§5. Forcing + admissible sets

Let $\mathbb{P} = \langle |\mathbb{P}|, \leq \rangle$ be a partially ordered structure (i.e. $\leq$ is a partial ordering whose field is $|\mathbb{P}|$). For reasons that will become apparent later, we refer to such structures as <u>systems of conditions</u> and to the elements of $\mathbb{P}$ as <u>conditions</u>. '$p \leq q$' is read 'the condition p <u>extends</u> the condition q'.

For $p \in \mathbb{P}$ set:

$$[p] =_{Df} \{q \mid q \leq p\}.$$

Topologize $\mathbb{P}$ by taking the collection of all $[p]$ as an open basis. By the <u>canonical Boolean algebra over</u> $\mathbb{P}$ ($BA(\mathbb{P})$) we mean the algebra of regular open sets in this topology. Let $\mathbb{B} = BA(\mathbb{P})$. $\mathbb{B}$ is a complete Boolean algebra, the operations being defined by

$$\neg b = \{p \mid \bigwedge p' \leq p \;\; p' \notin b\}$$

$$\bigcap^{\mathbb{B}} X = \bigcap X \text{ (the set theoretical intersection of } X\text{)}$$

for $X \subset \mathbb{B}$. Call $c \subset \mathbb{P}$ <u>dense</u> in a set $d \subset \mathbb{P}$ iff $\bigwedge p \in d \; \bigvee q \leq p \;\; q \in c$.

Then:

$$\bigcup^{\mathbb{B}} X = \neg \bigcap_{b \in X} \neg b = \{p \mid \bigcup X \text{ is dense in } [p]\}$$

Call p, q <u>compatible</u> iff they have a common extension. By $[c]$ we denote the smallest $b \in \mathbb{B}$ s.t. $c \subset \mathbb{B}$.

Then:

$$[c] = \bigcup_{p \in c}^{\mathbb{B}} [p]$$

$$= \{q \mid \bigwedge q' \leq q \; (q' \text{ is compatible with some } p \in c)\}$$

It is clear that the class $\mathbb{B}$ is p.r. in the parameter $\mathbb{P}$ and that the operations $\neg, \bigcap^{\mathbb{B}}, \bigcup^{\mathbb{B}}, [\;]$ are the restrictions of functions p.r. in $\mathbb{P}$ to classes p.r. in $\mathbb{P}$. In fact, these functions are <u>uniformly</u> p.r. in $\mathbb{P}$ - e.g. there is a p.r. f s.t. ~~f(\mathbb{P}~~ $f(\mathbb{P}, X) = \bigcup^{BA(\mathbb{P})} X$ whenever $\mathbb{P}$ is a system of conditions and $X \subset BA(\mathbb{P})$.

<u>$\mathbb{B}$-valued models</u>

$M = \langle |M| ; A_1, \ldots, A_m \rangle$ is called a $\mathbb{B}$-<u>valued model</u> if

$$A_i : |M|^{m_i} \longrightarrow \mathbb{B} \quad (i = 1, \ldots, m).$$

The M-<u>language</u> is, as usual, the first order language with predicates $\dot{A}_i$ and constants $\underset{\sim}{x}$ $(x \in M)$. We assign to each statement φ of the M-language a <u>truth</u> <u>value</u> $[\![\varphi]\!]_M$ in $\mathbb{B}$ as follows:

$$[\![\dot{A}_i \vec{\underset{\sim}{x}}]\!] = A_i(\vec{x}) \; ; \; [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \; ;$$

$$[\![\bigwedge v \varphi]\!] = \bigcap_{x \in M} [\![\varphi(^{v}/_{\underset{\sim}{x}})]\!] \; ;$$

$$[\![\neg \varphi]\!] = \neg [\![\varphi]\!] \, .$$

The forcing relation ($\Vdash_M$) between elements of $\mathbb{P}$ and M-statements is defined by:

$$p \Vdash \varphi \longleftrightarrow_{df} p \in [\![\varphi]\!].$$

Thus:

$$p \Vdash (\varphi \wedge \psi) \longleftrightarrow . \, p \Vdash \varphi \wedge p \Vdash \psi$$

$$p \Vdash \bigwedge v \varphi \longleftrightarrow . \bigwedge x \in M \quad p \Vdash \varphi(v/x)$$

$$p \Vdash \neg\varphi \longleftrightarrow \bigwedge p' \leq p \quad \neg p' \Vdash \varphi.$$

It is often technically easier and intuitively more enlightening to work with the forcing relation rather than directly with the Boolean evaluation. The intuition behind forcing may be understood as follows: We think of the conditions as embodying bits of information about a potential 2-valued model. If p extends q, then p contains at least as much information as q. If p, q are incompatible, then they contain conflicting information. The conditions contain sufficient information to eventually decide the truth value of every M-statement φ — i.e. each p has an extension which forces φ to be either true or false. Thus, if no extension of p forces φ to be true, p forces φ to be false.

