

§6 Forcing with classes of conditions

The more interesting applications of forcing to admissible sets M generally involve a class of conditions rather than an M -finite set. As an example:

Theorem 1 If $M = \langle |M|; \epsilon, A \rangle$ is a countable admissible structure and $A \subset \text{On}$, then there is an $F \subset \text{On}^3$ s.t. $N = \langle M[F]; A, F \rangle$ is admissible and every ordinal $\nu \in \text{On}_M$ is countable by a map $f \in F$.

proof.

Let ρ_ν ($\nu \in \text{On}$) enumerate monotonically the p.r. closed ordinals.

The F which we shall adjoin will have the form:

$$F = \{ \langle f_\nu^{(m)}, \rho_\nu, m \rangle \mid \nu \in \text{On}_M, m < \omega \},$$

where f_{ν} maps ω onto f_{ν} .

Our conditions are pieces of information which fix the function F at finitely many places:

$\mathbb{P} =_{pf}$ the set of finite maps p
 s.t. $\text{dom}(p) \subset \{f_{\nu} \mid \nu \in \mathbb{O}_n^M\} \times \omega$
 and $p(f_{\nu}, m) \leq f_{\nu}$ whenever defined.
 $p \leq q \iff_{pf} p \supset q$.

We also define:

$\mathbb{P}_{\nu} =_{pf} \{p \in \mathbb{P} \mid \text{dom}(p) \subset f_{\nu} \times \omega\}$.

Lemma 1 $A \subset \mathbb{P}_{\nu}$ is compatible in \mathbb{P}_{ν} ,
 then A is compatible in \mathbb{P} .

proof.

Let $p \in \mathbb{P}$. Then $p' = p \upharpoonright (f_{\nu} \times \omega)$ is
 compatible with some $q \in A$. It
 follows that p is compatible
 with q . □ E.D

For $G \subset \mathbb{P}$ set: $G_{\downarrow} =_{\text{pf}} G \cap \mathbb{P}_{\downarrow}$.

Lemma 2 If G is \mathbb{P} -generic over M , then G_{\downarrow} is \mathbb{P}_{\downarrow} -generic over M . More generally, if M', M'' are transitive structures s.t. \mathbb{P}_{\downarrow} is M'' -definable and \mathbb{P}, M'' are M' -definable, then if G is \mathbb{P} -generic over M' , \mathbb{P}_{\downarrow} is G_{\downarrow} -generic over M'' .

proof.

Let $C \subset \mathbb{P}_{\downarrow}$ be compatible in \mathbb{P}_{\downarrow} and M'' -definable. Then C is compatible in \mathbb{P} and M' -definable. Hence $G_{\downarrow} \cap C = G \cap C \neq \emptyset$.
QED

Now set: $\mathcal{M} =_{\text{pf}} \langle M^{\mathbb{P}}; \mathbb{I}, E, \check{A}, \check{G} \rangle$,

$$\text{where } \check{A}(x) = \bigcup_{y \in A} [x \equiv y]$$

$$\check{G}(x) = \bigcup_{p \in \mathbb{P}} [x \equiv \check{p}] \cap [p]$$

Clearly, if G is \mathbb{P} -generic, then $N = \mathcal{N}/G$ has the form:

$$N = \langle \mathcal{N}; A, G \rangle.$$

Define models M_ν ($\nu \in \mathcal{O}_n$) by:

$$M_\nu = \langle \mathcal{M}_\nu, A \cap \rho_\nu \rangle$$

where $\mathcal{M}_\nu = \{x \in \mathcal{M} \mid \pi_n(x) < \rho_\nu\}$.

\mathcal{M}_ν is p.r. closed.

Define $\text{BA}(\mathbb{P}_\nu)$ -valued structures

$$\mathcal{N}_\nu = \langle M_\nu^{\mathbb{P}_\nu}; I_\nu, E_\nu, \check{A}_\nu, \dot{G}_\nu \rangle$$

in analogy with \mathcal{N} , using \mathbb{P}_ν ,

$A \cap \rho_\nu$ in place of \mathbb{P}, A .

Then $N_\nu = \mathcal{N}_\nu / G_\nu$ has the form:

$$N_\nu = \langle \mathcal{N}_\nu; A \cap \rho_\nu, G_\nu \rangle.$$

Clearly: $G_\nu^*(x) = G^*(x)$ for $x \in M_\nu^{\mathbb{P}_\nu}$.

Hence $N_\nu \subseteq N_\kappa \subseteq N$ for $\nu \leq \kappa < \omega_{n_M}$.

For $\nu = \tau + 1$, we have: $\Pi_\nu \in M_\nu$,

hence: $|N_\nu| = M_\nu[G_\nu]$.

Thus:

Lemma 3 $|N| = M[G]$; $N_\nu = M_\nu[G_\nu]$

for $\nu < \omega_{n_M}$.

proof.

For $\nu = \tau + 1$, the theorem is proven.

$|N| \subseteq M[G]$, since $|N| = G^* \text{ " } M^{\mathbb{P}}$
and $f(x) = G^*(x)$ is p.r. in G .

But $M[G] = \bigcup_{\tau \in \omega_{n_M}} M_{\tau+1}[G_{\tau+1}] \subseteq |N|$.

