Preliminaries

ZF^- ("ZF without power set") consists of the axioms of extensionality and foundation together with:

(1) $\emptyset, \{x, y\}, \bigcup x$ are sets

(2) (Axiom of Subsets or "Aussonderungsaxiom")

$$x \cap \{z \mid \varphi(z)\} \text{ is a set}$$

(3) (Axiom of Collection)

$$\bigwedge x \bigvee y\, \varphi(x, y) \rightarrow \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y)$$

(4) (Axiom of infinity) ω is a set

<u>Note</u> (3) implies the usual replacement axiom, but cannot be derived from it without the power set axiom.

ZFC^- is ZF^- together with the strong form of the axiom of choice:

(5) Every set is enumerable by an ordinal.

<u>Note</u> The power set axiom is required to derive (5) from the weaker forms of choice.

The Levy hierarchy of formulae is defined in the usual way:
Σ_0 formulae are the formulae containing only bounded quantification — i.e.
Σ_0 = the smallest set of formulae containing the primitive formulae and closed under sentential operations and bounded quantification:

$$\bigwedge x \in y\, \varphi \ , \ \bigvee x \in y\, \varphi$$

(where $\bigwedge x \in y\, \varphi = \bigwedge x\,(x \in y \to \varphi)$ and
$\bigvee x \in y\, \varphi = \bigvee x\,(x \in y \wedge \varphi)$).
(In some contexts it is useful to introduce bounded quantifiers as primitive signs rather than defined operations.)

We set: $\Pi_0 = \Sigma_0$. Σ_{n+1} formulae are then the formulae of the form
$\bigvee x\, \varphi$, where φ is Π_n. Similarly
Π_{n+1} formulae have the form $\bigwedge x\, \varphi$,
where φ is Σ_n.

A relation R on the model $\mathfrak{A}$ is called $\Sigma_n(\mathfrak{A})$ ($\Pi_n(\mathfrak{A})$) iff it is definable over $\mathfrak{A}$ by a Σ_n (Π_n) formula.

R is $\Sigma_n(\mathfrak{A})$ ($\Pi_n(\mathfrak{A})$) in the parameters $p_1, \ldots, p_m$ iff it is Σ_n (Π_n) definable in the parameters $p_1, \ldots, p_m \in \mathfrak{A}$. It is $\underline{\Sigma}_n(\mathfrak{A})$ ($\underline{\Pi}_n(\mathfrak{A})$) iff it is Σ_n (Π_n) definable in some parameters. It is $\Delta_n(\mathfrak{A})$ iff it is $\Sigma_n(\mathfrak{A})$ and $\Pi_n(\mathfrak{A})$.