We call $M = \langle |M| ; I, A_1, \ldots, A_n \rangle$ an <u>equality model</u> if the axioms of identity logic hold in M, ~~intag~~ interpreting $\equiv$ by I; i.e.

$$[\![\underset{\sim}{x} \equiv \underset{\sim}{x}]\!] = 1$$

$$[\![\underset{\sim}{x} \equiv \underset{\sim}{z}]\!] \subset [\![\varphi(^v/_{\underset{\sim}{x}}) \longleftrightarrow \varphi(^v/_{\underset{\sim}{z}})]\!] .$$

(Obviously, it suffices that this hold for primitive φ).

We call M an <u>identity model</u> if, in addition, we have

$$[\![\underset{\sim}{x} \equiv \underset{\sim}{z}]\!] = 1 \rightarrow x = z .$$

For the most part we shall work with ~~identity~~ equality models rather than ~~equality~~ identity models, since these ~~can be obtained~~ prove to be more constructively definable.

(*Notational remark*: Where the context permits, we shall omit the underlining on constants $\underline{x}$, writing $\varphi(v/x)$ or $\varphi(x)$ instead of $\varphi(v/\underline{x})$).

The maximal $\mathbb{B}$-valued model

We now consider certain $\mathbb{B}$-valued models which are *classes* rather than sets in V. A *maximal $\mathbb{B}$-valued model of set theory* is a triple $\langle V^{\mathbb{B}}; I, E \rangle$, where I, E are $\mathbb{B}$-valued relations on the class $V^{\mathbb{B}}$ and (letting $\equiv, \varepsilon$ be interpreted by I, E):

(i) $V^{\mathbb{B}}$ is an equality model

(ii) $V^{\mathbb{B}}$ satisfies the axioms of extensionality + foundation.

(iii) $V^{\mathbb{B}}$ is <u>maximal</u> in the sense that, whenever f maps a subset u of $V^{\mathbb{B}}$ into $\mathbb{B}$, then there is an $x \in V^{\mathbb{B}}$ s.t.

$$[\![z \varepsilon x]\!] = \bigcup_{y \in u} ([\![z \equiv y]\!] \cap f(y))$$

(i.e. every $\mathbb{B}$-valued subset of $V^{\mathbb{B}}$ is represented by an element of $V^{\mathbb{B}}$).

(iv) If $x \in V^{\mathbb{B}}$, then there is a set $u \subset V^{\mathbb{B}}$ s.t. $[\![z \varepsilon x]\!] \subset \bigcup_{y \in u} [\![z \equiv y]\!]$

(i.e. ~~only~~ $\mathbb{B}$ valued <u>sets</u> and not proper classes are represented in $V^{\mathbb{B}}$).

It is known that:

(a) Any two maximal $\mathbb{B}$-valued models are elementarily equivalent; in fact, if they are identity models, they are isomorphic. (This generalizes the fact that any maximal 2-valued model is isomorphic to $\langle V, =, \in \rangle$

(b) If $V^{\mathbb{B}}$ is maximal, it satisfies all axioms of ZF.

There are many ways of constructing maximal $\mathbb{B}$-valued models. We shall find the following most convenient for our purposes:

By a $\mathbb{P}$-*set*, let us ~~denote~~ mean a relation whose ~~domain~~ range lies in $\mathbb{P}$. A *hereditary* $\mathbb{P}$-*set* is a set whose domain consists of $\mathbb{P}$-sets whose domains in turn consist of $\mathbb{P}$-sets ... etc. We shall take $V^{\mathbb{B}}$ (alternatively denoted by $V^{\mathbb{P}}$) as the collection of hereditary $\mathbb{P}$ sets. The formal definition reads:

$$V^{\mathbb{P}} = \{x \mid h(x) \text{ is a relation} \wedge \text{rng}(h(x)) \subseteq \mathbb{P}\},$$

where $h(x) = \tilde{h}^{\omega}(x)$, $\tilde{h}(x) = x \cup \bigcup \text{dom}(x)$.

