

## §6 Forcing with classes of conditions

The more interesting applications of forcing to admissible sets  $M$  generally involve a class of conditions rather than an  $M$ -finite set. As an example:

Theorem 1 If  $M = \langle |M|; E, A \rangle$  is a countable admissible structure and  $A \subset \text{On}$ , then there is an  $F \subset \text{On}^3$  s.t.  $N = \langle M[F]; A, F \rangle$  is admissible and every ordinal  $\nu \in \text{On}_M$  is countable by a map  $f \in F$ .

proof.

Let  $\rho_\nu$  ( $\nu \in \text{On}$ ) enumerate monotonically the p.r. closed ordinals.

The  $F$  which we shall adjoin will have the form:

$$F = \{ \langle f_\nu^{(m)}, \rho_\nu, m \rangle \mid \nu \in \text{On}_M, m < \omega \},$$

where  $f_{\nu}$  maps  $\omega$  onto  $f_{\nu}$ .

Our conditions are pieces of information which fix the function  $F$  at finitely many places:

$\mathbb{P} =_{pf}$  the set of finite maps  $p$   
 s.t.  $\text{dom}(p) \subset \{f_{\nu} \mid \nu \in \mathbb{O}_n^M\} \times \omega$   
 and  $p(f_{\nu}, m) \leq f_{\nu}$  whenever defined.  
 $p \leq q \iff_{pf} p \supset q$ .

We also define:

$\mathbb{P}_{\nu} =_{pf} \{p \in \mathbb{P} \mid \text{dom}(p) \subset f_{\nu} \times \omega\}$ .

Lemma 1  $A \subset \mathbb{P}_{\nu}$  is compatible in  $\mathbb{P}_{\nu}$ ,  
 then  $A$  is compatible in  $\mathbb{P}$ .

proof.

Let  $p \in \mathbb{P}$ . Then  $p' = p \upharpoonright (f_{\nu} \times \omega)$  is  
 compatible with some  $q \in A$ . It  
 follows that  $p$  is compatible  
 with  $q$ .  □ E.D

For  $G \subset \mathbb{P}$  set:  $G_{\downarrow} =_{\text{pf}} G \cap \mathbb{P}_{\downarrow}$ .

Lemma 2 If  $G$  is  $\mathbb{P}$ -generic over  $M$ , then  $G_{\downarrow}$  is  $\mathbb{P}_{\downarrow}$ -generic over  $M$ . More generally, if  $M', M''$  are transitive structures s.t.  $\mathbb{P}_{\downarrow}$  is  $M''$ -definable and  $\mathbb{P}, M''$  are  $M'$ -definable, then if  $G$  is  $\mathbb{P}$ -generic over  $M'$ ,  $\mathbb{P}_{\downarrow}$  is  $G_{\downarrow}$ -generic over  $M''$ .

proof.

Let  $C \subset \mathbb{P}_{\downarrow}$  be compatible in  $\mathbb{P}_{\downarrow}$  and  $M''$ -definable. Then  $C$  is compatible in  $\mathbb{P}$  and  $M'$ -definable. Hence  $G_{\downarrow} \cap C = G \cap C \neq \emptyset$ .  
QED

Now set:  $\mathcal{M} =_{\text{pf}} \langle M^{\mathbb{P}}; \mathbb{I}, E, \check{A}, \check{G} \rangle$ ,

$$\text{where } \check{A}(x) = \bigcup_{y \in A} [x \equiv y]$$

$$\check{G}(x) = \bigcup_{p \in \mathbb{P}} [x \equiv \check{p}] \cap [p]$$

Clearly, if  $G$  is  $\mathbb{P}$ -generic, then  $N = \mathcal{N}/G$  has the form:

$$N = \langle INI; A, G \rangle.$$

Define models  $M_\nu$  ( $\nu \in \text{On}_M$ ) by:

$$M_\nu = \langle IM_\nu, A \cap \rho_\nu \rangle$$

where  $IM_\nu = \{x \in M \mid \text{rn}(x) < \rho_\nu\}$ .

$IM_\nu$  is p.r. closed.

Define  $\text{BA}(\mathbb{P}_\nu)$ -valued structures

$$\mathcal{M}_\nu = \langle M_\nu^{\mathbb{P}_\nu}; I_\nu, E_\nu, \check{A}_\nu, \dot{G}_\nu \rangle$$

in analogy with  $\mathcal{M}$ , using  $\mathbb{P}_\nu$ ,

$A \cap \rho_\nu$  in place of  $\mathbb{P}, A$ .

Then  $N_\nu = \mathcal{M}_\nu / G_\nu$  has the form:

$$N_\nu = \langle IN_\nu, A \cap \rho_\nu, G_\nu \rangle.$$

Clearly:  $G_\nu^*(x) = G^*(x)$  for  $x \in M_\nu^{\mathbb{P}_\nu}$ .

Hence  $N_\nu \subseteq N_\kappa \subseteq N$  for  $\nu \leq \kappa < \omega_{n_M}$ .

For  $\nu = \tau + 1$ , we have:  $\Pi_\nu \in M_\nu$ ,

hence:  $|N_\nu| = M_\nu[G_\nu]$ .

Thus:

Lemma 3  $|N| = M[G]$ ;  $N_\nu = M_\nu[G_\nu]$

for  $\nu < \omega_{n_M}$ .

proof.

For  $\nu = \tau + 1$ , the theorem is proven.

$|N| \subseteq M[G]$ , since  $|N| = G^* \text{ " } M^{\mathbb{P}}$   
and  $f(x) = G^*(x)$  is p.r. in  $G$ .

But  $M[G] = \bigcup_{\tau \in \omega_{n_M}} M_{\tau+1}[G_{\tau+1}] \subseteq |N|$ .

