§3. The fine structure of the constructible hierarchy

Let u be a transitive set which is closed under the formation of finite sets (thus, in particular, u is closed under n-tuples and $\mathrm{Fml}_u \subset u$). In this section, we attempt to determine the p.r. closed levels $L_\alpha[u]$ of the constructible hierarchy over u at which interesting things happen – f.i. when is $L_\alpha[u]$ admissible? It turns out ~~that~~ that this is the case iff for no $\beta < \alpha$ ~~there is~~ a ~~map~~ Δ_1 map of $L_\beta[u]$ onto $L_\alpha[u]$ exists.

~~As a corollary we get: $L_\alpha[u]$ is admissible iff Δ_1 ... $L_\beta[u]$.~~

(Corollary: If α is a singular cardinal and $\overline{\overline{u}} < \alpha$, then $L_\alpha[u]$ is admissible.)

Throughout this section, u will be a fixed transitive set which is closed under finite subsets. $L_\alpha[u]$ will always be p.r. closed. 'p.r.' will always mean 'p.r. in parameters from $u \cup \{u\}$'. We begin with an observation on p.r. functions:

<u>Lemma 1</u> There is a p.r. function $\eta(x,y)$ which maps $u \times \alpha$ onto L_α.

proof.

There is a p.r. map $\langle\,\rangle : On^2 \longleftrightarrow On$ s.t. $\beta, \gamma \leq \langle \beta, \gamma \rangle$ (hence the inverses l, r, defined by $\alpha = \langle l(\alpha), r(\alpha) \rangle$ are p.r.). To see this, we order On^2 by:

$$\langle \alpha, \beta \rangle \, R \, \langle \gamma, \delta \rangle \longleftrightarrow . \max(\alpha,\beta) < \max(\gamma,\delta) \vee$$
$$\vee . \max = \max \wedge \alpha < \gamma . \vee .$$
$$. \vee . \max = \max \wedge \alpha = \gamma \wedge \beta < \delta .$$

Let $\langle\,\rangle : R \longleftrightarrow \in \Gamma On$. To see that $\langle\,\rangle$ is p.r., we first define the function $\langle\langle 0,\beta\rangle \mid \beta\in On\rangle$ by:

$$\langle 0,\beta\rangle = \sup_{\nu<\beta}\left(\langle 0,\nu\rangle + \nu\cdot 2\right),$$

and then set:

$$\langle \nu,\beta\rangle = \langle 0,\beta\rangle + \nu \quad \text{if } \nu<\beta$$

$$\langle \beta,\nu\rangle = \langle 0,\beta\rangle + \beta+\nu \quad \text{if } \nu\le\beta.$$

We can represent n-tuples of ordinals by:

$$\langle \beta_1,\dots,\beta_n\rangle = \langle \beta_1,\langle\beta_2,\dots,\beta_n\rangle\rangle.$$

Define a function $h(z,y)$ by:

$$h(\langle\varphi,n,i\rangle,\gamma) =$$

$$= \varphi\left({}^{v_{i_1},\dots,v_{i_n}}\big/ \underline{L_{\gamma_1}[u]},\dots,\underline{L_{\gamma_n}[u]}\right)$$

if $\varphi\in \mathrm{Fml}^{\varepsilon_0}_{u}$, $n<\omega$, $i<\omega$, $i=\langle i_1,\dots,i_n\rangle$, $\gamma=\langle\gamma_1,\dots,\gamma_n\rangle$.

$h(z,y) = \underline{0}\,\varepsilon\,\underline{0}$ otherwise.

Then h is a p.r. function which maps $u \times \alpha$ onto the set of Σ_0 fmlae φ containing only the constants $\underline{x}$ $(x \in u)$, $\underline{L_\nu[u]}$ $(\nu \in \alpha)$.

We have seen that every $y \in L_\alpha[u]$ has the form:

$$y = \{x \in L_\beta[u] \mid \models \varphi(v_0/\underline{x})\},$$

where $\beta < \alpha$ and φ is such a formula. Hence, we may define the desired function η by:

$$\eta(x, \langle\beta, \gamma\rangle) = \{z \in L_\beta[u] \mid \models h(x,\gamma)(v_0/\underline{z})\}$$

QED.

<u>Def</u> The function $r(\vec{x})$ <u>uniformises</u> the relation $Ry\vec{x}$ iff $\mathrm{dom}(r) = \mathrm{dom}(R)$ and $\bigwedge \vec{x}\ (\bigvee y\, Ry\vec{x} \longleftrightarrow Rr(\vec{x})\vec{x}$.

A structure $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is called Σ_n - <u>uniformisable</u> $(n \geq 1)$ iff each Σ_n relation R s.t. $\mathrm{rng}(R) \subset \mathrm{On}$ is uniformisable by a Σ_n function.

(This notion should really be called 'ordinal uniformisability'. We use it in preference to the stronger notion because the latter may fail for lack of a nice well ordering of $|M|$).

Thm 1. ~~Lemma 2~~ $L_\alpha^{[u]}$ is Σ_1 uniformisable.

proof. Let $R(\nu,\vec{x})$ be a Σ_1 relation s.t. $\mathrm{rng}(R) \subset On$. Let

$$R\nu\vec{x} \longleftrightarrow \bigvee y\, P_{\nu y}\vec{x}\ ,$$

where P is Σ_0. Set:

$$Q\nu\tau\vec{x} \underset{\mathrm{Df}}{\longleftrightarrow} \bigvee y\in u\ P_\nu\,\eta(\tau,y)\,\vec{x}\,.$$

Then Q is p.r. (in the parameters entering the Σ_1 definition of R) and:

$$R\nu\vec{x} \longleftrightarrow \bigvee\tau\ Q\nu\tau\vec{x}\,.$$

Set: $q(\vec{x}) \simeq \mu\gamma\ Q\,l(\gamma)\,r(\gamma)\,\vec{x}$

(where $\langle l(\gamma), r(\gamma)\rangle = \gamma$). Then q is Σ_1, since:

$$\gamma = q(\vec{x}) \longleftrightarrow Q\,l(\gamma)\,r(\gamma)\,\vec{x}\ \wedge$$
$$\wedge \bigwedge\delta<\gamma\ \neg Q\,l(\delta)\,r(\delta)\,\vec{x}\,.$$

Hence, $r(\vec{x}) \simeq l\,q(\vec{x})$ is a Σ_1 function which uniformises R. QED

(**Note**: By the proof of ~~Lemma 2~~ Thm 1, we may take r as being Σ_1 in the same parameters which enter the Σ_1 definition of R).

Def $X \prec_{\Sigma_n} L_\alpha[u]$ ("X is a Σ_n-elementary submodell of $L_\alpha[u]$) $\longleftrightarrow_{Df}$

$\longleftrightarrow_{Df}$ $u \cup \{u\} \subset X$ and for every ~~$\varphi \in Fml$~~ $\varphi \in Fml_X^{\Sigma_n}$:

$$\models_{\langle X, \in \rangle} \varphi \longleftrightarrow \models_{\langle L_\alpha[u], \in \rangle} \varphi .$$

In other words, $X \prec_{\Sigma_n} L_\alpha[u]$ iff $u \cup \{u\} \subset X$ and for each $R \subset L_\alpha[u]^{m+1}$ which is Σ_1 in parameters from X:

$$\bigvee y\, Ry\vec{x} \longleftrightarrow \bigvee y \in X\; Ry\vec{x} \qquad \text{for } \vec{x} \in X.$$

In particular, if $X \prec_{\Sigma_1} L_\alpha[u]$, then X is closed under Σ_1 functions definab[le] with parameters from X. Since p.r. functions are Σ_1 in the parameter ω, X is p.r. closed.