We associate with every $x \in V^{\mathbb{B}}$ a $\mathbb{B}$-valued function $[x]$ defined on $\text{dom}(x)$ by setting:

$$[x](y) =_{df} [\{p \mid \langle p, y \rangle \in x\}].$$

We must now define the $\mathbb{B}$-valued relations I, E. We wish to do this in such a way that x represents the $\mathbb{B}$-valued set $[x]$ in the sense of (iii) above; i.e.

$$(*) \quad [\![z \,\varepsilon\, x]\!] = \bigcup_{y \in \mathrm{dom}(x)} [\![z \equiv y]\!] \cap [x](y).$$

A consequence of $(*)$ is that bounded quantifiers are interpretable by:

$$[\![\bigwedge v \,\varepsilon\, x\, \varphi]\!] = \bigcap_{y \in \mathrm{dom}(x)} \left([x](y) \Rightarrow [\![\varphi(y)]\!] \right)$$

(writing $b \Rightarrow c$ for $\neg b \cup c$).

[<u>Note</u>: We, of course, originally interpret bounded quantifiers by: $[\![\bigwedge v \,\varepsilon\, x\, \varphi]\!] = \bigcap_{y \in V^{\mathbb{B}}} [\![y \,\varepsilon\, x \to \varphi(y)]\!]$]

By extensionality, $x \equiv y$ must be equivalent to:

$$\bigwedge x' \varepsilon\, x \bigvee y' \varepsilon\, y \;\; x' \equiv y' \wedge \bigwedge y' \varepsilon\, y \bigvee x' \varepsilon\, x \;\; y' \equiv x'.$$

Thus we must have:

$$(**)\quad I(x,y) = C(x,y) \wedge C(y,x)\ ,\ \text{where}$$

$$C(x,y) = \bigwedge_{x' \in dom(x)} \bigvee_{y' \in dom(y)} \left([x](x') \Rightarrow (I(x,y) \wedge [y](y'))\right).$$

By $(*)$, of course, we have:

$$(***)\quad E(x,y) = \bigvee_{z \in dom(y)} I(x,z) \wedge [y](z).$$

<u>Lemma 1</u> $(**)$, $(***)$ uniquely define functions I, E which are uniformly p.r. in the parameter $\mathbb{P}$.

proof.

Setting $f(\langle x,y\rangle) = I(x,y)$, it is obvious that $(**)$ can be written in the form:

$$f(z) = g(z, f \restriction h(z)),$$

where g is p.r. and

$$h(\langle x,y\rangle) = (dom(x) \times dom(y)) \cup \cup (dom(y) \times dom(x)).$$

$z' \in h(z)$ is well founded and h is manageable, since:

$$z' \in h(z) \rightarrow rn(z') < rn(z).$$

Thus I is p.r. E is then trivially p.r. QED (Lemma 1).

<u>Lemma 2</u> $\langle V^{B}, I\rangle$ is an ~~identity~~ equality model.

proof. We must show:

(a) $I(x,x)=1$

(b) $I(x\,y) = I(y,x)$

(c) $I(x,y)\wedge I(y,z) \subset I(x,z)$

(a) is easily proved by induction on $rn(x)$

(b) follows immediately by the definition.

(c) is proved by induction on ~~max~~ $\max(rn(x), rn(y), rn(z))$:

$$C(x,y)\wedge C(y,z) \subset$$

$$\subset \bigcap_{x'\in dom(x)} \bigcup_{y'\in dom(y)} \bigcup_{z'\in dom(z)} \big([x](x') \Rightarrow$$

$$\Rightarrow (I(x',y')\wedge I(y',z')\wedge [z](z')\big).$$

By the induction hypothesis:

$$I(x',y')\wedge I(y',z') \subset I(x',z').$$

Hence $C(x,y)\wedge C(y,z) \subset C(x,z)$. QED

<u>Lemma 3</u> $\langle V^{B}, I, E \rangle$ is an equality model.

proof. We must show:

(a) $I(x,y) \cap E(x,z) \subset E(y,z)$

(b) $I(x,y) \cap E(z,x) \subset E(z,y)$.

proof of (a):

$$I(x,y) \cap E(x,z) = \bigcup_{z' \in dom(z)} I(x,y) \cap I(x,z') \cap [z](z')$$

$$\subset \bigcup_{z' \in dom(z)} I(y,z') \cap [z](z') = E(y,z).$$

proof of (b):