Similarly we prove:  $|N_\lambda| = M_\lambda[G_\lambda]$   
for limit  $\lambda$ . QED

Let  $\Vdash_{\mathcal{M}}$  be the forcing relation of  $\mathcal{M}$  + let  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  be its restriction to  $\Sigma_0$  formulae. Similarly for  $\Vdash_{\mathcal{M}_1}$ ,  $\Vdash_{\mathcal{M}_1}^{\Sigma_0}$ . We do not yet know that  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  is  $M$ -definable, but we can show:

Lemma 4  $\{ \langle v, p, \varphi \rangle \mid p \Vdash_{\mathcal{M}_1}^{\Sigma_0} \varphi \}$  is  $\Delta_1$

proof. Since  $\Vdash_{\mathcal{M}_1}^{\Sigma_0}$  is uniformly p.r. in  $\check{A}, \check{C}$ , and the parameter  $\mathbb{P}_1$ , it suffices to note that the functions

$$f_0(v) = \mathbb{P}_1, \quad f_1(v, x) = \check{A}_1(x),$$

$$f_3(v, x) = \check{G}_1(x) \text{ are } \Delta_1. \quad \square \text{ EID}$$

The fact that  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  is  $\Delta_1$  now follows from:

Lemma 5 If  $\varphi$  is a  $\Sigma_0$  formula of  $\mathcal{M}$ , and if  $p \in \mathbb{P}$ , then

$$p \Vdash_{\mathcal{M}_p} \varphi \iff p \Vdash_{\mathcal{M}} \varphi.$$

proof.

By Lemma 2 and the completeness theorem, it suffices to show that for every  $G$  which is  $\mathbb{P}$ -generic over  $M$  we have:

$$G \Vdash_{\mathcal{M}} \varphi \iff G_p \Vdash_{\mathcal{M}_p} \varphi,$$

or equivalently:

$$\Vdash_{\mathcal{M}/G} \varphi^G \iff \Vdash_{\mathcal{M}_p/G_p} \varphi^{G_p}.$$

Since  $G^*(x) = G_p^*(x)$  for  $x \in M_p^{\mathbb{P}}$ ,

we may conclude:  $\varphi^G = \varphi^{G_p}$ .

By Lemma 3 we have:

$$\mathcal{M}/G = \langle M[G]; A, G \rangle$$

$$\mathcal{M}_\nu / G_\nu = \langle M_\nu[G_\nu]; A \cap P_\nu, G_\nu \rangle.$$

But  $A \cap M_\nu[G_\nu] = A \cap P_\nu$ ;  $G \cap M_\nu[G_\nu] = G_\nu$ ,

Since  $\varphi^G$  is  $\Sigma_0$ , the conclusion follows immediately. QED

An immediate corollary is:

Lemma 6  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  is  $\Delta_1$ .

proof.

$$P \Vdash_{\mathcal{M}} \varphi \iff P \Vdash_{\mathcal{M}}^{\eta(p, \varphi)} \varphi,$$

where  $\eta(p, \varphi) =$  the least  $\eta$   
s.t.  $p \in P_\eta$ ,  $\varphi \in \text{Fml } \mathcal{M}_\eta$ .

QED



We must now show that  $N = \mathcal{M}/G$  is admissible - i.e. that  $N$  satisfies the replacement axiom for  $\Sigma_0$ -formulae. In proving this, we shall make use of:

Lemma 7 Let  $D \subset \mathcal{P}$  be  $\Sigma_1$  and compatible in  $\mathcal{P}$ . Let  $\nu \in \mathcal{O}_n \mathcal{M}$ .

Then for some  $\beta \geq \nu$ ,  $D_\beta = D \cap \mathcal{P}_\beta$  is compatible in  $\mathcal{P}_\beta$  (hence in  $\mathcal{P}$ ).

(Note that  $D$  is not necessarily an element of  $\mathcal{M}$ , since  $D$  is  $\Sigma_1$  rather than  $\Delta_1$ ).

proof.

Set:  $\sigma(p) = \mu \tau \quad \forall q \in \mathcal{P}_\tau$  ( $p$  is compatible with  $q$ )

$$\eta(\tau) = \sup_{p \in \mathcal{P}_\tau} \sigma(p)$$

Then  $\beta = \eta^\omega(\nu)$  satisfies the conclusion of the lemma.  $\square$  QED

Lemma 8 If  $\varphi$  is a  $\Sigma_0$   $\mathcal{M}$ -formula,

then:

$$p \Vdash \forall x \varphi \rightarrow \forall x \in M^{\mathbb{P}} p \Vdash \forall x \varphi.$$

proof.

~~The direction~~ Let  $p \Vdash \forall x \varphi$ . Then

$$D = \tau[p] \cup \{q \leq p \mid \forall y \ q \Vdash \varphi(y)\}$$

is dense in  $\mathbb{P}$ . Let  $p \in \mathbb{P}_\nu$ . Then

there is a  $\beta \geq \nu$  s.t.  $D \cap \mathbb{P}_\beta$  is

dense in  $\mathbb{P}_\beta$ . Thus:

$$\bigwedge_{q \leq p} \bigwedge_{y \in \mathbb{P}_\beta} \bigwedge_{q' \leq q} q' \Vdash \varphi(y).$$

By admissibility there is a  $u \in M$

s.t.

$$\bigwedge_{q \leq p} \bigwedge_{y \in u} \bigwedge_{q' \leq q} q' \Vdash \varphi(y).$$

~~But since every  $p' \leq p$  is com-~~

~~We may conclude:~~

~~(\*)  $\bigwedge q$~~

Thus  $C = \{q \in \mathcal{P}_\beta \mid \forall y \in u \ q \Vdash \varphi(y)\}$   
is dense in  $[p] \cap \mathcal{P}_\beta$ , hence in  $\mathcal{P}$ .

Thus:

$$(*) \ \wedge q \leq p \ \forall q' \leq q \ \forall y \in u \ q' \Vdash \varphi(y).$$

Set:  $x = \{ \langle 1, y \rangle \mid y \in u \cap M^{\mathcal{P}} \}$ . Then:

$$p \Vdash \forall \varepsilon \in x \ \varphi. \quad \text{QED}$$

The fact that  $N$  is admissible follows by Lemma 8 and the fact that  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  is  $\Delta_1$ . We state this fact as a general lemma, for later reference:

Lemma 9. If  $M$  is an admissible structure,  $\mathcal{P}$  is a  $\Delta_1$  system of conditions and  $\mathcal{M} = \langle M^{\mathcal{P}}; I, E, A_1, \dots, A_n \rangle$  is a  $\text{BA}(\mathcal{P})$ -valued model s.t.  $\Vdash_{\mathcal{M}}^{\Sigma_0}$  is  $\Delta_1$  ~~this~~ and

if  $p \Vdash \forall x \varphi \rightarrow \forall u \ p \Vdash \forall x \varepsilon u \ \varphi$  for every  $\varphi \in \text{Fml}_{\mathcal{M}}^{\Sigma_0}$ , then  $\mathcal{M}/G$  is admissible for  $\mathbb{P}$ -generic  $G$ .

proof.