Clearly, $\in \restriction X$ satisfies the axiom of extensionality, since, if $x, y \in X$ and $x \neq y$, then $\bigvee z\,(z \in x \not\leftrightarrow z \in y)$, hence $\bigvee z \in X\,(z \in x \not\leftrightarrow z \in y)$. Therefore there exists a map π of X onto a transitive set υ s.t.

$$\pi : \langle X, \in \rangle \stackrel{\sim}{\longleftrightarrow} \langle \upsilon, \in \rangle.$$

If φ is a Σ_1 fmla (without constants), then

$$(+) \quad \models_{L_\alpha[u]} \varphi(\vec{x}) \longleftrightarrow \models_{\upsilon} \varphi(\underline{\pi(x)}) \quad \text{for } \vec{x} \in X.$$

We may conclude:

$$(++)\quad \pi f(\vec{x}) = f(\overrightarrow{\pi(x)}) \quad \text{for all p.r. } f,$$

since, by the stability lemma, each p.r. f has a Σ_1 definition which is absolute with respect to p.r. closed domains; i.e. there is a Σ_0 fmla φ_f (with constants $\underline{x}$ $(x \in u \cup \{u\} \cup \{\omega\})$) s.t.

$$y = f(\vec{x}) \longleftrightarrow \bigvee z \models \varphi_f(\underline{z}, \underline{y}, \underline{\vec{x}}) \quad \text{for all } y, \vec{x}$$

$$y = f(\vec{x}) \longleftrightarrow \bigvee z \in L_\alpha[u] \models \varphi_f(\underline{z}, \underline{y}, \underline{\vec{x}})$$

for $\vec{x} \in L_\alpha[u]$.

Hence, if $\vec{x} \in X$, we have:

$$y = f(\vec{x}) \longleftrightarrow \bigvee_{z \in L_\alpha[u]} \models \varphi_f(\underline{z}, \underline{y}, \underline{\vec{x}})$$

$$\longleftrightarrow \bigvee z \in \bar{u} \models \varphi_f(\underline{z}, \underline{\pi(y)}, \underline{\overrightarrow{\pi(x)}})$$

$$\longrightarrow \pi(y) = f(\overrightarrow{\pi(x)}). \quad \text{QED } (++).$$

(<u>Note</u> (++) implies that $\bar{u}$ is p.r. closed.

By (++), we get:

<u>Lemma 2</u> If $X \prec_{\Sigma_1} L_\alpha[u]$, $\pi : \langle X, \in\rangle \leftrightarrow \langle \sigma, \in\rangle$ and $\bigcup \sigma \subset \sigma$, then $\bigvee \beta \leq \alpha \quad \sigma = L_\beta$.

proof.
Since $\langle L_\nu[u] \mid \nu \in On\rangle$ is p.r., we have by (++): $\pi\, L_\nu[u] = L_{\pi(\nu)}[u]$.
For all $x \in X$, we have:

$$\bigvee \nu \in L_\alpha[u] \qquad x \in L_\nu[u], \text{ hence}$$
$$\bigvee \nu \in X \qquad x \in L_\nu[u], \text{ hence}$$
$$\bigvee \nu \in X \qquad \pi(x) \in L_{\pi(\nu)}[u].$$

Let β be the least ordinal not in σ.
Then $\beta = \pi\text{``}On \cap X$. Hence:
$$\sigma = \pi\text{``}X = \bigcup_{\nu \in X} L_{\pi(\nu)}[u] = L_\beta[u]$$
QED

Using the fact that $L_\beta[u] \subset L_\alpha[u]$, we can strengthen (++) to:

<u>Lemma 3</u> Let X, π be as in Lemma 2 and let f be a function which is Σ_1 without parameters (or at most parameters $x \in X$ s.t $\pi(x) = x$). Then, whenever $\vec{x} \in X$ and $f(\vec{x})$ is defined, so is $f(\overrightarrow{\pi(x)})$ and

$$\pi f(\vec{x}) = f(\overrightarrow{\pi(x)}).$$

proof.

Let φ be a Σ_1 fmla defining f in $L_\alpha[u]$ (containing at most constants $\underset{\sim}{x}$ s.t. $\pi(x) = x$). Then

$$y = f(\vec{x}) \longleftrightarrow \models_{L_\alpha[u]} \varphi(\underset{\sim}{y}, \vec{\underset{\sim}{x}})$$

$$\longleftrightarrow \models_{L_\beta[u]} \varphi(\underline{\pi(y)}, \underline{\overrightarrow{\pi(x)}})$$

$$\longrightarrow \models_{L_\alpha[u]} \varphi(\underline{\pi(y)}, \underline{\overrightarrow{\pi(x)}})$$

$$\longrightarrow \pi(y) = f(\overrightarrow{\pi(x)})$$

QED

<u>Lemma 4</u> There is a Σ_1 function h s.t. $\mathrm{dom}(h) \subset u \times L_\alpha[u]$ and

$$\bigwedge x \in L_\alpha[u] \quad x \in h``(u \times \{x\}) \prec_{\Sigma_1} L_\alpha[u].$$

proof.

Define $s(z,x)$ by:

$$s(\langle \varphi, i, j\rangle, x) = \varphi(v_i v_j / \underline{u}, \underline{x})$$

if $\varphi \in \mathrm{Fml}_u^{\Sigma_1}$, $i, j < \omega$, $i \neq j$.

$s(z,x) = \underline{0} \in \underline{0}$ otherwise.

Then s is p.r. and maps u onto $\mathrm{Fml}_u^{\Sigma_1}$. Note that, since $\models^{\Sigma_0}$ is p.r., $\models^{\Sigma_1}_{L_\alpha[u]}$ is Σ_1. Set:

$$R\nu\varphi \longleftrightarrow_{\mathrm{df}} \models^{\Sigma_1} \varphi(v_0/\underline{\nu}).$$

and let r uniformise R. Let r be Σ_1 in the parameter p and set: ~~$h(\langle z, w\rangle) = \eta$~~

$$h(\langle z, w\rangle, x) = \eta(z, r(s(w, \langle x, p\rangle)))$$

if $\langle z, w\rangle \in u$; (otherwise undefined).

Clearly, h is Σ_1.

Let $x \in L_\alpha[u]$. Set $X = h''(u \times \{x\})$.

<u>Claim</u> $X \prec_{\Sigma_1} L_\alpha[u]$

Let $A \subset L_\alpha[u]$ be Σ_1 in parameters from X. We must show:

$$\bigvee y\, Ay \longleftrightarrow \bigvee y \in X\, Ay.$$

Let $\vec{z} \in X$ be the parameters of A; since $z_i = h(w_i, x)$ $(w_i \in u)$, A is Σ_1 in parameters from $u \cup \{u, x, p\}$.

Assume: Ay. Then $y = \eta(z, \nu)$ for some $z \in u$, $\nu < \alpha$. The set $A' = \{\nu \mid A\eta(z, \nu)\}$ is Σ_1 in parameters from $u \cup \{u, x, p\}$. Hence there is a φ with constants from $u \cup \{u, x, p\}$ s.t.

$$A'\nu \longleftrightarrow \models^{\Sigma_1} \varphi(\underset{\sim}{\nu}).$$

Set: $y' = \eta(z, s(\varphi))$. Then Ay'.

But $y \in X$, since, letting $\varphi = s(w, \langle x, p\rangle)$,

$$y = \eta(z, r(s(w, \langle x, p\rangle))) = h(\langle z, w\rangle, x)$$

QED

<u>Thm 2</u> The following conditions are equivalent:

(i) There is a Σ_1 $a \subset u$ s.t. $a \notin L_\alpha[u]$

(ii) There is a Σ_1 map from a subset of u onto $L_\alpha[u]$.

proof.