~~By [illegible], if y c~~

If $x' \in dom(x)$, then:

$$I(x,y) \cap [x](x') \subset C(x,y) \cap [x](x')$$

$$\subset \bigcup_{y' \in dom(y)} I(x',y') \cap [y](y') = E(x',y).$$

Thus, in general:

$$I(x,y) \cap E(z,x) = \bigcup_{x' \in dom(x)} (I(x,y) \cap I(z,x') \cap [x](x'))$$

$$\subset \bigcup_{x' \in dom(x)} (I(z,x') \cap E(x',y))$$

$$\subset E(z,y) \text{ by (a)}$$

QED

<u>Lemma 4.</u> Let $\langle V^{\mathbb{B}}; I, E, A_1, \ldots, A_n \rangle$ be an equality model (i.e. $A_i : (V^{\mathbb{B}})^{m_i} \to \mathbb{B}$ s.t.

$$[\![\bigwedge_j x_j \equiv y_j]\!] \cap A_i(\vec{x}) \subset A_i(\vec{y}) \;).$$

Appoint predicates $\mathring{A}_i$ for A_i. Let $[\![\varphi]\!] = [\![\varphi]\!]_{A_1, \ldots, A_n}$ be the truth value of the formula φ in this ~~language~~ model. Then:

$$[\![\bigwedge v \varepsilon x \; \varphi]\!] = \bigcap_{y \in dom(x)} ([x](y) \Rightarrow [\![\varphi(y)]\!]).$$

proof.

$$[\![\bigwedge v \varepsilon x \varphi]\!] = \bigcap_{z} ([\![z \varepsilon x]\!] \Rightarrow [\![\varphi(z)]\!])$$

$$= \bigcap_{z} \bigcap_{y \in dom(x)} ([\![z \equiv y]\!] \cap [x](y) \Rightarrow [\![\varphi(z)]\!])$$

$$= \bigcap_{y \in dom(x)} ([x](y) \Rightarrow [\![\bigwedge v (v \equiv y \to \varphi)]\!])$$

$$= \bigcap_{y \in dom(x)} ([x](y) \Rightarrow [\![\varphi(y)]\!]).$$

QED

As a corollary to Lemma 4 we get:

<u>Lemma 5.</u> Let $\langle V^{\mathbb{B}}, I, E, A_1, \ldots, A_n \rangle$ be an equality model. Then

$$\langle [\![\varphi]\!]_{A_1 \ldots A_n} \mid \varphi \text{ is a } \Sigma_0 \text{ fmla} \rangle$$

uniformly
is p.r. in $A_1, \ldots, A_n$ ~~uniformly in~~ and the parameter $\mathbb{P}$.

proof.

$[\![\varphi]\!]$ may be defined by:

$[\![x \in y]\!] = E(x,y)$, $[\![x \equiv y]\!] = I(x,y)$,

$[\![\dot{A}_i \vec{x}]\!] = A_i(\vec{x})$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$,

$[\![\neg \varphi]\!] = \neg [\![\varphi]\!]$, ~~$\wedge v \varepsilon$~~

$$[\![\wedge v \varepsilon x\, \varphi]\!] = \bigcap_{y \in \mathrm{dom}(x)} ([x](y) \Rightarrow [\![\varphi(y)]\!]),$$

Setting $f(\varphi) = [\![\varphi]\!]$, we get a recursion $f(x) = g(x, f \upharpoonright h(x))$,

where: $h(\varphi) = \emptyset$ for primitive φ

$h(\varphi \wedge \psi) = \{\varphi, \psi\}$, $h(\neg\varphi) = \{\varphi\}$

$h(\wedge v \varepsilon x\, \varphi) = \{\varphi(y) \mid y \in \mathrm{dom}(x)\}$,

h is p.r. and $z \in h(w)$ is well founded. But h is also manageable, since

$$\varphi \in h(\psi) \to l(\varphi) < l(\psi),$$

where $l(\varphi)$ is the length of φ, defined by:

$l(\varphi) = 0$ for primitive φ

$l(\varphi \wedge \psi) = l(\varphi) + l(\psi) + 1$

$l(\neg \varphi) = l(\varphi) + 1$, $l(\bigwedge w \varepsilon x\, \varphi) = l(\varphi) + 1$.

QED

<u>Lemma 6</u> $\langle V^{\mathbb{B}}; E, I \rangle$ is a maximal $\mathbb{B}$-valued model of set theory.

proof.