It suffices to show that the  $\Sigma_0$  replacement axiom is forced; i.e. if  $\varphi$  is a  $\Sigma_0$  formula of  $\mathcal{M}$ , then

$$\Vdash (\forall x \forall y \varphi \rightarrow \forall u \forall v \forall x \varepsilon u \forall y \varepsilon v \varphi) ,$$

or equivalently:

$$p \Vdash \forall x \forall y \varphi \rightarrow p \Vdash \forall u \forall v \forall x \varepsilon u \forall y \varepsilon v \varphi$$

for  $p \in \mathbb{P}$ . Let  $p \Vdash \forall x \forall y \varphi$ . Then:

$$\forall x \in M^{\mathbb{P}} \forall v \in M^{\mathbb{P}} \quad p \Vdash \forall y \varepsilon v \ \varphi(x, y).$$

Let  $u \in \underline{\quad} M^{\mathbb{P}}$ . Then there is a  $w \in M^{\mathbb{P}}$  s.t.

$$\forall x \in \text{dom}(u) \forall v \in w \quad p \Vdash \forall y \varepsilon v \ \varphi(x, y).$$

Set:  $v' = \cup w$ . Then:

$$p \Vdash \forall x \varepsilon u \forall y \varepsilon v' \ \varphi. \quad \text{QED}$$

Thus  $N = \langle M[G], A, G \rangle$  is admissible,

The only remaining step ~~is~~ in the proof of Thm 1 is to show that

$N$  is equivalent to a structure

$N' = \langle M[F]; A, F \rangle$ , where  $F \subset \text{On}_M^3$

is as above.

(Equivalence ~~is~~, of course, means:

$|N| = |N'|$  and the  $\Delta_1$  relations of  $N$  are the  $\Delta_1$  relations of  $N'$ ).

$F = \cup G$  obviously has the required properties.  $F$  (as a relation) is p.r. in  $G$  since:

$$\langle x, y, z \rangle \in F \iff \{ \langle x, y, z \rangle \} \in G.$$

Conversely,  $G$  is p.r. in  $F$  since:

$$p \in G \iff p \in F.$$

This completes the proof of Theorem 1.

Def Call  $d$  admissible in

$A_1, \dots, A_m \subset On$  iff  $\clubsuit$

$\langle L_d[A_1, \dots, A_m]; A_1 \cap d, \dots, A_m \cap d \rangle$   
is an admissible structure.

As a corollary to Thm 1, we obtain:

Theorem 2 Let  $\clubsuit \lambda$  be a <sup>countable</sup> limit of  
admissible ordinals. Let  $A \clubsuit \subset \lambda$ ,

Then there is a  $B \subset \lambda$  s.t.

(a)  $\forall d < \lambda$  is admissible in  $C \subset d$   
and  $C$  is  $\langle L_\lambda[A], A \rangle$  - definable  
then  $d$  is admissible in  $B, C$ .

(Thus, in particular, if  $d < \lambda$  is  
admissible in  $A$ ,  $d$  is admissible  
in  $A, B$ ).

(b)  $\rho$  is countable by a map

$f \in L_{\rho_{\nu+1}}[B]$  for  $\nu < \lambda$ .

proof of Thm 2.

As before, force with conditions:

$\mathbb{P}$  = the set of finite maps  $p$

s.t.  $\text{dom}(p) \subset \{\rho_\nu \mid \nu < \lambda\} \times \omega$

and  $p(\rho_\nu, n) < \rho_\nu$  if defined.

Let  $G$  be  $\mathbb{P}$ -generic over  $\langle L_\lambda[A], A \rangle$ .

Let  $F = \cup G$ . Then

$$F = \{ \langle f_\nu(n), \rho_\nu, n \rangle \mid \nu < \lambda, n < \omega \},$$

where  $f_\nu$  maps  $\omega$  onto  $\rho_\nu$ .

$$\text{Set } B = \{ \langle \alpha, \beta, \gamma \rangle \mid \langle \alpha, \beta, \gamma \rangle \in F \}.$$

(b) follows immediately.

To prove (a), we note that

$G_d$  is  $\mathbb{P}_d$ -generic over  $\langle L_d[c], c \rangle$ .

Hence, by the proof of Thm 1,

$\langle L_d[c, B], c, B \cap d \rangle$  is admissible.

QED

We now give a second application of forcing ~~at~~ with a class of conditions to add, as before, a class of ordinals to an admissible structure  $M$ . This application will differ from the previous one, however, in that no new sets will be added to the structure.

Def  $d$  is Mahlo (or recursively Mahlo) in  $A \subset d$  iff  $d$  is admissible in  $A$  and every ~~normal~~ normal function which is  $\Delta_1$  in  $\langle L_d[A], A \rangle$  takes a value which is admissible in  $A$ .



Theorem 3 Let  $d$  be Mahlo in  $A$ ,

Assume that for all  $\beta < d$ ,  
 $\beta$  is countable in  $L_{\beta'}[A]$ ,  
where  $\beta'$  = the least p.r. closed  
 $\beta' > \beta$ . Then there is a  $B < d$   
s.t.

(i)  $\langle L_d[A], A, B \rangle$  is admissible

(ii) If  $\gamma < d$  is admissible in  $A$ ,  
then  $\gamma$  is admissible in  $A, B$

(iii)  $d$  is non Mahlo in  $A, B$ .

proof.

It suffices that  $B$  satisfy

(i) and

(iv)  $B$  is unbounded in  $d$ , but if  
 $\gamma < d$  is admissible in  $A$ , then

~~$B \cap \gamma$~~   $B \cap \gamma \in L_\gamma[A]$ .

(ii) obviously follows from (iv).

To see that (iii) follows,

note that the monotone enumeration of the limit pts. of  $B$  is a  $\Delta_1$  normal function (in  $\langle L_\alpha[A], A, B \rangle$ ) ~~containing no~~ none of whose values are admissible in  $A$ .

Our conditions will describe initial segments of the set  $B$  which we wish to add:

$\mathcal{P} =_{\text{pf}}$  the collection of bounded subsets  $p$  of  $d$  s.t.

(i) For all  $\beta < d$ ,  $p \cap \beta \in L_{\beta'}[A]$ , where  $\beta'$  = the least p.n. closed  $\beta' > \beta$ .