(ii) $\rightarrow$ (i) is trivial, since $a = \{x \mid x \notin f(x)\}$ is Σ_1 but not an element of $L_\alpha[u]$, for if not, we should have:

$$x \notin a \longleftrightarrow x \in f(z)$$

for some z; hence:

$$z \in a \longleftrightarrow z \notin a.$$

(i) $\rightarrow$ (iii). Let $a \subset u$ be Σ_1, $a \notin L_\alpha[u]$. Let a be Σ_1 in x. Set: $X = h''(u \times \{x\})$. Let $\pi : \langle X, \in \rangle \leftrightarrow \langle L_\beta[u], \in \rangle$. Then, if $\varphi(z, \underline{x})$ is the Σ_1 definition of a, we have:

$$z \in a \longleftrightarrow \models_{L_\alpha[u]} \varphi(\underline{z}, \underline{x})$$

$$\longleftrightarrow \models_{L_\beta[u]} \varphi(\underline{z}, \underline{\pi(x)}).$$

Hence a is Σ_1 in $L_\beta[u]$. But this means that $\beta = \alpha$, since otherwise $a \in L_{\beta+1}[u] \subset L_\alpha[u]$. Let h be Σ_1 in the parameter p; in particular let:

$$y = h(z, x) \longleftrightarrow H(p, y, z, x),$$

where H is Σ_1 without parameters.

~~Set: $h'(z,x) =_{Df}$ $\pi h(z, \pi^{-1}(x))$ By (+),~~ $\cancel{\pi h(z, \pi^{-1}(x))}$.

~~we get:~~

$\pi(y) \cancel{= h'(z, \pi(x)) \longleftrightarrow H(p\, \pi(p), \pi(y),)}$ z

Set: $h'(z,x) \simeq \pi h(z, \pi^{-1}(x))$. By (+):

$\pi(y) = h'(z, \pi(x)) \longleftrightarrow H(\pi(p), \pi(y), z, \pi(x))$.

Thus, h' is Σ_1 in $\pi(p)$, and

$h'``u \times \{\pi(x)\} = \pi``X = L_\alpha[u]$.

Set $f(z) \simeq h'(z, \pi(x))$. Then $\mathrm{dom}(f) \subset u$, f is Σ_1 and $f``u = L_\alpha[u]$. QED

As a corollary of Thm 2, we obtain:

<u>Thm 3</u> The following conditions are equivalent:

(a) There is a Δ_1 set $a \subset u$ s.t. $a \notin L_\alpha[u]$

(b) There is a Δ_1 map of u onto $L_\alpha[u]$.

proof.

(b) $\to$ (a) follows as before

We now prove (a) $\to$ (b)

By Thm 2, there exists a Σ_1 map f' s.t. $\mathrm{dom}(f') \subset u$ and $f''u = L_\alpha[u]$. We must replace f' by a Σ_1 map which is defined on the whole of u. Since A is Δ_1, we have:

$$z \in a \longleftrightarrow \bigvee y \; A_0 yz$$

$$z \notin a \longleftrightarrow \bigvee y \; \neg A_1 yz \; ,$$

where A_0, A_1 are Σ_0. In particular,

$$\bigwedge z \in u \; \bigvee y \; (A_0 yz \vee A_1 yz).$$

~~Set: $Gyz \longleftrightarrow A_0 yz \vee A_1 yz$.~~

~~Let g~~

Set: $G\nu z \longleftrightarrow \bigvee y \in L_\nu[u] \, (A_0 yz \vee A_1 yz)$.

Let g uniformise G. Then $g''u$ is unbounded in $L_\alpha[u]$, since if $g''u \subset L_\gamma[u]$, $\gamma < \alpha$, we would have: $a \in L_{\gamma+1}[u] \subset L_\alpha[u]$.

Since f' in Σ_1, we have:

$$y = f'(x) \longleftrightarrow \bigvee z\, Fzyx,$$

where F in Σ_0. Set:

$$\tilde{f}(\nu, x) = \begin{cases} y & \text{if } \bigvee z \in L_\nu[u]\ Fzyx \\ & \quad \text{and } y \in L_\nu[u] \\ 0 & \text{if not}. \end{cases}$$

Then $\tilde{f}$ is p.r.

Set: $f(\langle z, w\rangle) = \tilde{f}(g(z), w)$ if $\langle z, w\rangle \in u$

$f(x) = \emptyset$ otherwise.

Then $f``u = f'``u = L_\alpha[u]$. QED

Non projectible admissible sets

<u>Def</u> Call $M = \langle |M| ; \in, A_1, \ldots, A_n \rangle$ <u>non projectible</u> iff M is admissible and satisfies the stronger replace-ment axiom:

$$\bigwedge u \bigvee v \bigwedge x \in u \, (\bigvee y \, \varphi \longleftrightarrow \bigvee y \in v \, \varphi)$$

where φ is Σ_0.

One easily establishes the following

<u>Lemma</u> Let M be admissible; then the following are equivalent:

(a) M is non projectible

(b) $x \in M \longrightarrow x \cap A \in M$ for every Σ_1 set A.

(c) $x \in M \longrightarrow f''x \in M$ for every Σ_1 map f.

We wish to characterise the α s.t. $L_\alpha[u]$ is non projectible. Our major tool in this endeavour will be:

<u>Lemma 6</u> Let h be as in Lemma 4. Let h be Σ_1 in the parameter x.
~~Let $v \subset L_\alpha[u]$ be transit~~
Let $v \subset L_\alpha[u]$ be transitive, closed under finite sets, and let $u \cup \{x\} \subset v$. Then

$$\bigvee \beta \leq \alpha \quad h''u \times v = L_\gamma[u].$$

proof. Let $X = h''u \times v$.
Obviously, $X \prec_{\Sigma_1} L_\alpha[u]$. Let $\pi : X \overset{\sim}{\longleftrightarrow} L_\beta[u]$. Since $\pi \restriction v = id \restriction v$, we have: $\pi h(z,w) \simeq h(z,w)$ for $z \in u, w \in v$. Hence $\pi \restriction X = id \restriction X$;
$X = \pi''X = L_\beta[u]$. QED

Thm 4 $L_\alpha[u]$ is non projectible iff there is a normal function $\langle \alpha_\nu \mid \nu < \lambda \rangle$ ($Lim(\lambda)$) s.t. $\alpha = \sup_\nu \alpha_\nu$ and $L_{\alpha_\nu}[u] \prec_{\Sigma_1} L_\alpha[u]$ for $\nu < \lambda$.

proof.

($\leftarrow$) Let φ be a Σ_0 fmla. Let $v \in L_\alpha[u]$. Then $v \in L_{\alpha_\nu}[u]$ for some ν. For all $x \in v$, we have:

$$\models_{L_\alpha[u]} \bigvee y\, \varphi(y, \underline{x}) \longleftrightarrow \models_{L_{\alpha_\nu}[u]} \bigvee y\, \varphi(y, \underline{x}).$$

Hence, for $w = L_{\alpha_\nu}[u]$:

$$\models_{L_\alpha[u]} \bigwedge x \in \underline{v}\, (\bigvee y\, \varphi \longleftrightarrow \bigvee y \in \underline{w}\ \varphi).$$

The remaining admissibility axioms hold trivially by the fact that α is a limit ordinal.

($\rightarrow$) Since the set of $\beta < \alpha$ s.t. ~~β~~ $L_\beta[u] \prec_{\Sigma_1} L_\alpha[u]$ is closed, we need only show that it is unbounded. ~~i~~

Let $\nu < \alpha$. <u>Claim</u> There is $\beta < \alpha$ s.t. $\nu < \beta$ and $L_\beta[u] \prec_{\Sigma_1} L_\alpha[u]$.