We have seen that $V^{\mathbb{B}}$ is an equality model. The axiom of extensionality follows trivially from Lemma 4 and the definition of I. We show the axiom of foundation to hold as follows:

Suppose it to be false. Let $\varphi(u)$ be the fmla $\bigwedge x \varepsilon u \bigvee y \varepsilon u\; y \varepsilon x$. Then, for some u,

$$[\![\bigvee x\; x \varepsilon u \wedge \varphi(u)]\!] \neq 0.$$

Hence there exist x s.t. $[\![x \,\varepsilon\, u \wedge \varphi(u)]\!] \neq 0$. Among the x having this property choose an x_0 of minimal rank.

Then $[\![x_0 \,\varepsilon\, u \wedge \varphi(u)]\!] = [\![x_0 \,\varepsilon\, u \wedge \bigvee_{y \varepsilon u} y \,\varepsilon\, x_0 \wedge \varphi(u)]\!]$, hence there is a y s.t.

$$[\![x_0 \,\varepsilon\, u \wedge y \,\varepsilon\, u \wedge y \,\varepsilon\, x_0 \wedge \varphi(u)]\!] \neq 0.$$

But $[\![y \,\varepsilon\, x_0]\!] \leq \bigcup_{z \in \mathrm{dom}(x_0)} [\![y \equiv z]\!]$, hence there is a $z \in \mathrm{dom}(x_0)$ s.t.

$$[\![z \,\varepsilon\, u \wedge \varphi(u)]\!] \neq 0,$$

But $rn(z) < rn(x_0)$. Contradiction!

This proves (ii), (iii). To show that $V^{\mathbb{B}}$ is maximal, let $f: u \to \mathbb{B}$ where $u \subset V^{\mathbb{B}}$ and set: $x = \{\langle p, y\rangle \mid p \in f(y) \wedge y \in u\}$.

Then $[\![z \,\varepsilon\, x]\!] = \bigcup_{y \in u} [\![z \equiv y]\!] \wedge f(z)$.

(iv) follows trivially, since

$$[\![z \,\varepsilon\, x]\!] \subset \bigcup_{y \in \mathrm{dom}(x)} [\![z \equiv y]\!].$$

QED

Generic sets of conditions

Def Consider a structure ~~M~~ $M = \langle |M| ; \in, A_1, \ldots, A_m \rangle$, where $|M|$ is a transitive p.r. closed set. Let $\mathbb{P} = \langle |\mathbb{P}|, \leq \rangle$ be an ~~M~~ M-definable system of conditions (i.e. $|\mathbb{P}| \subset M$ and $|\mathbb{P}|$, $\leq$ are M-definable). We call $G \subset \mathbb{P}$ a $\mathbb{P}$-generic set of condititions over M iff

(i) $p \geq q \in G \rightarrow p \in G$

(ii) Any two elements of G are compatible.

(iii) G meets every M-definable dense set of conditions - i.e. if $\Delta \subset \mathbb{P}$ is M-definable and dense in $\mathbb{P}$, then $\Delta \cap G \neq \emptyset$.

<u>Note</u> Call $c \subset \mathbb{P}$ <u>compatible in</u> $d \subset \mathbb{P}$ iff every $p \in d$ is compatible with some $q \in c$. Call c <u>compatible</u> if c is compatible in $\mathbb{P}$. Then (iii) may be equivalently replaced by:

(iii)′ G meets every M-definable compatible set of conditions

<u>Note</u> Let $\mathbb{P} \in M$ and suppose every M-definable subset of $\mathbb{P}$ to be an element of M. Define:

$$\mathbb{G} = \{ b \in \mathbb{B} \cap M \mid b \cap G = \emptyset \}$$

(where $\mathbb{B} = BA(\mathbb{P})$).

Then $\mathbb{G}$ is an ultrafilter on $\mathbb{B}$ which preserves M-definable intersections; i.e.

$$b \notin \mathbb{G} \longleftrightarrow (\neg b) \in G \quad \text{for } b \in \mathbb{B} \cap M$$

$$\bigwedge X \in \mathbb{G} \longleftrightarrow X \subset \mathbb{G}$$

if $X \subset \mathbb{B} \cap M$ is M-definable.