(ii) If  $\beta \leq d$  is admissible in  $A$ , then  $p \cap \beta \in L_\beta[A]$ .

$$p \leq q \iff_{\text{pf}} q = p \cup \cup \{v+1 \mid v \in q\}$$

Set:  $\mathcal{M} = \langle L_d[A]^{\mathbb{P}}; \mathbb{I}, E, \check{A}, \check{B} \rangle,$

where:  $\check{A}(x) = \bigcup_{v \in A} [x \equiv v]$

$\check{B}(x) = \bigcup_{v < d} ([x \equiv v] \wedge \{p \in \mathbb{P} \mid v \in p\})$ .

Then  $N = \mathcal{M}/G = \langle |N|; A, B \rangle$ , for  $\mathbb{P}$ -generic  $G$ , where  $B$  is an unbounded subset of  $d$  satisfying (iv).

We must show that  $N$  is admissible and that  $|N| = L_d[A]$ .

Lemma 1.  $\forall G$  is  $\mathbb{P}$ -generic, then

$$|\mathcal{M}/G| = L_d[A],$$

proof.

Let  $\beta < d$  be p.r. closed.

Let  $p = \bigcup G \cap \beta$ . Then  $p \in G$ .

For  $x \in L_\beta[A]^{\mathbb{P}}$  set:

$$p^*(x) = \{p^*(y) \mid \forall q \geq p \langle q, y \rangle \in x\}.$$

Since  $G \cap L_\beta[A] = \{q \in L_\beta[A] \mid q \geq p\}$ ,

we have!

$$p^*(x) = G^*(x) \text{ for } x \in L_\beta[A]^{\mathbb{P}}.$$

But  $p^* \upharpoonright L_\beta[A]^{\mathbb{P}} \in L_\alpha[A]$ . QED

Lemma 2  $\mathbb{H}_{\mathcal{M}}^{\Sigma_0}$  is  $\Delta_1$  in  $\langle L_\alpha[A], A \rangle$ .

proof.

Let  $\beta < \alpha$  be ~~a limit~~ of p.r. closed,

Let  $\beta'$  be the next largest p.r. closed ordinal. Let

$p \in L_{\beta'}[A]$ ,  $p \in \mathbb{P}$  be s.t.

$\text{sup}(p) \geq \beta$ . Then for any  $\mathbb{P}$ -generic  $G$  s.t.  $p \in G$  we have:

(1)  $p^*(x) = G^*(x)$  for  $x \in L_\beta[A]^{\mathbb{P}}$

( $p^*$  being defined as in the proof of lemma 1)

(2)  $\mathbf{B} \cap \beta = p \cap \beta$  (where  $\mathbf{B} = G^*(\overset{\circ}{\mathbf{B}})$ ).

In particular, if  $\varphi$  is a  $\Sigma_0$  formula of  $\mathcal{M}$  containing only

constants  $x$  s.t.  $x \in L_\beta[A]^\mathbb{P}$ , then

$$G \Vdash \varphi \iff \mathbb{F} \langle L_\beta[A, p \cap \beta], A \cap \beta, p \cap \beta \rangle \varphi^{(p)}$$

(where  $\varphi^{(p)}$  is the result of replacing  $x$  by  $\underline{p^*(x)}$ ).

Since this holds for every  $\mathbb{P}$ -generic  $G \ni p$ , we conclude:

$$p \Vdash \varphi \iff \mathbb{F} \langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle \varphi^{(p)}$$

Now let  $p' \in \mathbb{P} \cap L_{\beta'}[A]$ . Every  $\mathbb{P}$ -generic  $G \ni p'$  contains a  $p \in L_\beta[A]$  s.t.  $p' \geq p$  and  $\text{sup}(p) \geq \beta$ . Hence:

$$p' \Vdash \varphi \iff \bigwedge p \in L_\beta[A] (p \leq p' \wedge \text{sup}(p) \geq \beta \rightarrow \mathbb{F} \langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle \varphi^{(p)})$$

$$\iff S(\varphi, p', \beta)$$

where  $S$  is  $\Delta_1$ .

Hence:

$$p \Vdash \varphi \iff S(\varphi, p, \eta(\varphi, p)),$$

where  $\eta(p, \varphi) =$  the least p.n. closed  $\beta$  s.t.  $p, \varphi \in L_\beta[A]$ . QED

Our main tool in showing that  $N = \mathbb{N}/G$  is admissible will be:

Lemma 3 Let  $D_i$  ( $i < \omega$ ) be a sequence of sets which are dense in  $\mathbb{P}$  and closed under extensions. Let  $\{ \langle p, i \rangle \mid p \in D_i \}$  be  $\Sigma_1$  in  $\langle L_\alpha[A], A \rangle$ . Then  $\bigcap_i D_i$  is dense in  $\mathbb{P}$ .

proof.

$$\text{Let } p \in D_i \iff \forall x \ Rxip,$$

where  $R$  is  $\Sigma_0$ .

For  $\delta < \alpha$  set:

$$D_i^\delta =_{\text{df}} \{ p \in \mathbb{P} \cap L_\delta[A] \mid \forall x \in L_\delta[A] \ Rxip \}$$

Claim There are arbitrarily large p.r. closed  $\delta < d$  s.t.

- (i)  $D_i^\delta$  is dense in  $\mathbb{P} \cap L_\delta[A]$  for  $i < \omega$
- (ii)  $\delta$  is not admissible in  $A$ .

proof.

Let  $\delta < d$  be s.t.  $\vec{x} \in L_\delta[A]$ , where  $\vec{x}$  are the constants occurring in the  $\Sigma_0$  definition of  $R$ .

Define  $\eta: \mathbb{P} \rightarrow d$  by:

$\eta(p) =$  the least  $\beta$  s.t.

$\Delta_{i < \omega} \forall q \leq p \forall x (q, x \in L_\beta[A] \wedge Rxiq)$ .

Then  $\eta$  is  $\Delta_1$ . Set:

$\bar{\eta}(\beta) =$  the least p.r. closed

$\beta' > \beta$  s.t.  $\Delta_{p \in \mathbb{P} \cap L_\beta[A]} \eta(p) < \beta'$ .

Then  $\bar{\eta}$  is  $\Delta_1$ . ~~Set~~

Let  $\delta = \bar{\eta}^\omega(\delta)$ .