~~Let $\gamma > \nu$ be a limit ordinal s.t. $x \in L_\gamma[u]$, where~~ Let h be as in Lemma 5 and let $\gamma > \nu$ be a limit ordinal s.t. $x \in L_\gamma[u]$, where h is Σ_1 in the parameter x. By Lemma 5:

$$h''(u \times L_\gamma[u]) = L_\beta[u] \prec_{\Sigma_1} L_\alpha[u]$$

for some $\beta \leq \alpha$. But, by the non projectibility of $L_\alpha[u]$:

$$h''(u \times L_\gamma[u]) \in L_\alpha[u],$$

hence $\beta < \alpha$. QED

Thm 5 $L_\alpha[u]$ is nonprojectible iff there is no Σ_1 function which, for some $\gamma < \alpha$, maps a subset of $L_\gamma[u]$ onto $L_\alpha[u]$.

proof.

($\rightarrow$) trivial

($\leftarrow$) Let $L_\alpha[u]$ be projectible

Then there is a Σ_0 relation R and a $v \in L_\alpha[u]$ s.t. for each $\nu < \alpha$ there is an $x \in v$ with:

$$\bigvee y \, Ryx \text{ but } \neg \bigvee y \in L_\nu[u] \; Ryx.$$

Let r uniformise the relation:

$$\bigvee y \in L_\nu[u] \quad Ryx.$$

Then $g``v$ is unbounded in α.

Let h be as in Lemma 4. Let h, r be Σ_1 in the parameter x and let $v, x \in L_\gamma[u]$, where

γ is a limit ordinal. By Lemma 5:

$$h``u \times L_\gamma[u] = L_\beta[u] \prec_{\Sigma_1} L_\alpha[u].$$

In particular, $g``v \subset L_\beta[u]$; hence $\beta = \alpha$, since $g``v$ is unbounded in α.

~~Set Then~~ Set $f(\langle x, y\rangle) \simeq h(x,y)$ for $x \in u$, $y \in L_\gamma[u]$. Then f is Σ_1; $\operatorname{dom}(f) \subset L_\gamma[u]$ and $\operatorname{rng}(f) = L_\alpha[u]$

QED

We now come to the Thm announced at the outset of this section:

<u>Thm 6</u> $L_\alpha[u]$ is admissible iff there is no Δ_1 function which, for some $\gamma < \alpha$, maps $L_\gamma[u]$ onto $L_\alpha[u]$.

proof.

$(\longrightarrow)$ trivial

($\longleftarrow$) Let $L_\alpha[u]$ not be admissible. Then there is a Σ_1 relation R s.t. $\bigwedge y \bigvee x\, Ryx$ but for some $v \in L_\alpha[u]$, there is no $\gamma < \alpha$ with: ~~$\bigwedge y \in v \bigvee x \in L_\gamma[u]$~~ $\bigwedge x \in v \bigvee y \in L_\gamma[u]\, Ryx$. Let r uniformize the relation

$$\bigvee y \in L_\nu[u]\, Ryx.$$

Then $r``v$ is unbounded in α. r is Σ_1 and defined everywhere. By Thm 5, there is a $\gamma < \alpha$ and a Σ_1 f s.t. $\mathrm{dom}(f) \subset L_\gamma[u]$ and $\mathrm{rng}(f) = L_\alpha[u]$. Let:

$$y = f(x) \longleftrightarrow \bigvee z\; Fzyx,$$

where F is Σ_0. Set:

$$\tilde{f}(\nu, x) =_{\mathrm{df}} \begin{cases} y & \text{if } y \in L_\nu[u] \text{ and } \\ & \bigvee z \in L_\nu[u]\; Fzyx \\ \emptyset & \text{if not} \end{cases}$$

Then $\tilde{f}$ is p.r.

~~Take γ as a limit ordinal large~~

Set: $\bar{f}(\langle x, y\rangle) = \tilde{f}(r(x), y)$; ~~$f(z) = 0$ otherw.~~

Then $\bar{f}$ is defined everywhere and $\bar{f}$"$v \times L_\gamma[u] = f$"$L_\gamma[u]$. As γ is a limit ordinal and $v \subset L_\gamma[u]$, then $\bar{f}$ maps $L_\gamma[u]$ onto $L_\delta[u]$. QED

The projectum

<u>Def</u> $\alpha^* =_{df}$ the least β s.t. there is a $\Sigma_1(L_\alpha[u])$ function mapping a subset of ~~β onto~~ $L_\beta[u]$ onto $L_\alpha[u]$.

α^* is called the projectum of α.

By Thm 5, $L_\alpha[u]$ is nonprojectible iff $\alpha = \alpha^*$.

<u>Thm</u> If $\alpha^* > 0$, then $L_{\alpha^*}[u]$ is ~~admissible~~ non projectible.

proof. ~~If $\alpha^* = \alpha$, $L_{\alpha^*}[u]$ is n~~

If $\alpha^* = \alpha$, the theorem is trivial.

Now let $\alpha^* < \alpha$. There is no $f \in L_\alpha[u]$ mapping a $\gamma < \alpha^*$ onto α^*, for then:

$$g(\langle z, \nu\rangle) = \eta(z, f(\nu)) \quad \text{if } z \in u, \nu < \gamma$$
$$g(z) = 0 \quad \text{if not}$$

would map $L_\gamma[u]$ onto $L_{\alpha^*}[u]$.

By composition, we would obtain a Σ_1 map of $L_\gamma[u]$ onto $L_\alpha[u]$.

But this means that α^* is p.r. closed, for, as we shall show in an appendix, whenever γ is p.r. closed and β is the first p.r. closed ordinal after γ, each $\nu < \beta$ is 1-1 mappable into γ by a map $f \in L_\beta$. If α^* were not p.r. closed, we should have $\gamma < \alpha^* < \beta \leq \alpha$ for such a pair γ, β; ~~$f \in L_\beta[u]$,~~ hence some $f \in L_\beta[u] \subset L_\alpha[u]$ would map γ onto β. But, since α^* is p.r. closed, we may apply Thm 5 to ~~cond~~ conclude that $L_{\alpha^*}[u]$ is non projectible, for otherwise there would be $\gamma < \alpha^*$ mappable onto α^* by an $f \in L_{\alpha^*+1}[u]$.

QED

Σ_n - admissibles

Def $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is called Σ_n- admissible $(n \geq 1)$ iff M is admissible and satisfies the replacement axiom:

$$\bigwedge x \bigvee y\, \varphi \longrightarrow \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\ \varphi$$

for Σ_{n-1} - fmlae φ.

(Thus, 'admissible' = 'Σ_1- admissible')

Def M is called Σ_n- non projectible iff M is admissible and satisfies:

$$\bigwedge u \bigvee v \bigwedge x \in u\, (\bigvee y\, \varphi \longleftrightarrow \bigvee y \in v\ \varphi)$$

for Σ_{n-1}- fmlae φ.

We can readily establish:

(1) M is Σ_n admissible iff $\langle M, \models_M^{\Sigma_{n-1}} \rangle$ is admissible

(2) M is Σ_n non-projectible iff $\langle M, \models_M^{\Sigma_{n-1}} \rangle$ is non-projectible

(3) If M is Σ_n-admissible, then R is Σ_n iff R is Σ_1 in Σ_{n-1} relations.

■ Thus, all the theorems of §1 carry over to Σ_n-admissibles. Some of the theorems in this section carry over. In particular, we shall obtain slightly weaker ~~analogs~~ analogues of Thm 4 – Thm 6.

<u>Lemma 7</u> If $\langle L_\alpha[u], A\rangle$ is admissible, then $\langle L_\alpha[u], A\rangle$ is Σ_2-uniformizable.

proof.

Let R be Σ_1, $rng(R) \subset \alpha$

Let $R\nu\vec{x} \longleftrightarrow \bigvee y\, P y \nu \vec{x}$,

where P is Π_1.