Such $\mathbb{G}$ is called a <u>generic filter</u> over M. If $\mathbb{G}$ is a generic filter, then a generic set G may be recovered by:

$$G = \{p \mid [p] \in \mathbb{G}\}.$$

The model M[G]

Let $M = \langle |M| ; \in, A_1, \ldots, A_n \rangle$, $\mathbb{P}$ be as above. Set:

$$M^{\mathbb{P}} =_{\mathrm{df}} M \cap V^{\mathbb{P}}.$$

Let $\mathfrak{M} = \langle M^{\mathbb{P}} ; I, E, \dot{B}_1, \ldots, \dot{B}_n \rangle$ be an equality model, where & the relations:

$$\{\langle p, x, y\rangle \mid p \in I(x, y)\}$$

$$\{\quad " \quad \mid p \in E(x, y)\}$$

$$\{\langle p, \vec{x}\rangle \mid p \in \dot{B}_i(\vec{x})\}$$

are M-definable. Then if $\varphi(\vec{v})$ is any $\mathfrak{M}$-fmla, the relation

$$\{\langle p, \vec{x}\rangle \mid p \Vdash \varphi(\vec{x})\}$$

will be M-definable (letting $\Vdash = \Vdash_{\mathfrak{M}}$ be the forcing relation of $\mathfrak{M}$).

Suppose G to be $\mathbb{P}$-generic over M. Set:

$$G \Vdash \varphi \underset{Df}{\longleftrightarrow} \bigvee p \in G \;\; p \Vdash \varphi .$$

for $\mathfrak{M}$-statements φ.

<u>Lemma 1</u>

$$G \Vdash \varphi \wedge \psi \longleftrightarrow G \Vdash \varphi \wedge G \Vdash \psi$$

$$G \Vdash \bigwedge v \varphi \longleftrightarrow \bigwedge x \in M^{\mathbb{P}} \; G \Vdash \varphi(x)$$

$$G \Vdash \neg \varphi \longleftrightarrow \neg G \Vdash \varphi$$

proof. We display a sample case of the proof. Let $G \Vdash \varphi(x)$ for all $x \in M^{\mathbb{P}}$. The set $D = \{p \mid p \Vdash \bigwedge v \varphi \vee \bigvee x \; p \Vdash \neg \varphi(x)\}$ is M-definable and dense in $\mathbb{P}$. Hence there is a $p \in D \cap G$. But p cannot force $\neg \varphi(x)$, since otherwise G would contain incompatible conditions. Hence $p \Vdash \bigwedge v \varphi$.

QED

<u>Def</u> $G^*: V^{\mathbb{P}} \to V$ is defined by:

$$G^*(x) = \{G^*(y) \mid \bigvee p \in G \ \langle p, y \rangle \in x\}.$$

(<u>Note</u> The function $f(G, x) = G^*(x)$ is p.r.)

<u>Lemma 2</u> If $x, y \in M^{\mathbb{P}}$, then

$$G^*(x) = G^*(y) \longleftrightarrow G \Vdash x \equiv y$$

$$G^*(x) \in G^*(y) \longleftrightarrow G \Vdash x \,\varepsilon\, y.$$

proof.

(a) If $x \in \mathrm{dom}(y)$, then

$$G^*(x) \in G^*(y) \longleftrightarrow G \cap [y](x) \neq \emptyset$$

The direction $(\rightarrow)$ is trivial.

$(\leftarrow)$ If $p \in G \cap [y](x)$, then $D = \{p' \mid \bigvee q \ \langle q, x \rangle \in y \wedge p \leq q\}$ is M-definable and dense in $[p]$. Hence $G \cap D \neq \emptyset$ (since $[p] \cap D . \cup \neg[p]$ is dense in $\mathbb{P}$).

(b) $G \Vdash \bigwedge v \,\varepsilon\, x \ \varphi \longleftrightarrow$

$$\longleftrightarrow \bigwedge y \in \mathrm{dom}(x) \left(G^*(y) \in G^*(x) \rightarrow G \Vdash \varphi\right)$$

(b) follows trivially from (a).

Using (b), we prove:

$$G^*(x) = G^*(y) \longleftrightarrow G \Vdash x \equiv y$$

by induction on $|x,y| = \max(\mathrm{rn}(x), \mathrm{rn}(y))$,

Let it hold for $\gamma < |x,y|$.