Then  $\delta > \delta$  is p.r. closed and satisfies (i).

To show that  $\delta$  satisfies (ii), we note that  $\eta \uparrow (\mathbb{P} \cap L_\delta[A])$  is  $\Delta_1$  in  $\langle L_\delta[A], A \cap \delta \rangle$ . Thus, if  $\bar{\sigma}$  were admissible in  $A$ ,  $\langle \bar{\eta}^i(\delta) \mid i < \omega \rangle$  would be  $\Delta_1$  in  $\langle L_\delta[A], A \cap \delta \rangle$ . Contradiction!

QED (Claim)

Now let  $p \in \mathbb{P}$ . We must show that there is  $q \leq p$  with  $q \in \bigcap_i D_i$ .

Pick  $\delta$  s.t.  $p \in L_\delta[A]$ ,  $D_i^\delta$  is dense in  $\mathbb{P} \cap L_\delta[A]$  and  $\delta$  is not admissible in  $A$ .

~~Define  $q_i$  ( $i < \omega$ ) by:~~

~~$q_0 = p$~~

~~$q_{i+1} =$  the least  $q \leq q_i$  (in the canonical well ordering of  $L[A]$ ) s.t.  $\sup(q)$~~



Let  $f: \omega \leftrightarrow \delta$ ,  $f \in L_{\delta'}[A]$ ,  
where  $\delta'$  is the next largest  
p.r. closed ordinal. Define  
 $q_i \in \mathbb{P} \cap L_{\delta}[A]$  ( $i < \omega$ ) by;

$$q_0 = p$$

$q_{i+1}$  = The least  $q \leq q_i$  (in the  
canonical well ordering of  
 $L[A]$ ) s.t.  $q \in D_i^{\delta}$  and  
 $\sup(q) \geq f(i)$ .

Set  $q = \bigcup_i q_i$ . Then  $q \in \mathbb{P}$ ,  
since  $\sup(q) = \delta$  and  $q \in L_{\delta'}[A]$ .  
Hence  $q \leq p \wedge q \in \bigcap_i D_i$ .

QED

We are now ready to show:

Lemma 4 If  $G$  is  $\mathbb{P}$ -generic,  
then  $\mathcal{M}/G$  is admissible.  
proof.

Since all sets are countable  
in  $\mathcal{M}/G$ , it suffices to show  
that  $\mathcal{M}/G$  satisfies:

$\bigwedge i < \omega \forall y R_{iy} \rightarrow \forall u \bigwedge i < \omega \forall y \in u R_{iy}$   
for  $\Sigma_0$   $\mathbb{P}$ . That is, we must  
prove:

$\Vdash \bigwedge x \in \check{\omega} \forall y \varphi \rightarrow \forall u \bigwedge x \in \check{\omega} \forall y \in u \varphi$   
for  $\Sigma_0$   $\mathcal{M}$ -formulae  $\varphi$ .

Let  $p \Vdash \bigwedge x \in \check{\omega} \forall y \varphi$ .

Set:  $D_i = \{q \in p \mid \forall y \ q \Vdash \varphi(i, \underline{y})\} \cup$   
 $\cup \neg[p]$

for  $i < \omega$ . Then  $D_i$  is dense  
in  $\mathbb{P}$  and  $\{\langle i, x \rangle \mid x \in D_i\}$  is  $\Sigma_1$ .

Hence  $\bigcap_i D_i$  is dense in  $\mathbb{P}$ .

This means that for every  $p' \in \mathbb{P}$  there is a  $p'' \leq p'$  s.t.

$$\bigwedge i < \omega \quad \forall y \quad p'' \Vdash \varphi(i, \underline{y}).$$

By the replacement axiom there is a ~~set~~  $\sigma$  s.t.

$$\bigwedge i < \omega \quad \forall y \in \sigma \quad p'' \Vdash \varphi(i, \underline{y}).$$

$$\text{Set: } \tilde{\sigma} = \{ \langle i, y \rangle \mid y \in \sigma \cap \mathbb{P} \}.$$

Then:

$$p'' \Vdash \bigwedge x \in \check{\omega} \quad \forall y \in \tilde{\sigma} \quad \varphi.$$

$$\text{Hence } p \Vdash \forall \sigma \bigwedge x \in \check{\omega} \quad \forall y \in \sigma \quad \varphi.$$

QED

We now turn to the major theorem of this section:

Theorem 4 Let  $B \subset \text{On}$ . Let  $d_\nu$  ( $\nu < \delta$ ) be a countable sequence of countable ordinals s.t.  $d_\nu > \omega$  and  $d_\nu$  is admissible in  $B \cap d_\nu$ ,  $\{d_\nu \mid \nu < \delta\}$  for  $\nu < \delta$ . Then there is a bcw s.t.

- (i)  $d_\nu$  is the  $\nu$ -th  $d > \omega$  s.t.  $d$  is admissible in  $b$
- (ii)  $B \cap d_\nu$  is  $\Delta_1$  in  $L_{d_\nu}[b]$  for  $\nu < \delta$ .

Thm 4 follows immediately from the conjunction of the following two theorems:

Theorem 4.1 Let  $\lambda$  be a countable limit of admissible ordinals, let  $B \subset \lambda$ . Let  $A \subset \lambda$  be s.t.  $\omega \notin A$  and each  $d \in A$  is admissible in  $B \cap d, A \cap d$ . Then there is a  $C \subset \lambda$  s.t. for all  $d < \lambda$ ;

- (i)  $d$  is admissible in  $C$  iff  $d \in A$
- (ii) If  $d$  is admissible in  $C$ , then  $B \cap d, A \cap d$  are  $\Delta_1$  in  $\langle L_d[C], C \cap d \rangle$ .
- (iii) If  $d$  is p.r. closed and  $d'$  is the next largest p.r. closed ordinal, then  $d$  is countable in  $L_{d'}[C \cap d]$ .

Theorem 4.2 Let  $\lambda$  be a countable limit of admissible ordinals and let  $C \subset \lambda$  satisfy (iii) of Thm 4.1. Then there is a  $b < \omega$  s.t. for all  $d < \lambda$  s.t.  $d > \omega$ :

(ii) If  $d$  is admissible in  $C$ , then  
 $d$  is admissible in  $b$

(iii) If  $d$  is admissible in  $b$ , then  
 $C \cap d$  is  $\Delta_1$  in  $L_d[b]$

(Hence  $d$  is admissible in  $b$  iff  
 $d$  is admissible in  $C$ ).