Set: $p(\vec{x}) \simeq \mu\gamma \; \bigvee z \in L_\gamma \; P z\, l(\gamma)\, \vec{x}$,

where $\langle l(\gamma), r(\gamma)\rangle = \gamma$.

Then p is Σ_2, since:

$$\gamma = p(\vec{x}) \longleftrightarrow \underbrace{\bigvee z \in L_\gamma \;\; P z\, l(\gamma)\, \vec{x}}_{\Pi_1} \;\wedge$$

$$\wedge \underbrace{\bigwedge \tau < \gamma \; \bigwedge z \in L_\tau \;\; \neg P z\, l(\tau)\, \vec{x}}_{\Sigma_1} \;,$$

Set: $r(\vec{x}) \simeq l(p(\vec{x}))$. Then r uniformizes R.

QED

(Note: This proof also goes thru on the assumption: Σ_1 in $\Sigma_1 = \Sigma_2$)

Since, if $L_\alpha[u]$ is Σ_n-admissible, $\langle L_\alpha[u], \models^{\Sigma_{n-1}} \rangle$ is admissible and $\Sigma_1(\langle L_\alpha[u], \models^{\Sigma_{n-1}} \rangle) = \Sigma_n(L_\alpha[u])$, we get:

<u>Corollary 7a</u> If $L_\alpha[u]$ is Σ_n admissible, then $L_\alpha[u]$ is Σ_{n+1} uniformisable.

<u>Lemma 8</u> If $L_\alpha[u]$ is Σ_n-uniformisable, then there is a Σ_n function h s.t. $\mathrm{dom}(h) \subset u \times L_\alpha[u]$ and

$$\bigwedge x \, (x \in h''(u \times \{x\}) \prec_{\Sigma_n} L_\alpha[u]).$$

~~Lemma 8~~

Lemma 8 is proved exactly like Lemma 4, which is a special case of it.

Lemma 2 obviously holds with Σ_n in place of Σ_1 $(n \geq 1)$, since $X \prec_{\Sigma_n} L_\alpha[u]$ implies $X \prec_{\Sigma_1} L_\alpha[u]$.

The analogue of Lemma 3 does not hold, but we do get the weaker form:

<u>Lemma 9</u> If $X \prec_{\Sigma_n} L_\alpha[u]$ and $\pi : \langle X, \in \rangle \overset{\sim}{\longleftrightarrow} \langle L_\alpha[u], \in \rangle$, then for every Σ_1 f (which is Σ_1 in parameter $x \in X$ s.t. $\pi(x) = x$):

$$\pi f(\vec{x}) \simeq f(\overrightarrow{\pi(x)}) \text{ for } \vec{x} \in X.$$

The proof is obvious.

Using Lemmas 8, 9 in place of Lemmas 4, 3, we get

<u>Thm 7</u> If $L_\alpha[u]$ is Σ_n-uniformisable, then the Σ_n analogues of Thm 2, Thm 3 hold.

The proofs of Thm 2, Thm 3 can be repeated word for word to obtain Thm 7.

By ~~Condition~~ Lemma 7, then, the Σ_{n+1} analogues of Thms 2, 3 hold whenever $L_\alpha[u]$ is Σ_n admissible.

We shall show later that this result can be greatly strengthened: The hypothesis of Thm 7 is <u>always</u> satisfied. But first we turn to the question of criteria for Σ_n admissibility + non ~~project~~ projectibility.

The Σ_n analogue of Thm 6 does not hold. Fr. ins. letting $L_{\omega_\omega}[u]$ admits no function mapping an element onto the entire domain, yet

$L_{\omega_\omega}[u]$ is not admissible, since $\langle \omega_n \mid n<\omega \rangle$ is Σ_2 (understanding ω_ω in the sense of $L[u]$).

The analogues of Thms 4, 5, 6 do hold, however, on the assumption that, for some $\beta<\alpha$, $L_\beta[u]$ can be mapped onto each $x \in L_\alpha[u]$ by an $f \in L_\alpha[u]$. Since $L_\alpha[u] = L_\alpha[L_\beta[u]]$, it suffices to prove this for the case: $\beta = 0$ ($L_\beta[u] = u$).

<u>Def</u> $L_\alpha[u]$ is u-<u>dense</u> iff for all $\gamma<\alpha$ there is an $f \in L_\alpha[u]$ mapping u onto γ.

By Lemma 1, u-density is equivalent to the condition, that u can be mapped onto each $x \in L_\alpha[u]$ by an $f \in L_\alpha[u]$.

<u>Lemma 40</u> If $L_\alpha[u]$ is u-dense and $X \prec_{\Sigma_n} L_\alpha[u]$, then

$$\bigvee_{\beta \leq \alpha} X = L_\beta[u].$$

proof. By Lemma 2 it suffices to show that X is transitive. Let $x \in X$. We wish to show: $x \subset X$. The statement:

$$\bigvee f \quad f : u \xrightarrow{\text{onto}} x$$

holds in $\langle L_\alpha[u], \in \rangle$, hence in $\langle X, \in \rangle$. Thus there is an $f \in X$ s.t. $f : u \xrightarrow{\text{onto}} x$. But then $f(z) \in X$ for each $z \in u$; hence: $x = f``u \subset X$. QED

Using Lemma 10 in place of Lemma 6, we can repeat the proofs of Thms 4, 5, 6 to obtain:

(*) If $L_\alpha[u]$ is Σ_n uniformizable, then the Σ_n analogues of ~~Thms 4~~ Thms 4, 5, 6 hold.

(The proofs can be repeated word for word).

But this enables us to prove the Σ_n analogues of those Thms outright. We use induction on n. For $n=1$ the Thms are proven. Now suppose the Thms to hold for n. Then either n is admissible, or else the Thms hold trivially for all $m \geq n$. But if n is admissible, then by Lemma 7 $L_\alpha[u]$ is $n+1$ uniformizable and the Thms hold for $n+1$ by (*).

Thus:

<u>Thm 8</u> If $L_\alpha[u]$ is u-dense, then the Σ_n analogues of Thms 4, 5, 6 hold for $n \geq 1$.

. . . . . .

u-<u>uniformizability</u>

<u>Def</u> A function $r(z, \vec{x})$ is called a u-<u>uniformization</u> of a relation $Ry\vec{x}$ iff $\mathrm{dom}(r) = u \times \mathrm{dom}(R)$, $\mathrm{rng}(r) \subset \mathrm{rng}(R)$ and $\bigvee y \, Ry\vec{x} \longleftrightarrow \bigvee z \in u \; Rr(z, \vec{x})\vec{x}$.

<u>Def</u> $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ (s.t. $u \in M$) is Σ_n u-<u>uniformizable</u> iff every Σ_n relation is u-uniformizable by a Σ_n function.

§3 Until now we have worked with the notion of ordinal uniformizability (i.e. uniformizability of relations with ordinal range) rather than u-uniformizability. ~~([illegible])~~ However, ordinal uniformizability implies u-uniformizability for $L_\alpha[u]$ (and, indeed, the efficacy of ordinal uniformizability as a tool depends on this fact).

<u>Lemma 11</u> ~~[illegible]~~ If $L_\alpha[u]$ is Σ_n ordinal uniformizable, then $L_\alpha[u]$ is u-uniformizable.

~~proof.~~

~~Let $Ry\vec{x}$ be Σ_n. Let~~

$Ry\vec{x} \leftrightarrow \bigvee z\, Pzy\vec{x}$, where P is Π_{n-1}

~~[illegible]~~

Set: $P'v\vec{x} \leftrightarrow_{df} P(v)_0 (v)_1 \vec{x}$, where $(\langle x, y\rangle)_0 = x$, $(\langle x, y\rangle)_1 = y$. Set:

$G\nu z\vec{x} \leftrightarrow_{df} P'(z,\nu)\vec{x} \wedge z \in u.$

Th

proof of Lemma 11.