Similarly we prove: $|N_\lambda| = M_\lambda[G_\lambda]$
for limit λ . QED

Let $\Vdash_{\mathcal{M}}$ be the forcing relation of \mathcal{M} + let $\Vdash_{\mathcal{M}}^{\Sigma_0}$ be its restriction to Σ_0 formulae. Similarly for $\Vdash_{\mathcal{M}_1}$, $\Vdash_{\mathcal{M}_1}^{\Sigma_0}$. We do not yet know that $\Vdash_{\mathcal{M}_1}^{\Sigma_0}$ is M -definable, but we can show:

Lemma 4 $\{ \langle v, p, \varphi \rangle \mid p \Vdash_{\mathcal{M}_1}^{\Sigma_0} \varphi \}$ is Δ_1

proof. Since $\Vdash_{\mathcal{M}_1}^{\Sigma_0}$ is uniformly p.r. in \check{A}, \check{C} , and the parameter \mathbb{P}_1 , it suffices to note that the functions

$$f_0(v) = \mathbb{P}_1, \quad f_1(v, x) = \check{A}_1(x),$$

$$f_3(v, x) = \check{G}_1(x) \text{ are } \Delta_1. \quad \square \text{ EID}$$

The fact that $\Vdash_{\mathcal{M}}^{\Sigma_0}$ is Δ_1 now follows from:

Lemma 5 If φ is a Σ_0 formula of \mathcal{M} , and if $p \in \mathbb{P}$, then

$$p \Vdash_{\mathcal{M}_p} \varphi \iff p \Vdash_{\mathcal{M}} \varphi.$$

proof.

By Lemma 2 and the completeness theorem, it suffices to show that for every G which is \mathbb{P} -generic over M we have:

$$G \Vdash_{\mathcal{M}} \varphi \iff G_p \Vdash_{\mathcal{M}_p} \varphi,$$

or equivalently:

$$\Vdash_{\mathcal{M}/G} \varphi^G \iff \Vdash_{\mathcal{M}_p/G_p} \varphi^{G_p}.$$

Since $G^*(x) = G_p^*(x)$ for $x \in M_p$, \mathbb{P}_p ,

we may conclude: $\varphi^G = \varphi^{G_p}$.

By Lemma 3 we have:

$$\mathcal{M}/G = \langle M[G]; A, G \rangle$$

$$\mathcal{M}_\nu / G_\nu = \langle M_\nu[G_\nu]; A \cap P_\nu, G_\nu \rangle.$$

But $A \cap M_\nu[G_\nu] = A \cap P_\nu$; $G \cap M_\nu[G_\nu] = G_\nu$,

Since φ^G is Σ_0 , the conclusion follows immediately. QED

An immediate corollary is:

Lemma 6 $\Vdash_{\mathcal{M}}^{\Sigma_0}$ is Δ_1 .

proof.

$$P \Vdash_{\mathcal{M}} \varphi \iff P \Vdash_{\mathcal{M}}^{\eta(p, \varphi)} \varphi,$$

where $\eta(p, \varphi) =$ the least η
s.t. $p \in P_\eta$, $\varphi \in \text{Fml } \mathcal{M}_\eta$.

QED

We must now show that $N = \mathcal{M}/G$ is admissible - i.e. that N satisfies the replacement axiom for Σ_0 -formulae. In proving this, we shall make use of:

Lemma 7 Let $D \subset \mathcal{P}$ be Σ_1 and compatible in \mathcal{P} . Let $\nu \in \mathcal{O}_n \mathcal{M}$.

Then for some $\beta \geq \nu$, $D_\beta = D \cap \mathcal{P}_\beta$ is compatible in \mathcal{P}_β (hence in \mathcal{P}).

(Note that D is not necessarily an element of \mathcal{M} , since D is Σ_1 rather than Δ_1).

proof.

Set: $\sigma(p) = \mu \tau \quad \forall q \in \mathcal{P}_\tau$ (p is compatible with q)

$$\eta(\tau) = \sup_{p \in \mathcal{P}_\tau} \sigma(p)$$

Then $\beta = \eta^\omega(\nu)$ satisfies the conclusion of the lemma. \square QED

Lemma 8 If φ is a Σ_0 \mathcal{M} -formula,

then:

$$p \Vdash \forall x \varphi \rightarrow \forall x \in M^{\mathbb{P}} p \Vdash \forall x \varphi.$$

proof.

~~The direction~~ Let $p \Vdash \forall x \varphi$. Then

$$D = \tau[p] \cup \{q \leq p \mid \forall y \ q \Vdash \varphi(y)\}$$

is dense in \mathbb{P} . Let $p \in \mathbb{P}_\nu$. Then

there is a $\beta \geq \nu$ s.t. $D \cap \mathbb{P}_\beta$ is

dense in \mathbb{P}_β . Thus:

$$\bigwedge_{q \leq p} \forall y \ \bigvee_{q' \leq q} q' \Vdash \varphi(y).$$

By admissibility there is a $u \in M$

s.t.

$$\bigwedge_{q \leq p} \forall y \in u \ \bigvee_{q' \leq q} q' \Vdash \varphi(y).$$

~~But since every $p' \leq p$ is com-~~

~~We may conclude:~~

~~(*) $\bigwedge q$~~

Thus $C = \{q \in \mathcal{P}_\beta \mid \forall y \in u \ q \Vdash \varphi(y)\}$
 is dense in $[p] \cap \mathcal{P}_\beta$, hence in \mathcal{P} .

Thus:

$$(*) \ \wedge q \leq p \ \forall q' \leq q \ \forall y \in u \ q' \Vdash \varphi(y).$$

Set: $x = \{ \langle 1, y \rangle \mid y \in u \cap M^{\mathcal{P}} \}$. Then:

$$p \Vdash \forall \varepsilon \in x \ \varphi. \quad \text{QED}$$

The fact that N is admissible follows by Lemma 8 and the fact that $\Vdash_{\mathcal{M}}^{\Sigma_0}$ is Δ_1 . We state this fact as a general lemma, for later reference:

Lemma 9. If M is an admissible structure, \mathcal{P} is a Δ_1 system of conditions and $\mathcal{M} = \langle M^{\mathcal{P}}; I, E, A_1, \dots, A_n \rangle$ is a $\text{BA}(\mathcal{P})$ -valued model s.t. $\Vdash_{\mathcal{M}}^{\Sigma_0}$ is Δ_1 ~~this~~ and

if $p \Vdash \forall x \varphi \rightarrow \forall u \ p \Vdash \forall x \varepsilon u \ \varphi$ for every $\varphi \in \text{Fml}_{\mathcal{M}}^{\Sigma_0}$, then \mathcal{M}/G is admissible for \mathbb{P} -generic G .

proof.