Then:

$$G \Vdash x \equiv y \longleftrightarrow G \Vdash \bigwedge x' \varepsilon x \bigvee y' \varepsilon y \; x' \equiv y' \wedge$$

$$\wedge \; G \Vdash \bigwedge y' \varepsilon y \bigvee x' \varepsilon x \; y' \equiv x'$$

$$\longleftrightarrow \bigwedge x' \in G^*(x) \bigvee y' \in G^*(y) \; x' = y' \wedge$$

$$\wedge \bigwedge y' \in G^*(y) \bigvee x' \in G^*(x) \; y' = x'$$

(by (b))

$$\longleftrightarrow x = y .$$

But then:

$$G \Vdash x \varepsilon y \longleftrightarrow G \Vdash \bigvee z \varepsilon y \; x \equiv z$$

$$\longleftrightarrow \bigvee z \in G^*(y) \; x = z \qquad \text{(by (b))}$$

$$\longleftrightarrow x \in y \qquad \text{QED}$$

Let $N = \langle |N| ; \in, B_1, \dots, B_m \rangle$ be defined by: $|N| = G^{*} \text{“} M^{\mathbb{P}}$;

$$B_i = G^{*}(\dot{B}_i) = \{ \langle G^{*}(\vec{x}) \rangle \mid G \Vdash \dot{B}_i \vec{x} \} .$$

If φ is an $\mathfrak{M}$-fmla, let φ^G be the result of replacing $\underset{\sim}{x}$ by $\underline{G^{*}(x)}$ everywhere in φ. (Hence $f(G, \varphi) = \varphi^G$ is p.r.)
N is also denoted by $\mathfrak{M}/G$.

<u>Lemma 3</u> $G \Vdash \varphi \longleftrightarrow \Vdash_N \varphi^G$

proof. By Lemmas 1, 2.

<u>Lemma 4</u> $G^{*}(\check{x}) = x$; hence $|M| \subset |N|$

proof. By induction on x

Now let us strengthen our assumptions on $M, \mathfrak{M}$ by:

$\mathbb{P} \in M$; M is p.r. closed in $B_1, \dots, B_m$

<u>Lemma 5</u> $G^{*}(\dot{G}) = G$, where $\dot{G} = \{ \langle p, \check{p} \rangle \mid p \in \mathbb{P} \}$; hence $G \in N$.

proof. By Lemma 4.

<u>Def</u> Let $\mathcal{U}$ be p.r. closed and transitive. $\mathcal{U}[A_1, \dots, A_m]$ denotes the <u>constructible closure of $\mathcal{U}$ relative to</u> $A_1, \dots, A_m$, defined by:

$$\mathcal{U}[\vec{A}] =_{\text{Df}} \bigcup_{\substack{x \in \mathcal{U} \\ \nu < rn(\mathcal{U})}} L_\nu[x; A_1, \dots, A_m].$$

(<u>Note</u>: If $A_i \subset x \in \mathcal{U}$, then $A_i \in \mathcal{U}[\vec{A}]$)

<u>Lemma 6</u> $\mathcal{U}[A_1, \dots, A_m]$ = the p.r. closure of $\mathcal{U}$ in $A_1, \dots, A_m$.

proof.

($\subset$) $f(x, \nu) = L_\nu[x, \vec{A}]$ is p.r.

($\supset$) By Carol Karp's stability lemma

QED

<u>Lemma 7</u> $N = M[G] = M[G, B_1, \dots, B_m]$.

proof.

$N \subset M[G]$: $N = f''\{G\} \times M^{\mathbb{P}}$, where f is the p.r. fcn ~~f~~ $f(G, x) = G^*(x)$,

$M[G, \vec{B}] \subset N$:

We can assume without loss of generality that ~~$\dot{B}_1(x) = [\dot{G}](x)$~~ $B_1(x) = [\![x \in \dot{G}]\!]$, where $\dot{G}$ is as in Lemma 5. Hence it suffices to show:

<u>Claim</u> If $u \in M$ is transitive and $\nu \in M$, then $L_\nu[u, \vec{B}] \in N$.

proof.

Call $x \in M^{\mathbb{P}}$ <u>nice</u> iff

(i) $y \in \mathrm{dom}(x) \rightarrow \mathrm{dom}(y) \subset \mathrm{dom}(x)$

(ii) $y \in \mathrm{dom}(x) \rightarrow \langle 1, y \rangle \in x$.

~~(Note that $\check{x}$ is nice, for $x \in M$).~~

Then, if x is nice, we have:

$$G^*(x) = G^{*}{}''\mathrm{dom}(x) \; ; \; \bigcup G^*(x) \subset G^*(x).$$

Then:

<u>(a)</u> There is a p.r. fcn f s.t. if x is nice and φ is a Σ_0 fmla in $\mathfrak{M}$, then

$$G^*(f(x, \varphi)) = \{y \in G^*(x) \mid \models \varphi^G(y)\}.$$

proof:

$$f(x, \varphi) = \{\langle p, z \rangle \mid z \in \mathrm{dom}(x) \wedge p \Vdash \varphi(z)\}.$$

(b) There is a p.r. fcn d s.t. if x is nice, then $d(x)$ is nice and

$$G^*(d(x)) = \mathrm{Def}(\langle G^*(x); \in, \vec{B}\rangle).$$

proof.