. . . . .

We begin with the proof of  
Thm 4.1.

Lemma 1 There is an  $A < \lambda$  s.t. for  $d < \lambda$

(a)  $d$  is admissible in  $B \cap d$ , and iff  
 $d$  is admissible in  $A$

(b) If  $d$  is admissible in  $A$ , then  $B \cap d$ ,  
and are  $\Delta_1$  in  $\langle L_d[A], A \rangle$

(c) If  $d$  is p.r. closed, then  $d$  is count-  
able in  $L_{d'}[A]$ , where  $d'$  is the next  
largest p.r. closed ordinal.

proof of Lemma 1.

By Thm 2 there is an  $A' < \lambda$  satisfying (iii) s.t., whenever  $d < \lambda$  is admissible in  $A \cap d, B \cap d$ , then  $d$  is admissible in  $A \cap d, B \cap d, A' \cap d$ . Set:

$$A = \{3 \cdot v \mid v \in A\} \cup \{3 \cdot v + 1 \mid v \in B\} \cup \{3 \cdot v + 2 \mid v \in A'\}$$

Then  $A$  satisfies (i) - (iii). QED

Now ~~let~~ let  $A^*$  be the set of  $d < \lambda$  s.t.  $d$  is admissible in  $A$ , (Hence  $A \subset A^*$ ).

Lemma 2 There is an  $f: A^* \rightarrow \lambda$  s.t.

(i)  $f(d) < d$  for  $d \in A^*$

(ii)  $f$  is 1-1

(iii)  $\{d \in A^* \mid f(d) < \kappa\}$  is finite for  $\kappa < \lambda$

(iv) If  $d \in A^*$ , then  $d$  is admissible in  $A, f \cap d, \text{rng}(f \cap d)$ . (i.e.

$\langle L_d[A, f \cap d, \text{rng}(f \cap d)]; A \cap d, f \cap d, \text{rng}(f \cap d) \rangle$   
is admissible)

proof of Lemma 2.

Let  $\rho_\nu$  ( $\nu \leq \lambda$ ) be the monotone enumeration of the p.r. closed ordinals  $\leq \lambda$ . Let  $\Theta_\nu$  ( $\nu \leq \lambda$ ) be the set of maps  $f$  defined on  $A^* \wedge \rho_\nu$  s.t.  $f$  satisfies (i)-(iv) and  $f \in L_{\rho_{\nu+1}}[A]$ . Set:

$$f \leq f' \iff_{\text{pt}} \forall \nu \leq \tau \ ( \nu \leq \tau \wedge f \in \Theta_\nu \wedge f' \in \Theta_\tau \wedge f \subset f' \wedge \text{rng}(f' \setminus f) \subset \rho_\tau \setminus \rho_\nu ).$$

By induction on  $\kappa < \lambda$  we prove:

Claim  $\Theta_\kappa \neq \emptyset$ ; moreover, if  $\nu < \kappa$ ,  $\nu \notin A^*$  and  $f \in \Theta_\nu$ , then there is an  $f' \in \Theta_\kappa$  s.t.  $f \leq f'$ .

proof.

Case 1  $\kappa = 0$ :  $\emptyset \in \Theta_0$ .

Case 2  $\kappa = \tau + 1$ ;  $\tau \notin A^*$ . Then  $\Theta_\kappa = \Theta_\tau$

Case 3  $\kappa = d + 1$ ;  $d \in A^*$ .

$\forall \nu < d$ ,  $\nu \notin A^*$ ,  $f \in \Theta_\nu$ , then



pick  $\bar{f} \in \Theta_\alpha$  s.t.  $\bar{f} \geq f$ . ~~Let  $\bar{f} \in \Theta_\alpha$~~

Let  $u \in \alpha \setminus \beta$  s.t.  $u \notin \text{rng}(f)$ .

Set:  $f' = f \cup \{\langle u, \alpha \rangle\}$ .

Case 4  $\text{Lim}(\kappa); \kappa \notin A^*$ .

Since  $\beta$  is countable in  $L_{\beta_{\kappa+1}}[A]$ , there

is a  $g \in L_{\beta_{\kappa+1}}[A]$  s.t.  $g: \omega \rightarrow \kappa$  is

monotone + cofinal in  $\kappa$ . We may assume:  $g(0) = \nu$  and  $g(i) \notin A^*$  for

$i < \omega$ . Define  $f_i$  ( $i < \omega$ ) by:

$$f_0 = f$$

$f_{i+1} =$  the least  $f' \in \Theta_{g(i+1)}$  ~~s.t.~~

(in the canonical well ordering of  $L[A]$ ) s.t.

$$f' \geq f_i.$$

Set:  $f' = \bigcup_i f_i$ . Then  $f' \in \Theta_\kappa$  and

$$f' \geq f.$$

Case 5  $\kappa = d \in \mathcal{A}^*$ .

By Thm 3 there is a  $B \in \mathcal{A}$  s.t.

$d$  is admissible in  $A, B$ ,  $B$  is unbounded in  $d$ ,  $B \cap d'$  is bounded in

$d'$  for every  $d' < d$  s.t.  $d' \in \mathcal{A}^*$  and

$B \cap \rho_{\nu} \in L_{\rho_{\nu+1}}[A]$  for  $\nu < d$ . From the

proof of Thm 3 it is clear that

we may assume:  $B \in L_{\rho_{d+1}}[A]$ , for

$L_d[A]$  is countable in  $L_{\rho_{d+1}}[A]$ ;

hence we can find  $G \in L_{\rho_{d+1}}[A]$  s.t.

$G$  is  $\mathbb{P}$ -generic over  $\langle L_d[A], A \rangle$ ,

$\mathbb{P}$  being the system of conditions used in the proof of Thm 3.

Assume:  $B \in L_{\rho_{d+1}}[A]$  and let

$g: d \rightarrow \mathcal{A}$  be the monotone enumeration of the limit points of  $B$ .