It suffices to show that the Σ_0 replacement axiom is forced; i.e. if φ is a Σ_0 formula of \mathcal{M} , then

$$\Vdash (\forall x \forall y \varphi \rightarrow \forall u \forall v \forall x \varepsilon u \forall y \varepsilon v \varphi) ,$$

or equivalently:

$$p \Vdash \forall x \forall y \varphi \rightarrow p \Vdash \forall u \forall v \forall x \varepsilon u \forall y \varepsilon v \varphi$$

for $p \in \mathbb{P}$. Let $p \Vdash \forall x \forall y \varphi$. Then:

$$\forall x \in M^{\mathbb{P}} \forall v \in M^{\mathbb{P}} \quad p \Vdash \forall y \varepsilon v \ \varphi(x, y).$$

Let $u \in \underline{\quad} M^{\mathbb{P}}$. Then there is a $w \in M^{\mathbb{P}}$ s.t.

$$\forall x \in \text{dom}(u) \forall v \in w \quad p \Vdash \forall y \varepsilon v \ \varphi(x, y).$$

Set: $v' = \cup w$. Then:

$$p \Vdash \forall x \varepsilon u \forall y \varepsilon v' \ \varphi. \quad \text{QED}$$

Thus $N = \langle M[G], A, G \rangle$ is admissible,

The only remaining step ~~is~~ in the proof of Thm 1 is to show that

N is equivalent to a structure

$N' = \langle M[F]; A, F \rangle$, where $F \subset \text{On}_M^3$

is as above.

(Equivalence ~~is~~, of course, means:

$|N| = |N'|$ and the Δ_1 relations of N are the Δ_1 relations of N').

$F = \cup G$ obviously has the required properties. F (as a relation) is p.r. in G since:

$$\langle x, y, z \rangle \in F \iff \{ \langle x, y, z \rangle \} \in G.$$

Conversely, G is p.r. in F since:

$$p \in G \iff p \in F.$$

This completes the proof of Theorem 1.

Def Call d admissible in

$A_1, \dots, A_m \subset On$ iff \clubsuit

$\langle L_d[A_1, \dots, A_m]; A_1 \cap d, \dots, A_m \cap d \rangle$
is an admissible structure.

As a corollary to Thm 1, we obtain:

Theorem 2 Let $\clubsuit \lambda$ be a ^{countable} limit of
admissible ordinals. Let $A \clubsuit \subset \lambda$,

Then there is a $B \subset \lambda$ s.t.

(a) If $d < \lambda$ is admissible in $C \subset d$
and C is $\langle L_\lambda[A], A \rangle$ - definable
then d is admissible in B, C .

(Thus, in particular, if $d < \lambda$ is
admissible in A , d is admissible
in A, B).

(b) f is countable by a map
 $f \in L_{\nu+1}^\rho[B]$ for $\nu < \lambda$.

proof of Thm 2.

As before, force with conditions:

\mathbb{P} = the set of finite maps p

s.t. $\text{dom}(p) \subset \{\rho_\nu \mid \nu < \lambda\} \times \omega$

and $p(\rho_\nu, n) < \rho_\nu$ if defined.

Let G be \mathbb{P} -generic over $\langle L_\lambda[A], A \rangle$.

Let $F = \cup G$. Then

$$F = \{ \langle f_\nu(n), \rho_\nu, n \rangle \mid \nu < \lambda, n < \omega \},$$

where f_ν maps ω onto ρ_ν .

$$\text{Set } B = \{ \langle \alpha, \beta, \gamma \rangle \mid \langle \alpha, \beta, \gamma \rangle \in F \}.$$

(b) follows immediately.

To prove (a), we note that

G_d is \mathbb{P}_d -generic over $\langle L_d[c], c \rangle$.

Hence, by the proof of Thm 1,

$\langle L_d[c, B], c, B \cap d \rangle$ is admissible.

QED

We now give a second application of forcing ~~at~~ with a class of conditions to add, as before, a class of ordinals to an admissible structure M . This application will differ from the previous one, however, in that no new sets will be added to the structure.

Def d is Mahlo (or recursively Mahlo) in $A \subset d$ iff d is admissible in A and every ~~normal~~ normal function which is Δ_1 in $\langle L_d[A], A \rangle$ takes a value which is admissible in A .

Theorem 3 Let d be Mahlo in A ,

Assume that for all $\beta < d$,
 β is countable in $L_{\beta'}[A]$,
where $\beta' =$ the least p.r. closed
 $\beta' > \beta$. Then there is a $B < d$
s.t.

(i) $\langle L_d[A], A, B \rangle$ is admissible

(ii) If $\gamma < d$ is admissible in A ,
then γ is admissible in A, B

(iii) d is non Mahlo in A, B .

proof.

It suffices that B satisfy

(i) and

(iv) B is unbounded in d , but if
 $\gamma < d$ is admissible in A , then

~~$B \cap \gamma$~~ $B \cap \gamma \in L_\gamma[A]$.

(ii) obviously follows from (iv).

To see that (iii) follows,

note that the monotone enumeration of the limit pts. of B is a Δ_1 normal function (in $\langle L_\alpha[A], A, B \rangle$) ~~containing no~~ none of whose values are admissible in A .

