§1 Almost Subcomplete Forcing

§1.1 Preliminaries

Def Let $N = L_\tau^A = \langle L_\tau[A], A\rangle$ be a transitive ZFC$^-$ model. Let $\mathbb{B} \in N$ be a complete BA in the sense of N. Let G be an ultrafilter on $\mathbb{B}$.

- G is <u>well founded</u> iff
  $$\{\langle s,t\rangle \mid G \Vdash^N_{\mathbb{B}} s \in t\} \text{ is well founded}$$
- G is <u>weakly</u> $\mathbb{B}$-<u>generic</u> over N iff $G \cap \Delta \neq \emptyset$ whenever $\Delta \in N$ is predense in $\mathbb{B}$ (i.e. $\bigcup \Delta = 1$) and $\overline{\overline{\Delta}} \leq \omega_1$ in N.

(Equivalently: Whenever $\Vdash_{\mathbb{B}} t < \omega_1$, then there is $b \in G$ with $b \Vdash t = \check{\nu}$ for a $\nu < \omega_1^N$.)

It is easily seen that:

<u>Fact 1</u> Let $\pi : N \prec N'$, $\mathbb{B}' = \pi(\mathbb{B})$. Then

(a) If $G' \subset \mathbb{B}'$ is well founded over N' and $G = \pi^{-1}{}''G'$, then G is well founded over N,

(b) If $G' \subset \mathbb{B}'$ is weakly generic over N', $G = \pi^{-1}{}''G'$, and $\omega_1^N = \omega_1^{N'}$, then G is weakly generic over N

Clearly every G which is fully generic over N is also well founded and weakly generic.

<u>Note</u> We shall generally take the forcing language for N as including the class predicates:

$\check{V}$ where $[\![s \in \check{V}]\!] = \bigcup_{x \in N} [\![s = \check{x}]\!]$

$\check{A}$ where $N = L_\tau^A$ and $[\![s \in \check{A}]\!] = \bigcup_{x \in A} [\![s = \check{x}]\!]$

(similarly for $N = L_\tau^{A_1, \ldots, A_n}$).

Let $\tilde{N}$ be the model

$\langle N^{\mathbb{B}}, E, I, \check{V}, \check{A} \rangle$, where:

$s \, E \, t \longleftrightarrow G \Vdash s \in t$

$s \, I \, t \longleftrightarrow G \Vdash s = t$

$\check{V} t \longleftrightarrow G \Vdash s \in \check{V}$

$\check{A} t \longleftrightarrow G \Vdash t \in \check{A}$

where $|\tilde{N}| =$ the set of $s \in N^{\mathbb{B}}$ s.t. $\Vdash s \in \check{V}$

Then: Set $\tilde{N} = \langle |\tilde{N}|, E, I, \check{A} \rangle$

<u>Fact 2</u> $\tilde{N} \models \varphi(\vec{s}) \leftrightarrow [\![\varphi(\vec{s})]\!] \in G$

proof.

By ind. on φ, using:

$[\![\bigvee v \, \varphi(v, \vec{s})]\!] = [\![\varphi(t, \vec{s})]\!]$ for a $t \in N^{\mathbb{B}}$,

QED (Fact 2)

(Hence $\tilde{N} \models ZFC^-$. This holds for any ultrafilter G on $\mathbb{B}$.)

Now suppose that G is well founded.

Then there is an isomorphism

$i_G : \tilde{N}/I \xleftrightarrow{\sim} N^*$

where N^* is transitive.

If we then set:

$s^G = i_G(s/I^G)$, we get:

<u>Fact 3</u> $t^G = \{s^G \mid G \Vdash s \in t\}$ and

$N^* \models \varphi(\vec{s}) \longleftrightarrow [\![\varphi(\vec{s})]\!] \in G$.

Set: $\check{V}^G = \{t^G \mid \Vdash t \in \check{V}\}$

$\check{A}^G = \{t^G \mid \Vdash t \in \check{A}\}$

It is easily seen that

$N^* = \langle |N^*|, \check{V}^G, \check{A}^G \rangle$.

Set: $\hat{N} = \langle \check{V}^G, \check{A}^G \rangle$.

Then $\hat{N}$ is a transitive submodel of N^* and there is an embedding $e : N \prec \hat{N}$ defined by: $e(x) =_{Df} \check{x}^G$.

<u>Def</u> $\langle e, \hat{N} \rangle$ is the G-<u>extension</u> of N

<u>Note</u> If G is fully generic, then $\hat{N} = N$ and $e = id$.

<u>Fact 4</u> Let G be well founded. Then G is weakly generic iff $\omega_1^N = \omega_1^{\hat{N}}$.

<u>Def</u> $\hat{G} = \dot{G}^G$, where $\dot{G}$ is the canonical $\mathbb{B}$-generic name.

Since $\Vdash \dot{G}$ is generic over $\check{V}$, we have:

$N^* \models \hat{G}$ is generic over $\check{V}^{\dot{G}}$. Hence:

<u>Fact 5</u> $\hat{G}$ is $\hat{\mathbb{B}}$-generic over $\hat{N}$ where $\hat{\mathbb{B}} = e(\mathbb{B})$.