Let Fml_x = the set of M-fmlae containing only ~~unbounded quan~~ ~~tif~~ constants from $\mathrm{dom}(x)$.

Let: $\varphi_{(x)}$ = the result of bounding all unbounded quantifiers in φ by $\underline{x}$. (Fml_x, $\varphi_{(x)}$ are p.r.).

~~Set: $d(x) = \{\langle 1, f(x, \varphi_{(x)}$~~

$$d(x) = x \cup \{\langle 1, f(x, \varphi_{(x)})\rangle \mid \varphi \in Fml_x\},$$

where f is as in (a). QED (b)

Since $\check{u}$ is nice for transitive $u \in M$, we can show by induction on ν that: $G^*(d^\nu(\check{u})) = L_\nu[u; \vec{B}]$.

QED

Lemma 8 If M is admissible and $B_1, \ldots, B_m$ are Δ_1, then N is admissible.

proof.

Let $\varphi(x,y)$ be a Σ_0 $\mathfrak{N}$-fmla.

Claim $\Vdash \bigwedge x \bigvee y\, \varphi \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi$.

[Note: $\Vdash \varphi$ means: $1 \Vdash \varphi$].

proof.

Let $p \Vdash \bigwedge x \bigvee y\, \varphi$; let $u \in M^{\mathbb{P}}$.

Claim Ex $v \in M^{\mathbb{P}}$ s.t.

$$p \Vdash \bigwedge x \in \underline{u} \bigvee y \in \underline{v}\, \varphi.$$

proof.

$\bigwedge x \in \mathrm{dom}(u) \bigwedge p' \leq p \bigvee p'' \leq p \bigvee y\ p'' \Vdash \varphi(\underline{x}, \underline{y})$.

Hence there is w s.t.

$\bigwedge x \in \mathrm{dom}(u) \bigwedge p' \leq p \bigvee p'' \leq p \bigvee y \in w\ p'' \Vdash \varphi(\underline{x}, \underline{y})$.

Setting: $v = \{\langle 1, y\rangle \mid y \in w\}$,

we get: $\bigwedge x \in \mathrm{dom}(u)\ p \Vdash \bigvee y \in \underline{v}\ \varphi(\underline{x}, y)$,

hence $p \Vdash \bigwedge x \in \underline{u} \bigvee y \in \underline{v}\ \varphi(x,y)$.

QED

The completeness theorem for countable M :

Lemma 10. If M is countable, then

$\bigwedge p \bigvee G$ (G is $\mathbb{P}$-generic over $M \wedge$
$\wedge\ p \in G$).

Hence;

$$p \Vdash \varphi \longleftrightarrow \bigwedge_{\substack{G \ni p \\ \mathbb{P}\text{-generic}}} G \Vdash \varphi .$$

Cohen generic reals :

Let us now give a concrete example of an extension by forcing. We wish to adjoin to M a new real number a (a is called a real number if $a \subset \omega$).

As conditions, we take bits of information which fix the characteristic function of a at finitely many places:

$\mathbb{P} = \{ p \mid p$ is a finite map from a subset of ω to $2 \}$.

$$p \le q \longleftrightarrow_{Df} p \supset q .$$

Let G be $\mathbb{P}$-generic. Then $\bigcup G$ is the characteristic function which we shall call a. Then

$$G = \{ p \mid p \subset \chi_a \}$$

($\chi_b =_{Df}$ the characteristic fcn of b), since, if $p \notin G$, then p is incompatible with some $p' \in G$; Thus $p \cup p'$ is not a function $\rightarrow$ $\rightarrow p \cup \chi_a$ is not a function $\rightarrow p \not\subset \chi_a$.

Hence $M[G] = M[a]$, since $a \in M[G]$ and ~~a is p.r. in~~ G is p.r. in a.

. . . .

This $\mathbb{P}$ is known as the set of <u>Cohen conditions</u>. $a \subset \omega$ is called <u>Cohen-generic over</u> M iff $G_a =_{Df} \{ p \mid p \subset \chi_a \}$ is $\mathbb{P}$-generic over M. $\langle a_1, \ldots, a_n \rangle$ is called a <u>Cohen generic</u> <u>n-tuple</u> iff $G_{a_1} \times \ldots \times G_{a_n}$ is $\mathbb{P}^n$-generic over M.