Our conditions will describe initial segments of the set B which we wish to add:

$\mathcal{P} =_{\text{pf}}$ the collection of bounded subsets p of d s.t.

(i) For all $\beta < d$, $p \cap \beta \in L_{\beta'}[A]$, where β' = the least p.n. closed $\beta' > \beta$.

(ii) If $\beta \leq d$ is admissible in A , then $p \cap \beta \in L_\beta[A]$.

$$p \leq q \iff_{\text{pf}} q = p \cup \cup \{v+1 \mid v \in q\}$$

Set: $\mathcal{M} = \langle L_d[A]^{\mathbb{P}}; \mathbb{I}, E, \check{A}, \check{B} \rangle,$

where: $\check{A}(x) = \bigcup_{v \in A} [x \equiv \check{v}]$

$\check{B}(x) = \bigcup_{v < d} ([x \equiv \check{v}] \wedge \{p \in \mathbb{P} \mid v \in p\})$.

Then $N = \mathcal{M}/G = \langle |N|; A, B \rangle$, for \mathbb{P} -generic G , where B is an unbounded subset of d satisfying (iv).

We must show that N is admissible and that $|N| = L_d[A]$.

Lemma 1. \uparrow G is \mathbb{P} -generic, then

$$|\mathcal{M}/G| = L_d[A],$$

proof.

Let $\beta < d$ be p.r. closed.

Let $p = \bigcup G \cap \beta$. Then $p \in G$.

For $x \in L_\beta[A]^{\mathbb{P}}$ set:

$$p^*(x) = \{p^*(y) \mid \forall q \geq p \langle q, y \rangle \in x\}.$$

Since $G \cap L_\beta[A] = \{q \in L_\beta[A] \mid q \geq p\}$,

we have!

$$p^*(x) = G^*(x) \text{ for } x \in L_\beta[A]^{\mathbb{P}}.$$

But $p^* \upharpoonright L_\beta[A]^{\mathbb{P}} \in L_\alpha[A]$. QED

Lemma 2 $\mathbb{H}_{\mathcal{M}}^{\Sigma_0}$ is Δ_1 in $\langle L_\alpha[A], A \rangle$.

proof.

Let $\beta < \alpha$ be ~~a limit~~ of p.r. closed,

let β' be the next largest p.r. closed ordinal. Let

$p \in L_{\beta'}[A]$, $p \in \mathbb{P}$ be s.t.

$\text{sup}(p) \geq \beta$. Then for any \mathbb{P} -generic G s.t. $p \in G$ we have:

(1) $p^*(x) = G^*(x)$ for $x \in L_\beta[A]^{\mathbb{P}}$

(p^* being defined as in the proof of lemma 1)

(2) $\mathbf{B} \cap \beta = p \cap \beta$ (where $\mathbf{B} = G^*(\overset{\circ}{B})$).

In particular, if φ is a Σ_0 formula of \mathcal{M} containing only

constants x s.t. $x \in L_\beta[A]^\mathbb{P}$, then

$$G \Vdash \varphi \iff \mathbb{F} \langle L_\beta[A, p \cap \beta], A \cap \beta, p \cap \beta \rangle \varphi^{(p)}$$

(where $\varphi^{(p)}$ is the result of replacing x by $\underline{p^*(x)}$).

Since this holds for every \mathbb{P} -generic $G \ni p$, we conclude:

$$p \Vdash \varphi \iff \mathbb{F} \langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle \varphi^{(p)}$$

Now let $p' \in \mathbb{P} \cap L_{\beta'}[A]$. Every \mathbb{P} -generic $G \ni p'$ contains a $p \in L_\beta[A]$ s.t. $p' \geq p$ and $\text{sup}(p) \geq \beta$. Hence:

$$p' \Vdash \varphi \iff \bigwedge p \in L_\beta[A] (p \leq p' \wedge \text{sup}(p) \geq \beta \rightarrow \mathbb{F} \langle L_\beta[A, p], A \cap \beta, p \cap \beta \rangle \varphi^{(p)})$$

$$\iff S(\varphi, p', \beta)$$

where S is Δ_1 .

Hence:

$$p \Vdash \varphi \iff S(\varphi, p, \eta(\varphi, p)),$$

where $\eta(p, \varphi) =$ the least p.n. closed β s.t. $p, \varphi \in L_\beta[A]$. QED

Our main tool in showing that $N = \mathbb{N}/G$ is admissible will be:

Lemma 3 Let D_i ($i < \omega$) be a sequence of sets which are dense in \mathbb{P} and closed under extensions. Let $\{ \langle p, i \rangle \mid p \in D_i \}$ be Σ_1 in $\langle L_\alpha[A], A \rangle$. Then $\bigcap_i D_i$ is dense in \mathbb{P} .

proof.

$$\text{Let } p \in D_i \iff \forall x \ Rxip,$$

where R is Σ_0 .

For $\delta < \alpha$ set:

$$D_i^\delta =_{\text{df}} \{ p \in \mathbb{P} \cap L_\delta[A] \mid \forall x \in L_\delta[A] \ Rxip \}$$

Claim There are arbitrarily large p.r. closed $\delta < d$ s.t.

- (i) D_i^δ is dense in $\mathbb{P} \cap L_\delta[A]$ for $i < \omega$
- (ii) δ is not admissible in A .

proof.

Let $\delta < d$ be s.t. $\vec{x} \in L_\delta[A]$, where \vec{x} are the constants occurring in the Σ_0 definition of R .

Define $\eta: \mathbb{P} \rightarrow d$ by:

$\eta(p) =$ the least β s.t.

$$\Delta_{i < \omega} \forall q \leq p \forall x (q, x \in L_\beta[A] \wedge Rxiq).$$

Then η is Δ_1 . Set:

$\bar{\eta}(\beta) =$ the least p.r. closed

$\beta' > \beta$ s.t. $\Delta_{p \in \mathbb{P} \cap L_\beta[A]} \eta(p) < \beta'$.

