q_i >, << \nu_i, l_i >>, < \eta_i >, < \pi_{ij} >, T \rangle.$$

We now carry out the proof just as that of showing that M_ξ is defined by defining $\sigma_i : Q_i \to N_{\xi_i}$ in such a way that σ_{i+1} is Σ_0 preserving and cardinal preserving if $i + 1$ is simple in $\mathcal{T}$ and σ_{i+1} is $\Sigma_0^{(n_i)}$ preserving and cardinal preserving (if $n_i = 0$) otherwise, where $n_i = m(crit(E^{Q_i}_{\nu_i,l_i}), Q_{T(i+1)})$, i.e., $n_i = \omega$ if $crit(E^{Q_i}_{\nu_i,l_i}) < \omega\rho^{\omega}_{Q_{T(i+1)}}$, and n_i is the unique n such that $\omega\rho^{n+1}_{Q_{T(i+1)}} \leq crit(E^{Q_i}_{\nu_i,l_i}) < \omega\rho^{n}_{Q_{T(i+1)}}$.

There is no any complication involved in modifying that proof to this new environment. This shows indeed that N is a bicephalus. Hence $F = G$.

This finishes that proof of the uniqueness of next extender.

□

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Institute of Mathematics
Academy of Mathematics and System Sciences
The Chinese Academy of Sciences
Beijing 100080, China
`feng@logic.math.ac.cn`
and
Department of Mathematics
National University of Singapore
Singapore 117543
`matqfeng@math.nus.edu.sg`

Institute of Mathematics
Humboldt University at Berlin
Unter den Linden 6
10099 Berlin, Germany
`jensen@mathematik.hu-berlin.de`