We first show that each Π_{n-1} relation is u-uniformisable by a Σ_n function. Let R be Π_{n-1}.

Set: $G\nu\vec{x} \underset{\text{Df}}{\longleftrightarrow} \bigvee z \in u \; R\,\eta(z,\nu)\,\vec{x}$.

G is Σ_n. Let g uniformise G.

We may assume w.l.o.g. that $R \neq \emptyset$, hence that $y \in \mathrm{rng}(R)$.

Set:

$$r(z,\vec{x}) \simeq \begin{cases} \eta(z,g(\vec{x})) & \text{if } R\,\eta(z,g(\vec{x}))\,\vec{x} \\ y & \text{if } \neg R\,\eta(z,g(\vec{x}))\,\vec{x} \end{cases}.$$

Then r uniformises R. ~~Now l~~

Now let R be Σ_n. Let:

$$Ry\vec{x} \longleftrightarrow \bigvee z \; Pzy\vec{x},$$

where P is Π_{n-1}. Set:

$$P'\langle z,y\rangle\vec{x} \longleftrightarrow Pzy\vec{x}$$

and let p u-uniformise P'. Set:

$$r(w,\vec{x}) \simeq (p(w,\vec{x}))_1$$

(where $(\langle z, y\rangle)_0 = z$, $(\langle z, y\rangle)_1 = y$).

Then r u-uniformises R QED

All previous theorems in which ordinal uniformisability was mentioned as an assumption hold on the (apparently) weaker assumption of u-uniformisability. In ~~part~~ particular:

<u>Lemma 12</u> If $L_\alpha[u]$ is Σ_n u-uniformisable, then there is a Σ_n function h s.t. $\mathrm{dom}(h) \subset u \times L_\alpha[u]$ and

$$\bigwedge x \in L_\alpha[u] \; (x \in h''(u \times \{x\}) \prec_{\Sigma_n} L_\alpha[u])$$

proof. We imitate the proof of Lemma 4, letting $s(z,x)$ s.t. $s : u \times \{x\} \xrightarrow{\text{onto}} \mathrm{Fml}^{\Sigma_n}_{u \cup \{u,x\}}$ be as before, we set

$$R x \varphi \longleftrightarrow \models^{\Sigma_n}_{L_\alpha[u]} \varphi(v_0/\underline{x})$$

and let r uniformise R. Set:

$$h(\langle z, w\rangle, x) \simeq r(z, s(w, \langle x, p\rangle)),$$

where p in Σ is the parameter p. QED

Carrying through the earlier proofs, again virtually without change, we get:

<u>Thm 9</u> Let $L_\alpha[u]$ be Σ_n u-uniformizable. Then the following are equivalent

(a) There is a Σ_n set $a \subset u$ s.t. $a \notin L_\alpha[u]$

(b) There is a Σ_n map f s.t. $\mathrm{dom}(f) \subset u$ and $f``u = L_\alpha[u]$.

<u>Thm 10</u> Let $L_\alpha[u]$ be Σ_n u-uniformizable. Then the following are equivalent

(a) There is a Δ_n set $a \subset u$ s.t. $a \notin L_\alpha[u]$

(b) There is a Δ_n map of u onto $L_\alpha[u]$.

We now prove:

<u>Thm 11</u> L_α is Σ_n u-uniformizable $(n \geq 1)$.

The proof of Thm 11 extends over several lemmas. From now on, we shall write 'uniformizable' to mean 'u-uniformizable'.

Lemma 13 Let $L_\alpha[u]$ be admissible and let $A \subset L_\alpha[u]$ be s.t.

$$x \in L_\alpha[u] \to A \cap x \in L_\alpha[u].$$

Then $\models^{\Sigma_0}_A$ is Δ_1 in $\langle L_\alpha[u], A\rangle$. Moreover, R is Σ_1 in $\langle L_\alpha, A\rangle$ iff R is Σ_1 in $\langle L_\alpha[u], \models^{\Sigma_0}_u\rangle$.

proof.

We first show that $\models^{\Sigma_0}_A$ is Δ_1.

Set: $a(x) =_{Df} A \cap x$.

$L_\alpha[u]$ is closed under a. a is Σ_1 since

$$y = a(x) \longleftrightarrow . \, y \subset x \wedge \bigwedge z \in x \, (z \in y \longleftrightarrow Az).$$

Thus $\models^{\Sigma_0}_A$ is Δ_1, since

$$\models^{\Sigma_0}_{A} \varphi \longleftrightarrow \models^{\Sigma_0}_{\langle C(\varphi),\, a(C(\varphi))\rangle} \varphi \quad .$$

But, by the same argument,

$\models^{\Sigma_0}_{\models^{\Sigma_0}_{A}}$ is Δ_1, since:

$$\models_{\models^{\Sigma_0}_{A}} \psi \longleftrightarrow \models_{\left\langle C(\psi),\, \models^{\Sigma_0}_{\langle C(\psi),\, a(C(\psi))\rangle} \right\rangle} \psi$$

This establishes the second part of the lemma. QED

Using Lemma 13, we can repeat the proofs of Thm 1 and Lemma 11 ~~Lemma 4~~ to obtain the analogues:

<u>Lemma 14.</u> If $L_\alpha[u]$, A are as in Lemma 13, then $\langle L_\alpha[u], A\rangle$ is Σ_1 uniformizable.

Since the only two facts used in the proof of Lemma 12 were: Σ_n uniformizability and the Σ_n definability of $\models^{\Sigma_n}$, we may repeat the proof to obtain:

<u>Lemma 15.</u> If $L_\alpha[u]$, A are as in Lemma 13 and if $\langle L_\alpha[u], A\rangle$ is Σ_n uniformizable, then there is a Σ_n Skolem function (i.e. an h s.t. $\mathrm{dom}(h) \subset u \times L_\alpha[u]$, and $\bigwedge x\ (x \in h``(u \times L_\alpha[u]) \prec_{\Sigma_n} \langle L_\alpha[u], A\rangle)$.

In particular, by Lemma 14, there is a Σ_1 Skolem function.

<u>Def</u> $\langle L_\alpha[u], A\rangle$ is called <u>feasible</u> iff for every Δ_1 set B we have:
$$x \in L_\alpha[u] \rightarrow B \cap x \in L_\alpha[u]$$
$\langle L_\alpha[u], A\rangle$ is called Σ_n-<u>feasible</u> if this holds for every Δ_n B.

<u>Lemma 16</u> Let $\langle L_\alpha[u], A\rangle$ be Σ_n-feasible but not Σ_n admissible. Let $\langle L_\alpha[u], A\rangle$ be Σ_n uniformisable. Then a relation R is Σ_1 in Σ_n iff R is [redacted] Σ_{n+1}.

proof.

($\leftarrow$) trivial, since each Σ_{n+1} relation is Σ_1 in Σ_n

($\rightarrow$) Since $\langle L_\alpha[u], A\rangle$ is not Σ_n admissible, there is a Π_{n-1} relation R and a $\beta < \alpha$ s.t. $\bigwedge x \bigvee y\, Ryx$ but for each $\gamma < \alpha$: $\bigvee x \in L_\beta[u]\ \neg \bigvee y \in L_\gamma[u]\ Ryx$.

Set: $G\nu x \underset{Df}{\longleftrightarrow} \bigvee y \in L_\nu[u]\ Ryx$

and let g uniformise G. Then g is Δ_n, $\mathrm{dom}(g) = u \times L_\beta[u]$ and $g``u \times L_\beta[u] = L_\alpha[u]$. Let h be a Σ_n Skolem function for $\langle L_\alpha[u], A\rangle$.

(h exists by Lemma 15).