Then $\bar{\eta}$ is Δ_1 . ~~Set~~

Let $\delta = \bar{\eta}^\omega(\delta)$.

Then $\delta > \delta$ is p.r. closed and satisfies (i).

To show that δ satisfies (ii), we note that $\eta \uparrow (\mathbb{P} \cap L_\delta[A])$ is Δ_1 in $\langle L_\delta[A], A \cap \delta \rangle$. Thus, if $\bar{\sigma}$ were admissible in A , $\langle \bar{\eta}^i(\delta) \mid i < \omega \rangle$ would be Δ_1 in $\langle L_\delta[A], A \cap \delta \rangle$. Contradiction!

QED (Claim)

Now let $p \in \mathbb{P}$. We must show that there is $q \leq p$ with $q \in \bigcap_i D_i$.

Pick δ s.t. $p \in L_\delta[A]$, D_i^δ is dense in $\mathbb{P} \cap L_\delta[A]$ and δ is not admissible in A .

~~Define q_i ($i < \omega$) by:~~

~~$q_0 = p$~~

~~q_{i+1} = the least $q \leq q_i$ (in the canonical well ordering of $L[A]$) s.t. $\sup(q)$~~

Let $f: \omega \leftrightarrow \delta$, $f \in L_{\delta'}[A]$,
where δ' is the next largest
p.r. closed ordinal. Define
 $q_i \in \mathbb{P} \cap L_{\delta}[A]$ ($i < \omega$) by;

$$q_0 = p$$

q_{i+1} = The least $q \leq q_i$ (in the
canonical well ordering of
 $L[A]$) s.t. $q \in D_i^{\delta}$ and
 $\sup(q) \geq f(i)$.

Set $q = \bigcup_i q_i$. Then $q \in \mathbb{P}$,
since $\sup(q) = \delta$ and $q \in L_{\delta'}[A]$.
Hence $q \leq p \wedge q \in \bigcap_i D_i$.

QED

We are now ready to show:

Lemma 4 If G is \mathbb{P} -generic,
then \mathcal{M}/G is admissible.
proof.

Since all sets are countable
in \mathcal{M}/G , it suffices to show
that \mathcal{M}/G satisfies:

$\bigwedge i < \omega \forall y R_{iy} \rightarrow \forall u \bigwedge i < \omega \forall y \in u R_{iy}$
for Σ_0 \mathbb{P} . That is, we must
prove:

$\Vdash \bigwedge x \in \check{\omega} \forall y \varphi \rightarrow \forall u \bigwedge x \in \check{\omega} \forall y \in u \varphi$
for Σ_0 \mathcal{M} -formulae φ .

Let $p \Vdash \bigwedge x \in \check{\omega} \forall y \varphi$.

Set: $D_i = \{q \in p \mid \forall y \ q \Vdash \varphi(i, \underline{y})\} \cup$
 $\cup \neg[p]$

for $i < \omega$. Then D_i is dense
in \mathbb{P} and $\{\langle i, x \rangle \mid x \in D_i\}$ is Σ_1 .

Hence $\bigcap_i D_i$ is dense in \mathbb{P} .

This means that for every $p' \in \mathbb{P}$ there is a $p'' \leq p'$ s.t.

$$\bigwedge i < \omega \quad \forall y \quad p'' \Vdash \varphi(i, \underline{y}).$$

By the replacement axiom there is a ~~set~~ σ s.t.

$$\bigwedge i < \omega \quad \forall y \in \sigma \quad p'' \Vdash \varphi(i, \underline{y}).$$

$$\text{Set: } \tilde{\sigma} = \{ \langle i, y \rangle \mid y \in \sigma \cap \mathbb{P} \}.$$

Then:

$$p'' \Vdash \bigwedge x \in \check{\omega} \quad \forall y \in \tilde{\sigma} \quad \varphi.$$

$$\text{Hence } p \Vdash \forall \sigma \bigwedge x \in \check{\omega} \quad \forall y \in \sigma \quad \varphi.$$

QED

We now turn to the major theorem of this section:

Theorem 4 Let $B \subset \text{On}$. Let d_ν ($\nu < \delta$) be a countable sequence of countable ordinals s.t. $d_\nu > \omega$ and d_ν is admissible in $B \cap d_\nu$, $\{d_\nu \mid \nu < \delta\}$ for $\nu < \delta$. Then there is a b.c.w. s.t.

- (i) d_ν is the ν -th $d > \omega$ s.t. d is admissible in b
- (ii) $B \cap d_\nu$ is Δ_1 in $L_{d_\nu}[b]$ for $\nu < \delta$.

Thm 4 follows immediately from the conjunction of the following two theorems:

Theorem 4.1 Let λ be a countable limit of admissible ordinals, let $B \subset \lambda$. Let $A \subset \lambda$ be s.t. $\omega \notin A$ and each $d \in A$ is admissible in $B \cap d, A \cap d$. Then there is a $C \subset \lambda$ s.t. for all $d < \lambda$;

- (i) d is admissible in C iff $d \in A$
- (ii) If d is admissible in C , then $B \cap d, A \cap d$ are Δ_1 in $\langle L_d[C], C \cap d \rangle$.
- (iii) If d is p.r. closed and d' is the next largest p.r. closed ordinal, then d is countable in $L_{d'}[C \cap d]$.

Theorem 4.2 Let λ be a countable limit of admissible ordinals and let $C \subset \lambda$ satisfy (iii) of Thm 4.1. Then there is a $b < \omega$ s.t. for all $d < \lambda$ s.t. $d > \omega$;

(ii) If d is admissible in C , then
 d is admissible in b

(iii) If d is admissible in b , then
 $C \cap d$ is Δ_1 in $L_d[b]$

(Hence d is admissible in b iff
 d is admissible in C).