~~Then  $d$  is admissible~~

Then  $g$  is a normal function

taking no ~~admissible~~ values in  $A^*$ ,

$g \in L_{\rho_{d+1}}[A]$  and  $d$  is admissible

in  $A, g$ . Moreover,  $g \upharpoonright \rho_{\nu} \in L_{\rho_{\nu+1}}[A]$  for

$\nu < d$ . ~~Let~~

let  $\nu < d$ ; ~~let~~  $f \in \Theta_{\nu}$ . We may assume without loss of generality that  $g(0) = \nu$ . Define a sequence

$f_{\nu} (\nu < d)$  by:

$$f_0 = f$$

$f_{\nu+1}$  = the least  $f' \in \Theta_{\nu+1}$  (in the canonical well ordering of  $L[A]$ )  
 s.t.  $f' \geq f_{\nu}$ ,

$$f_{\delta} = \bigcup_{\nu < \delta} f_{\nu} \quad \text{for limit } \delta$$

(hence  $f_{\delta} \in \Theta_{\delta}$ ;  $f_{\delta} \geq f_{\nu}$  for  $\nu < \delta$ ).

~~Let~~ The sequence  $f_{\nu} (\nu < d)$  is

$\Delta_1$  in  $\langle L_d[A, g], A, g \rangle$ .

$$\text{Set: } f' = \bigcup_{\nu} f_{\nu}.$$

Then  $f'$  is  $\Delta_1$  in  $\langle L_\alpha[A, g], A, g \rangle$

since:

$$v = f'(\tau) \iff v = f_{\tau+1}(\tau).$$

$\text{rng}(f')$  is  $\Delta_1$  in  $\langle L_\alpha[A, g], A, g \rangle$

since:

$$v \in \text{rng}(f') \iff v \in \text{rng}(f_{i+1}).$$

Hence

~~B~~  $f' \geq f$  and  $f' \in \Theta_\alpha$ .

QED (Claim).

Lemma 2 follows almost immediately from the claim: Let  $\lambda_i$  ( $i < \omega$ ) be monotone + cofinal in  $\lambda$ . Assume:  $\lambda_i \notin A^*$ . Select

$f_i$  ( $i < \omega$ ) s.t.  $f_i \in \Theta_{\lambda_i}$  and  $f_{i+1} \geq f_i$ . Set:  $f = \bigcup_i f_i$ .

Then  $f$  satisfies (i) - (iv) of Lemma 2. QED

Now let  $f$  be as in Lemma 2 and define a map  $\tilde{f}; (A^* \setminus A) \times \omega \rightarrow \lambda$  by:  $\tilde{f}(d, i) = g_d(i)$ , where:

$g_d =$  the least  $g$  (in the canonical well ordering of  $L[A]$ ) s.t.  $g$  is a monotone map from  $\omega$  to  $d$  and  $\sup_{i < \omega} g(i) = d$ .

Set:

$$D = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in A^* \setminus A, i < \omega \},$$

and

$$D_d = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in (A^* \setminus A) \cap d, i < \omega \}$$

for  $d < \lambda$ . Then if  $d \in A^*$ ,  $D_d$  is

$\Delta_1$  in  $\langle L_d[A], \text{And}, f \upharpoonright d, \text{rng}(f \upharpoonright d) \rangle$ .

It follows that, if  $d \in A$ ,  $D \cap d$  is

also  $\Delta_1$ , since  $D \cap d$  differs from  $D_d$

by at most finitely many points.

On the other hand, if  $d \in A^* \setminus A$ ,

$d$  is not admissible in  $D \cap d$ .

since, if it were, we should have:  
 $\text{rng}(g_d) = \{v \mid \langle f(d), v \rangle \in D \cap d\}$  is  
 $\Delta_1$  in  $\langle L_d[D], D \cap d \rangle$ ; hence  $g_d$ , being  
 the monotone enumeration of  $\text{rng}(g_d)$ ,  
 would be  $\Delta_1$ . But this violates the  
 replacement axiom.

We now define the set  $C$ , whose  
 existence is asserted in Theorem 4.1.

$$C = \{4 \cdot d \mid d \in A\} \cup \{4 \cdot \langle d, \beta \rangle + 1 \mid d = f(\beta)\} \cup \\ \cup \{4 \cdot d + 2 \mid d \in \text{rng}(f)\} \cup \{4d + 3 \mid d \in D\}.$$

For  $d < \lambda$  we define  $C_d$  similarly,  
 using  $A \cap d$ ,  $f \upharpoonright d$ ,  $\text{rng}(f \upharpoonright d)$ ,  $D_d$  in  
 place of  $A$ ,  $f$ ,  $\text{rng}(f)$ ,  $D$ . Clearly,  
 if  $d \in A^*$ , then  $C_d$  is  $\Delta_1$  in  
 $\langle L_d[A]; A \cap d, f \upharpoonright d, \text{rng}(f \upharpoonright d) \rangle$ . Hence  
 $C \cap d$  is  $\Delta_1$  for  $d \in A$ , since  $C_d$  differs  
 from  $C_d$  by at most finitely many  
 elements. Thus  $d$  is admissible

in  $C$  if  $d \in A$ . On the other hand, if  $d \notin A$ , then  $d$  is not admissible in  $C$ , since, ~~if it were, we should have  $D \cap d$  is  $\Delta_1$  in  $L_d$~~ , if it were, we should have:  $d \in A^* \setminus A$ , since  $A \cap d$  is  $\Delta_1$  in  $\langle L_d[C], C \cap d \rangle$ . But then  ~~$D \cap d$  is~~  $d$  is not admissible in  $C$ , since  $D \cap d$  is  $\Delta_1$  in  $\langle L_d[C], C \cap d \rangle$  and  $d$  is not admissible in  $D \cap d$ . This establishes (i) of Theorem 4.1, (ii) follows from Lemma 1 and the fact that  $A \cap d$  is  $\Delta_1$  in  $\langle L_d[C], C \cap d \rangle$ , (iii) is established by the following lemma:

Lemma 3  $\rho_d$  is countable in  $L_{\rho_{d+1}}[C \cap d]$

for  $d < \lambda$ .

proof of Lemma 3.

We assume the following facts:

(1)  $f_0$  is countable in  $L_{f_1}$

(2)  $f_\nu$  is mappable onto  $f_{\nu+1}$  by a function  $g \in L_{f_{\nu+2}}$ .

By (2) we get:

(3) If  $\text{Lim}(d)$  and  $f_\alpha > d$ , then  $d$  is mappable onto  $f_\alpha$  by a function  $g \in L_{f_\alpha}$ .

proof of (3).