Let:

$$y = h(z,x) \longleftrightarrow \bigvee w \; H w y z x ,$$

where H is Π_{n-1}. Set:

$$h^*(\langle w, z\rangle, x) = \begin{cases} y \text{ if } y \in L_{g(w)} \wedge \\ \quad \wedge \bigvee v \in L_{g(w)} \; H v y z x \\ \emptyset \text{ if not} \end{cases}$$

Then $\operatorname{dom}(h^*) = (u \times L_\rho[u]) \times L_\alpha[u]$ and

$$h^{*}{}''(u \times L_\rho[u]) \times \{x\} = h'' u \times \{x\}$$

for all $x \in L_\alpha[u]$.

For $\gamma < \alpha$ set:

$$\bar{\gamma} = (u \times L_\rho[u]) \times L_\gamma[u]$$

$$e(\gamma) = \{\langle x, y\rangle \mid x, y \in \bar{\gamma} \wedge h^*(x) \in h^*(y)\}$$

$$a(\gamma) = \{x \mid x \in \bar{\gamma} \wedge A\, h^*(x)\}.$$

Since, for each $\gamma < \alpha$, $e(\gamma)$, $a(\gamma)$ are Δ_n subsets of $\bar{\gamma}$, we have:

$$e(\gamma),\ a(\gamma) \in L_\alpha[u] \quad \text{for } \gamma < \alpha.$$

Let $m(\gamma) =_{Df} \langle v, \in \restriction v, a \rangle$, where $\bigcup v \subset v$ and for some π:

$$\pi : \langle \bar{\gamma}, e(\gamma), a(\gamma) \rangle \xrightarrow{\sim} m(\gamma).$$

By the admissibility of $L_\alpha[u]$ (Thm 3, Thm 6), we may conclude that $m(\gamma)$, π are elements of $L_\alpha[u]$. This follows by the recursion theorem, since the factorisation of $e(\gamma)$ by extensional equivalence is certainly in $L_\alpha[u]$ and the ~~factoized~~ factorised $e(\gamma)$ is well founded.

Thus, $L_\alpha[u]$ is closed under the function $m(\gamma)$. We show now that m is Δ_{n+1}. e is Σ_{n+1} since:

$$\underbrace{y = e(\gamma) \leftrightarrow . \, y \subset \bar{\gamma}^2 \wedge \bigwedge z, w \in \bar{\gamma} \underbrace{(\langle z, w \rangle \in y \leftrightarrow h^*(z) \in h^*(y))}_{\Delta_n}}_{\Pi_n}$$

Similarly, $a(\gamma)$ is Δ_{n+1}.

This means that m is Δ_{n+1}, since:

$$y = m(\gamma) \longleftrightarrow \bigvee \pi \, (\pi : \langle \bar{\gamma}, e(\gamma), a(\gamma) \rangle \overset{\sim}{\longleftrightarrow} y)$$

$\pi : x \overset{\sim}{\longleftrightarrow} y$ being Δ_1.

To establish the lemma, we need only show that relations Σ_0 in Σ_n relations are Σ_{n+1}.

Let the fmla φ be Σ_0 in Σ_n (i.e. built up from Σ_n fmlae by sentential operations and bounded ~~quantifiers~~ quantifications).

Then

$$\models_{\langle L_\alpha[u], A \rangle} \varphi \longleftrightarrow \bigvee \gamma < \alpha \, (\operatorname{Lim}(\gamma) \wedge \wedge\, \varphi \in L_\gamma[u] \wedge \models_{m(\gamma)} \varphi)$$

QED

Note that the assumption: $\Sigma_{n+1} = \Sigma_1$ in Σ_n can be used alternatively to Σ_n admissibility to carry out the proof of Lemma 7; hence:

<u>Lemma 17</u> If $\langle L_\alpha[u], A\rangle$ is Σ_n feasible, ~~then $\langle L_\alpha[u], A\rangle$~~ and Σ_n-uniformisible, then $\langle L_\alpha[u], A\rangle$ is Σ_{n+1} uniformisable.

We are now ready to prove Thm 11. We proceed by induction on n. For $n=1$ the theorem is proven. We now suppose it to hold for n and prove it for $n+1$.

<u>Case</u> I. $L_\alpha[u]$ is Σ_n- feasible.

The conclusion follows by Lemma 17.

If Case I fails, there is a $\beta < \alpha$ s.t. a Δ_n $a \subset L_\beta[u]$ exists with $a \notin L_\alpha[u]$. Let β be the least such. By Thm 10, there is a Δ_1 map from $L_\beta[u]$ onto $L_\gamma[u]$.

<u>Case II</u>. $\beta = 0$ (hence $L_\beta[u] = u$).

We first show that each Δ_n relation is uniformizable by a Σ_n function. Let $Ry\vec{x}$ be Δ_n. Assume (w.l.o.g.) $y \in \mathrm{rng}(R)$.

Set: $$r(z,\vec{x}) = \begin{cases} f(z) & \text{if } Rf(z)\vec{x} \\ y & \text{if } \neg Rf(z)\vec{x} \\ & \quad \wedge \bigvee y\, Ry\vec{x} \end{cases}$$

Then r uniformizes R. If R is Σ_n, there is Π_{n-1} P s.t.

$$Ry\vec{x} \leftrightarrow \bigvee z\, Pzy\vec{x}.$$

Set $P'\langle z,y\rangle\vec{x} \leftrightarrow_{df} Pzy\vec{x}$.

Let p uniformize P' and set:

$$r(w,\vec{x}) \simeq (p(w,\vec{x}))_1 \qquad \text{QED Case II}$$

Case III $\beta > 0$.

Then $L_\beta[u]$ will be admissible by the same argument which demonstrated that the projectum of α is admissible.

<u>Lemma 18</u> If A is $\Delta_n(L_\alpha[u])$ and $A \subset L_\beta[u]$, then each $R \subset L_\beta[u]^m$ which is $\Sigma_n(\langle L_\beta[u], A\rangle)$ is $\Sigma_{n+1}(L_\alpha[u])$.

proof.

It suffices to show: If R is $\Sigma_0(\langle L_\beta, A\rangle)$, then R is $\Sigma_{n+1}(L_\alpha)$.

Let φ be a Σ_0 fmla of $\langle L_\beta, A\rangle$.

Then

$$\models_A \varphi \longleftrightarrow \models_{\langle C(\varphi), A \cap C(\varphi) \rangle} \varphi \, .$$

But $a(u) = A \cap u$ is a $\Sigma_2(L_\alpha[u])$ function which is defined on all $L_\beta[u]$. Hence $\models_A^{\Sigma_0}$ is $\Sigma_2(L_\alpha[u])$

QED

Letting $f : L_\beta \xrightarrow{\text{onto}} L_\alpha$ be $\Delta_n(L_\alpha)$, pick $A \subset L_\beta$ in such a way that:

$\{\langle x, y \rangle \mid f(x) \in f(y)\}$, $f^{-1} \restriction L_\beta[u]$,

$$\models_f^{\Pi_n} =_{\mathrm{Df}} \{ \varphi \in \mathrm{Fml}_{L_\beta}^{\Pi_{n-1}} \mid \models_{L_\alpha} \bar{f}(\varphi) \}$$

are Σ_0 in $\langle L_\beta, A \rangle$. (Setting $\bar{f}(\varphi(\vec{x})) =_{\mathrm{Df}} \varphi(\overrightarrow{f(x)})$).

Then every $\Sigma_n(L_\alpha[u])$ relation $R \subset L_\beta[u]^m$ is $\Sigma_1(\langle L_\beta, A \rangle)$.

Using an obvious abbreviation, we have then:

$$\Sigma_n(L_\alpha) \subset \Sigma_1(\langle L_\beta, A\rangle) \subset \Sigma_{n+1}(L_\alpha).$$

On which side, if any, of this chain of conclusions does the identity lie?