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We begin with the proof of
Thm 4.1.

Lemma 1 There is an $A < \lambda$ s.t. for $d < \lambda$

(a) d is admissible in $B \cap d$, and iff
 d is admissible in A

(b) If d is admissible in A , then $B \cap d$,
and are Δ_1 in $\langle L_d[A], A \rangle$

(c) If d is p.r. closed, then d is count-
able in $L_{d'}[A]$, where d' is the next
largest p.r. closed ordinal.

proof of Lemma 1.

By Thm 2 there is an $A' < \lambda$ satisfying (iii) s.t., whenever $d < \lambda$ is admissible in $A \cap d, B \cap d$, then d is admissible in $A \cap d, B \cap d, A' \cap d$. Set:

$$A = \{3 \cdot v \mid v \in A\} \cup \{3 \cdot v + 1 \mid v \in B\} \cup \{3 \cdot v + 2 \mid v \in A'\}$$

Then A satisfies (i) - (iii). QED

Now ~~let~~ let A^* be the set of $d < \lambda$ s.t. d is admissible in A , (Hence $A \subset A^*$).

Lemma 2 There is an $f: A^* \rightarrow \lambda$ s.t.

(i) $f(d) < d$ for $d \in A^*$

(ii) f is 1-1

(iii) $\{d \in A^* \mid f(d) < \kappa\}$ is finite for $\kappa < \lambda$

(iv) If $d \in A^*$, then d is admissible in $A, f \cap d, \text{rng}(f \cap d)$. (i.e.

$\langle L_d[A, f \cap d, \text{rng}(f \cap d)]; A \cap d, f \cap d, \text{rng}(f \cap d) \rangle$
is admissible)

proof of Lemma 2.

Let ρ_ν ($\nu \leq \lambda$) be the monotone enumeration of the p.r. closed ordinals $\leq \lambda$. Let Θ_ν ($\nu \leq \lambda$) be the set of maps f defined on $A^* \wedge \rho_\nu$ s.t. f satisfies (i)-(iv) and $f \in L_{\rho_{\nu+1}}[A]$. Set:

$$f \leq f' \iff_{\text{pt}} \forall \nu \leq \tau \ (\nu \leq \tau \wedge f \in \Theta_\nu \wedge f' \in \Theta_\tau \wedge f \subset f' \wedge \text{rng}(f' \setminus f) \subset \rho_\tau \setminus \rho_\nu).$$

By induction on $\kappa < \lambda$ we prove:

Claim $\Theta_\kappa \neq \emptyset$; moreover, if $\nu < \kappa$, $\nu \notin A^*$ and $f \in \Theta_\nu$, then there is an $f' \in \Theta_\kappa$ s.t. $f \leq f'$.

proof.

Case 1 $\kappa = 0$: $\emptyset \in \Theta_0$.

Case 2 $\kappa = \tau + 1$; $\tau \notin A^*$. Then $\Theta_\kappa = \Theta_\tau$

Case 3 $\kappa = d + 1$; $d \in A^*$.

$\forall \nu < d, \nu \notin A^*, f \in \Theta_\nu$, then

pick $\bar{f} \in \Theta_\alpha$ s.t. $\bar{f} \geq f$. ~~Let $\bar{f} \in \Theta_\alpha$~~

Let $u \in \alpha \setminus \beta$ s.t. $u \notin \text{rng}(f)$.

Set: $f' = f \cup \{\langle u, \alpha \rangle\}$.

Case 4 $\text{Lim}(\kappa); \kappa \notin A^*$.

Since β is countable in $L_{\beta_{\kappa+1}}[A]$, there

is a $g \in L_{\beta_{\kappa+1}}[A]$ s.t. $g: \omega \rightarrow \kappa$ is

monotone + cofinal in κ . We may assume: $g(0) = \nu$ and $g(i) \notin A^*$ for

$i < \omega$. Define f_i ($i < \omega$) by:

$$f_0 = f$$

$f_{i+1} =$ the least $f' \in \Theta_{g(i+1)}$ ~~s.t.~~

(in the canonical well ordering of $L[A]$) s.t.

$$f' \geq f_i.$$

Set: $f' = \bigcup_i f_i$. Then $f' \in \Theta_\kappa$ and

$$f' \geq f.$$

Case 5 $\kappa = d \in \mathcal{A}^*$.

By Thm 3 there is a $B \in \mathcal{d}$ s.t.

\mathcal{d} is admissible in A, B , B is unbounded in \mathcal{d} , $B \cap \mathcal{d}'$ is bounded in

\mathcal{d}' for every $\mathcal{d}' < \mathcal{d}$ s.t. $\mathcal{d}' \in \mathcal{A}^*$ and

$B \cap \rho_{\nu} \in L_{\rho_{\nu+1}}[A]$ for $\nu < \mathcal{d}$. From the

proof of Thm 3 it is clear that

we may assume: $B \in L_{\rho_{\mathcal{d}+1}}[A]$, for

$L_{\mathcal{d}}[A]$ is countable in $L_{\rho_{\mathcal{d}+1}}[A]$;

hence we can find $G \in L_{\rho_{\mathcal{d}+1}}[A]$ s.t.

G is \mathbb{P} -generic over $\langle L_{\mathcal{d}}[A], A \rangle$,

\mathbb{P} being the system of conditions used in the proof of Thm 3.

Assume: $B \in L_{\rho_{\mathcal{d}+1}}[A]$ and let

$g: \mathcal{d} \rightarrow \mathcal{d}$ be the monotone enumeration of the limit points of B .