Suppose not. Let  $\kappa$  be the least  $\kappa \leq d$  s.t.  $d$  is not mappable onto  $f_\kappa$  by a map  $g \in L_{f_{\kappa+1}}$ . For  $\nu < \kappa$  let

$g_\nu : d \rightarrow f_\nu$  be the least  $g \in L_{f_{\nu+1}}$  (in

the canonical well ordering of  $L$ )

s.t.  $g$  maps  $d$  onto  $f_\nu$ .  $\kappa$  is a limit

ordinal by (2); hence the function

$$g(\langle \nu, u \rangle) = g_\nu(u)$$



maps  $\kappa \times d$  onto  $f_\kappa$ . It is easily seen

that  $g \in L_{f_{\kappa+1}}$ . But since  $\kappa \leq d$

there is an  $h \in L_{f_{\kappa+1}}$  which maps

~~$\kappa \times d$  onto  $d$~~ ,  $d$  onto  $\kappa \times d$ . Hence!

$g \circ h \in L_{f_{\kappa+1}}$  and  $g \circ h$  maps  $d$  onto  $f_\kappa$ .

Contradiction! QED (3).

Now suppose Lemma 3 to be false.

Let  $d$  be the least ordinal for which

Lemma 3 fails. Then  $d = f_d$  by (1)-(3).

(hence  $L_{f_d}[Cnd] = L_d[C]$ ).

(4)  $d$  is admissible in  $C$ . Moreover,

if  $R \in L_d[C]$  and  $R \in L_{f_{d+1}}[Cnd]$ ,

then:

(\*)  $\forall x \forall y Rxy \rightarrow \exists u \forall v \exists x \in u \forall y \in v Rxy$

holds in ~~the~~  $L_d[C]$ .

proof of (4).

Since each  $x \in L_d[C]$  is countable in  $L_d[C]$ , it suffices to show:

$$(**) \quad \bigwedge_{m < \omega} \forall y \in R_{xy} \rightarrow \forall \alpha \bigwedge_{m < \omega} \forall y \in \alpha \ R_{my}.$$

Suppose not. Set:

$$h(m) = \mu \delta \forall y \in L_\delta[C] \ R_{my} \quad (m < \omega).$$

Then  $\sup_{m < \omega} h(m) = d$ . Set:

$g_m = \nu$  the least  $g \in L_d[C]$  (in the canonical well ordering of  $L[C]$ ) s.t.  $g$  maps  $\omega$  onto  $h(m)$ .

Then  $g(\langle m, m \rangle) = g_m(m)$  maps  $\omega$  onto  $d$  and  $g \in L_{\delta}^{d+1}[C]$ . Contradiction!

QED

Using (4), we get:

(5)  $d$  is Mahlo in ~~and~~  $C$ .

proof of (5)

Let  $D$  be a closed, unbounded subset of  $d$  which is  $\Delta_1$  in  $\langle L_d[C], \text{Cnd} \rangle$ .

We must show:  $\beta \in D$  for some  $\beta < d$  which is admissible in  $C$ .

Let  $\vec{x}$  be all constants occurring in the  $\Delta_1$  definition of  $D$ . Let  $u$  be the set of  $x \in L_d[C]$  which are

$\langle L_d[C], \text{Cnd} \rangle$ -definable in the parameters  $\vec{x}$ . Since there is an  $\langle L_d[C], \text{Cnd} \rangle$ -definable well ordering of  $L_d[C]$ ,

we have:  ~~$\langle u, \text{Cnd} \rangle$  is an~~

~~elementary submodel of~~

(\*)  $\langle u, \text{Cnd} \rangle$  is an elementary submodel of  $\langle L_d[C], \text{Cnd} \rangle$ .

It follows that  $u$  is transitive, for by (\*), if  $x \in u$  ~~then~~ there is ~~an~~ ~~for~~

a  $g \in u$  s.t.  $g$  maps  $w$  onto  $x$ .

Hence:  $x = g^{\omega} w \in u$ .

Let  $\beta = d \cap u$ . By the condensation lemma we have:  $u = L_\beta[C]$ .

By (\*)  $\langle L_\beta[C], C \cap \beta \rangle$  satisfies the admissibility axioms, since  $\langle L_\alpha[C], C \cap \alpha \rangle$  does. Hence  $\beta$  is admissible in  $C$ . We now claim:

$\beta < \alpha$ . To see this, we let  $\tilde{u}$  be the set of formulae (with  $\varepsilon, \exists, \dot{C}$ ) containing at most the constants  $\vec{x}$  and the free vbl  $v_0$ . Set:

$h(\varphi) =_{\text{df}}$  the least  $y \in L_\alpha[C]$  (in the canonical well ordering of  $L[C]$ ) s.t.  $\models \varphi(v_0/y)$   
 $\langle L_\alpha[C], C \cap \alpha \rangle$

if such  $y$  exists

$=_{\text{df}} \emptyset$  if not.

Then  $\tilde{u} \in L_\alpha[C]$  and  $h \in L_{\alpha+1}[C \cap \alpha]$ .

By (4) there is a  $v \in L_\alpha[C]$  s.t.

$u = h''\tilde{u} \subset v$ . Hence  $\beta = d \cap u < \alpha$ .

By (\*),  $D \cap \beta$  is closed and unbounded in  $\beta$ . Hence  $\beta$  is a limit point of  $D$ .

Hence  $\beta \in D$

QED (5).

(5) leads to a contradiction, however, since:

~~(6)  $d < \lambda$  is not Mahlo in  $C$ .~~

(6)  $d < \lambda$  is not Mahlo in  $C$ .

proof.

We may assume:  $d \in \mathcal{A}$ . Since  $f \upharpoonright d$ ,  $\text{rng}(f \upharpoonright d)$  are  $\Delta_1$  in  $\langle L_d[C], C \cap d \rangle$ ,

we can define a  $\Delta_1$  normal function  $g$  by:

$$g(0) = 0$$

$$g(v+1) = \text{the least } \beta > g(v) \text{ s.t.}$$

$$f^{-1}(v) < \beta \text{ for all } v \in \text{rng}(f \upharpoonright d) \cap g(v)$$

$$g(\delta) = \sup_{v < \delta} g(v) \text{ for limit } \delta.$$

Clearly,  $g$  takes no values in  $\mathcal{A}^*$ .

QED

This completes the proof of Thm 4.1.