We consider two cases:

<u>Case 1</u> There is a $\gamma < \beta$ and a $\Sigma_n(L_\alpha)$ function g s.t. $\mathrm{dom}(g) = L_\gamma[u]$ and $g``L_\gamma[u]$ is unbounded in α.

In this case we prove that, for an appropriate ~~choice~~ choice of A:

$$\Sigma_1(\langle L_\beta, A\rangle) = \Sigma_{n+1}(L_\alpha).$$

But, by Lemma 14, $\langle L_\beta, A\rangle$ is Σ_1 uniformisable.

Case 2 Case 1 fails.

In this case we show that

$$\Sigma_n(L_\alpha) = \Sigma_1(\langle L_\beta, A\rangle).$$

But then $\langle L_\beta, A\rangle$ is feasible and, by Lemmas 14, 17, $\langle L_\beta, A\rangle$ is Σ_2 uniformisable, whereby:

$$\Sigma_{n+1}(L_\alpha) = \Sigma_2(\langle L_\beta, A\rangle).$$

In either case, we may conclude that, if $R \subset L_\beta[u]^{m+1}$ is $\Sigma_{n+1}(L_\alpha)$, then R is uniformisable by a $\Sigma_{n+1}(L_\alpha)$ function. Now let $R \subset L_\alpha[u]^{m+1}$.

Set: $R'y\vec{x} \leftrightarrow Rf(y)\,\vec{f(x)}$.

Let r' uniformise R'. Let f' uniformise: $f(x) = y$ and set:

$$r(\langle z, \vec{w}\rangle, \vec{x}) \simeq f r'(z, f'(w_1, x_1), \ldots, f'(w_m, x_m)).$$

Then r uniformises R.

Thus, it remains only to prove the assertions made in Cases 1, 2.

<u>Lemma 19</u>. Let $n < \beta$ and let there be a $\Delta_n(L_\alpha)$ function g which maps $L_\kappa[u]$ onto $L_\alpha[u]$. Then A can be so chosen that every $R \subset L_\beta[u]^m$ which is $\Sigma_{n+1}(L_\alpha)$ is $\Sigma_1(\langle L_\beta, A\rangle)$.

proof. It suffices to show: If $R \subset L_\beta[u]^m$ is $\Sigma_n(L_\alpha)$, then R is $\Delta_n(\langle L_\beta, A\rangle)$. For this, it suffices that

$$\{\varphi \in Fml^{\Sigma_n}_{L_\beta[u]} \mid \models_{L_\beta[u]} \varphi\}$$

is $\Delta_1(\langle L_\beta, A\rangle)$. Let h be a Σ_n Skolem function for $L_\alpha[u]$.

~~Let $\varphi \ h(i,x) \longleftrightarrow V \ H \ y$~~

Let $y = h(z, x) \longleftrightarrow \bigvee v\ Hvyzx$,

where H is $\Pi_{n-1}(L_\alpha[u])$. Define:

$$h^*(\langle w, z\rangle, x) = \begin{cases} y \text{ if } y \in L_{g(w)}[u] \text{ and} \\ \quad \bigvee v \in L_{g(w)}\ Hvyzx \\ 0 \text{ if not} \end{cases}$$

$$h^*(z, x) = 0 \text{ in all other cases.}$$

Then $\mathrm{dom}(h^*) = L_\kappa[u] \times L_\alpha[u]$ and

$\bigwedge x \quad h^{*}{}''(L_\kappa[u] \times \{x\}) = h''(u \times \{x\})$.

Choose A in such a way that R

is $\Sigma_0 (\langle L_\beta, A\rangle)$, where:

$$Rxy \longleftrightarrow x, y \in L_\kappa \times L_\beta \wedge h^*(x) \in h^*(y).$$

R is $\Delta_n(L_\alpha)$.

Set: $\left.\begin{array}{l} \bar{\gamma} = L_\kappa \times L_\gamma \\ e(\gamma) = \bar{\gamma}^2 \cap R \end{array}\right\}$ for $\gamma < \beta$.

Then $\bigwedge \gamma < \beta\ \ e(\gamma) \in L_\beta[u]$, since

$e(\gamma)$ is a $\Delta_n(L_\alpha)$ subset of $\bar{\gamma}^2$.

The function $e(\gamma)$ is $\Sigma_1(L_\beta, A)$, since:

$$e = e(\gamma) \longleftrightarrow e \subset \bar{\gamma}^2 \wedge \bigwedge x y \in \bar{\gamma}\ (\langle x, y\rangle \in e \longleftrightarrow Rxy).$$

Set: $m(\gamma) =$ that $\langle v, \in \restriction v\rangle$ s.t. $\bigcup v \subset v$ and

$$\langle v, \in\rangle \overset{\sim}{\longleftrightarrow} \langle \bar{\gamma}, e(\gamma)\rangle \overset{\sim}{\longleftrightarrow} \langle h^{*}{}'' \bar{\gamma}, \in\rangle.$$

Imitating the methods of the proof of Lemma 16, we get: The function m is $\Sigma_1(\langle L_\beta, A\rangle)$ and is defined everywhere. But this means that $\{\varphi \in \mathrm{Fml}^{\Sigma_n}_{L_\beta} \mid \models_{L_\alpha} \varphi\}$ is ~~$\Delta_1(L_\beta, A)$~~, $\Delta_1(\langle L_\beta, A\rangle)$, since, setting $\gamma(\varphi) = \mu\gamma\,(\mathrm{lim}(\gamma) \wedge \varphi \in L_\gamma[u])$, we have:

$$\models^{\Sigma_n}_{L_\alpha} \varphi \longleftrightarrow \models^{\Sigma_n}_{m(\gamma(\varphi))} \varphi$$

for $L_\beta[u]$-fmlae φ. QED

<u>Lemma 20</u> If the hypothesis of Lemma 19 fails, then every $\Sigma_1(\langle L_\beta, A\rangle)$ relation is $\Sigma_n(L_\alpha)$ (hence $\langle L_\beta, A\rangle$ is feasible).

proof.

It suffices to show: If R is $\Sigma_0(\langle L_\beta, A\rangle)$, then R is $\Delta_n(L_\alpha)$. We show this by induction on the defining fmla of R, using

(*) If $Ry\vec{x}$ is $\Sigma_n(L_\alpha)$, then so is:

$$\bar{R}y\vec{x} \underset{Df}{\longleftrightarrow} y \in L_\beta[u] \wedge \bigwedge z \in y\, Rz\vec{x}.$$

proof of (*):

Let $Ry\vec{x} \longleftrightarrow \bigvee u\, Puy\vec{x}$, where P is Π_{n-1}. For $y \in L_\beta[u]$, we have:

$\cancel{\bigwedge z \in y \bigvee u\, Puy\vec{x} \rightarrow \bigvee \delta < \alpha \bigwedge z \in y \bigvee u \in L}$

$$\bigwedge z \in y \bigvee v\, P v z \vec{x} \longrightarrow \bigvee \gamma < \alpha \bigwedge z \in y \bigvee v \in L_\gamma\, P v z \vec{x}.$$

since otherwise, letting $p(w, z)$ uniformize the relation:

$$P'\nu z \underset{df}{\longleftrightarrow} \bigvee v \in L_\nu[u]\, P v z \vec{x},$$

p would map uxy unboundedly into α. (Contradiction!)

Hence:

$$\bigwedge z \in y\, P z \vec{x} \longleftrightarrow \bigvee w \bigwedge z \in y \bigvee v \in w\, P v z \vec{x}.$$

We apply the same reduction to $\bigvee v \in w\, P v z \vec{x}$ etc. until we are left with a Σ_n fmla. QED