~~Then \mathcal{d} is admissible~~

Then g is a normal function

taking no ~~admissible~~ values in A^* ,

$g \in L_{\rho_{d+1}}[A]$ and d is admissible

in A, g . Moreover, $g \upharpoonright \rho_{\nu} \in L_{\rho_{\nu+1}}[A]$ for

$\nu < d$. ~~Therefore~~

Let $\nu < d$; ~~let~~ $f \in \Theta_{\nu}$. We may assume without loss of generality that $g(0) = \nu$. Define a sequence

$f_{\nu} (\nu < d)$ by:

$$f_0 = f$$

$f_{\nu+1}$ = the least $f' \in \Theta_{\nu+1}$ (in the canonical well ordering of $L[A]$)
 s.t. $f' \geq f_{\nu}$

$$f_{\delta} = \bigcup_{\nu < \delta} f_{\nu} \quad \text{for limit } \delta$$

(hence $f_{\delta} \in \Theta_{\delta}$; $f_{\delta} \geq f_{\nu}$ for $\nu < \delta$).

~~Set~~ The sequence $f_{\nu} (\nu < d)$ is

Δ_1 in $\langle L_d[A, g], A, g \rangle$.

$$\text{Set: } f' = \bigcup_{\nu} f_{\nu}$$

Then f' is Δ_1 in $\langle L_\alpha[A, g], A, g \rangle$

since:

$$v = f'(\tau) \iff v = f_{\tau+1}(\tau).$$

$\text{rng}(f')$ is Δ_1 in $\langle L_\alpha[A, g], A, g \rangle$

since:

$$v \in \text{rng}(f') \iff v \in \text{rng}(f_{i+1}).$$

Hence

~~B~~ $f' \geq f$ and $f' \in \Theta_\alpha$.

QED (Claim).

Lemma 2 follows almost immediately from the claim: Let λ_i ($i < \omega$) be monotone + cofinal in λ . Assume: $\lambda_i \notin A^*$. Select

f_i ($i < \omega$) s.t. $f_i \in \Theta_{\lambda_i}$ and

$f_{i+1} \geq f_i$. Set: $f = \bigcup_i f_i$.

Then f satisfies (i) - (iv) of Lemma 2. QED

Now let f be as in Lemma 2 and define a map $\tilde{f}; (A^* \setminus A) \times \omega \rightarrow \lambda$ by: $\tilde{f}(d, i) = g_d(i)$, where:

$g_d =$ the least g (in the canonical well ordering of $L[A]$) s.t. g is a monotone map from ω to d and $\sup_{i < \omega} g(i) = d$.

Set:

$$D = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in A^* \setminus A, i < \omega \},$$

and

$$D_d = \{ \langle f(\beta), \tilde{f}(\beta, i) \rangle \mid \beta \in (A^* \setminus A) \cap d, i < \omega \}$$

for $d < \lambda$. Then if $d \in A^*$, D_d is

Δ_1 in $\langle L_d[A], \text{And}, f \upharpoonright d, \text{rng}(f \upharpoonright d) \rangle$.

It follows that, if $d \in A$, $D \cap d$ is

also Δ_1 , since $D \cap d$ differs from D_d

by at most finitely many points.

On the other hand, if $d \in A^* \setminus A$,

d is not admissible in $D \cap d$.

since, if it were, we should have:
 $\text{rng}(g_d) = \{v \mid \langle f(d), v \rangle \in D \cap d\}$ is
 Δ_1 in $\langle L_d[D], D \cap d \rangle$; hence g_d , being
 the monotone enumeration of $\text{rng}(g_d)$,
 would be Δ_1 . But this violates the
 replacement axiom.

We now define the set C , whose
 existence is asserted in Theorem 4.1.

$$C = \{4 \cdot d \mid d \in A\} \cup \{4 \cdot \langle d, \beta \rangle + 1 \mid d = f(\beta)\} \cup \\ \cup \{4 \cdot d + 2 \mid d \in \text{rng}(f)\} \cup \{4d + 3 \mid d \in D\}.$$

For $d < \lambda$ we define C_d similarly,
 using $A \cap d$, $f \upharpoonright d$, $\text{rng}(f \upharpoonright d)$, D_d in
 place of A , f , $\text{rng}(f)$, D . Clearly,
 if $d \in A^*$, then C_d is Δ_1 in
 $\langle L_d[A]; A \cap d, f \upharpoonright d, \text{rng}(f \upharpoonright d) \rangle$. Hence
 $C \cap d$ is Δ_1 for $d \in A$, since C_d differs
 from C_d by at most finitely many
 elements. Thus d is admissible

in C if $d \in A$. On the other hand, if $d \notin A$, then d is not admissible in C , since, ~~if it were, we should have $D \cap d$ is Δ_1 in L_d~~ , if it were, we should have: $d \in A^* \setminus A$, since $A \cap d$ is Δ_1 in $\langle L_d[C], C \cap d \rangle$. But then ~~$D \cap d$ is~~ d is not admissible in C , since $D \cap d$ is Δ_1 in $\langle L_d[C], C \cap d \rangle$ and d is not admissible in $D \cap d$. This establishes (i) of Theorem 4.1. (ii) follows from Lemma 1 and the fact that $A \cap d$ is Δ_1 in $\langle L_d[C], C \cap d \rangle$. (iii) is established by the following lemma:

Lemma 3 ρ_d is countable in $L_{\rho_{d+1}}[C \cap d]$

for $d < \lambda$.

proof of Lemma 3.

We assume the following facts:

(1) f_0 is countable in L_{f_1}

(2) f_ν is mappable onto $f_{\nu+1}$ by a function $g \in L_{f_{\nu+2}}$.

By (2) we get:

(3) If $\text{Lim}(d)$ and $f_\alpha > d$, then d is mappable onto f_α by a function $g \in L_{f_\alpha}$.

proof of (3).

Suppose not. Let κ be the least $\kappa \leq d$ s.t. d is not mappable onto f_κ by a map $g \in L_{f_{\kappa+1}}$. For $\nu < \kappa$ let

$g_\nu : d \rightarrow f_\nu$ be the least $g \in L_{f_{\nu+1}}$ (in

the canonical well ordering of L)

s.t. g maps d onto f_ν . κ is a limit

ordinal by (2); hence the function

$$g(\langle \nu, u \rangle) = g_\nu(u)$$

maps $\kappa \times d$ onto f_κ . It is easily seen

that $g \in L_{f_{\kappa+1}}$. But since $\kappa \leq d$

there is an $h \in L_{f_{\kappa+1}}$ which maps

~~$\kappa \times d$ onto d~~ , d onto $\kappa \times d$. Hence!

$g \circ h \in L_{f_{\kappa+1}}$ and $g \circ h$ maps d onto f_κ .

Contradiction! QED (3).

Now suppose Lemma 3 to be false.

Let d be the least ordinal for which

Lemma 3 fails. Then $d = f_d$ by (1)-(3).

(hence $L_{f_d}[Cnd] = L_d[C]$).

(4) d is admissible in C . Moreover,

if $R \in L_d[C]$ and $R \in L_{f_{d+1}}[Cnd]$,

then:

(*) $\forall x \forall y Rxy \rightarrow \exists u \forall v \exists x \in u \forall y \in v Rxy$

holds in ~~the~~ $L_d[C]$.

proof of (4).

Since each $x \in L_d[C]$ is countable in $L_d[C]$, it suffices to show:

$$(**) \quad \bigwedge_{m < \omega} \forall y \in R_{xy} \rightarrow \forall \alpha \bigwedge_{m < \omega} \forall y \in \alpha \ R_{my}.$$

Suppose not. Set:

$$h(m) = \mu \delta \forall y \in L_\delta[C] \ R_{my} \quad (m < \omega).$$

Then $\sup_{m < \omega} h(m) = d$. Set:

$g_m = \nu$ the least $g \in L_d[C]$ (in the canonical well ordering of $L[C]$) s.t. g maps ω onto $h(m)$.

Then $g(\langle m, m \rangle) = g_m(m)$ maps ω onto d and $g \in L_{\delta}^{d+1}[C]$. Contradiction!

QED

Using (4), we get:

(5) d is Mahlo in ~~and~~ C .

proof of (5)

Let D be a closed, unbounded subset of d which is Δ_1 in $\langle L_d[C], \text{Cnd} \rangle$.

We must show: $\beta \in D$ for some $\beta < d$ which is admissible in C .

Let \vec{x} be all constants occurring in the Δ_1 definition of D . Let u be the set of $x \in L_d[C]$ which are

$\langle L_d[C], \text{Cnd} \rangle$ -definable in the parameters \vec{x} . Since there is an $\langle L_d[C], \text{Cnd} \rangle$ -definable well ordering of $L_d[C]$,

we have: ~~$\langle u, \text{Cnd} \rangle$ is an~~

~~elementary submodel of~~

(*) $\langle u, \text{Cnd} \rangle$ is an elementary submodel of $\langle L_d[C], \text{Cnd} \rangle$.

It follows that u is transitive, for by (*), if $x \in u$ ~~then~~ there is ~~an~~ ~~for~~

a $g \in u$ s.t. g maps w onto x .

Hence: $x = g^{\omega} w \in u$.

Let $\beta = d \cap u$. By the condensation lemma we have: $u = L_\beta[C]$.

By (*) $\langle L_\beta[C], C \cap \beta \rangle$ satisfies the admissibility axioms, since $\langle L_\alpha[C], C \cap \alpha \rangle$ does. Hence β is admissible in C . We now claim:

$\beta < \alpha$. To see this, we let \tilde{u} be the set of formulae (with $\varepsilon, \exists, \dot{C}$) containing at most the constants \vec{x} and the free vbl v_0 . Set:

$h(\varphi) =_{\text{df}}$ the least $y \in L_\alpha[C]$ (in the canonical well ordering of $L[C]$) s.t. $\models \varphi(v_0/y)$
 $\langle L_\alpha[C], C \cap \alpha \rangle$

if such y exists

$=_{\text{df}} \emptyset$ if not.

Then $\tilde{u} \in L_\alpha[C]$ and $h \in L_{\alpha+1}[C \cap \alpha]$.

By (4) there is a $v \in L_\alpha[C]$ s.t.

$u = h''\tilde{u} \subset v$. Hence $\beta = d \cap u < \alpha$.

By (*), $D \cap \beta$ is closed and unbounded in β . Hence β is a limit point of D .

Hence $\beta \in D$

QED (5).

(5) leads to a contradiction, however, since:

~~(6) $d < \lambda$ is not Mahlo.~~

(6) $d < \lambda$ is not Mahlo in C .

proof.

We may assume: $d \in \mathcal{A}$. Since $f \upharpoonright d$, $\text{rng}(f \upharpoonright d)$ are Δ_1 in $\langle L_d[C], C \cap d \rangle$,

we can define a Δ_1 normal function g by:

$$g(0) = 0$$

$$g(v+1) = \text{the least } \beta > g(v) \text{ s.t.}$$

$$f^{-1}(v) < \beta \text{ for all } v \in \text{rng}(f \upharpoonright d) \cap g(v)$$

$$g(\delta) = \sup_{v < \delta} g(v) \text{ for limit } \delta.$$

Clearly, g takes no values in \mathcal{A}^* .

QED

This completes the proof of Thm 4.1.