Since $\Vdash t = \check{t}^{\dot{G}}$, we have:

$$t^G = ((\check{t})^G)^{\dot{G}^G} = e(t)^{\hat{G}} \text{ in } N^*,$$

hence, since $|N^*| = \{t^G \mid t \in N^{\mathbb{B}}\}$:

<u>Fact 6</u>

- $t^G = e(t)^{\hat{G}}$ for $t \in N^{\mathbb{B}}$
- $N^* = \hat{N}[\hat{G}]$

<u>Def</u> $\hat{G}$ is the <u>completion</u> of G w.r.t. N.

We also say that $\langle e, \hat{N}, \hat{G} \rangle$ is the <u>completion</u> of N, G.

<u>Fact 6</u> $\langle e', N', G' \rangle$ is the completion of N, G iff

(a) $e' : N \prec N'$

(b) $e'``G \subset G'$

(c) G' is $\mathbb{B}' = e'(\mathbb{B})$-generic over N'

(d) $N' = \{e'(x)^{G'} \mid x \in N^{\mathbb{B}} \wedge \Vdash x \in \check{V}\}$.

proof:

$(\rightarrow)$ is immediate by the previous facts.

We prove $(\leftarrow)$. Let $\tilde{N}, N^*$ be as above.

Then $\tilde{N} \models \varphi(\vec{x}) \leftrightarrow [\![\varphi(\vec{x})]\!] \in G \leftrightarrow$
$\leftrightarrow e'([\![\varphi(u)]\!]^N_{\mathbb{B}}) = [\![\varphi(e'(\vec{x}))]\!]^{N'}_{\mathbb{B}'} \in G' \leftrightarrow$
$\leftrightarrow N'[G'] \models \varphi(e'(\vec{x})^{G'})$.
Hence there is $\sigma : \tilde{N}/I^G \xrightarrow{\sim} N'[G']$
defined by: $\sigma(u/I^G) = e'(u)^{G'}$.
But then $N'[G'] = N^*$ is the transitivation of $\tilde{N}/I^G$ and σ the transitivizing function. Moreover, $e'(\mathring{G})$ is the canonical $e'(\mathbb{B})$ - generic name for N', since $N \models \bigwedge b \in \mathbb{B}\ b = [\![\check{b} \in \dot{G}]\!]$; hence $N' \models \bigwedge b \in e'(\mathbb{B})\ b = [\![\check{b} \in e'(\dot{G})]\!]$. Hence $G' = e(\dot{G})^{G'} = \sigma(\dot{G}/I^G) = \dot{G}^G = \hat{G}$. Finally we have: $e'(\check{x}) = \check{e'}(x)$; hence
$e'(x) = e'(\check{x})^{G'} = \sigma(\check{x}/I^G) = \check{x}^G = e(x)$,

QED (Fact 6)

The <u>interpolation</u> <u>lemma</u>

<u>Fact 8</u> Let $\sigma : N \prec N'$. Let $G = \sigma^{-1}{}''G'$, where G' is $\mathbb{B}' = \sigma(\mathbb{B})$ - generic over N'. (Hence G is well founded over N.) Let $\langle e, \hat{N}, \hat{G} \rangle$ be the completion of N, G. There is a unique $\pi : \hat{N}[\hat{G}] \prec N[G]$ s.t. $\pi(\hat{G}) = G$ and $\pi e = \sigma$.

proof.
We first show existence. Let $t_1, \ldots, t_n \in N^{\mathbb{B}}$.
Then $\hat{N}[\hat{G}] \models \varphi(\vec{t}^{\,G}) \leftrightarrow G \Vdash^{N}_{\mathbb{B}} \varphi(\vec{t}) \leftrightarrow$
$\leftrightarrow G' \Vdash^{N'}_{\mathbb{B}'} \varphi(\sigma(\vec{t})) \leftrightarrow N'[G'] \models \varphi(\sigma(\vec{t})^{G'})$.
Hence there is $\pi : \hat{N}[\hat{G}] \prec N'[G']$ defined by: $\pi(t^G) = \sigma(t)^{G'}$. Clearly $\pi e(x) =$
$= \pi(\check{x}^G) = \check{\sigma}(x)^{G'} = \sigma(x)$, so $\pi(e) = \sigma$.
Moreover $\pi(\hat{G}) = \pi(\dot{G}^{\hat{G}}) = \sigma(\dot{G})^{G'} =$
$= G'$, since $\sigma(\dot{G})$ is the canonical $\mathbb{B}'$-generic name.
To prove uniqueness, let π' be another such embedding. Then $\pi'(t^G) =$
$\pi'(e(t)^{\hat{G}}) = \sigma(t)^{G'} = \pi(t^G)$.

QED (Fact 8)

<u>Def</u> If π is as in Fact 8, we call it the <u>interpolant</u> of σ, G' and denote it by: $\mathrm{int}(\sigma, G')$.

We leave it to the reader to show:

<u>Fact 9</u>. Let $\pi = \mathrm{int}(\sigma, G')$. Then $\pi \restriction \hat{N}$ is the unique $\pi' : \hat{N} \prec N$ s.t. $\pi' e = \sigma$ and $\pi'{}''\hat{G} \subset G$.

We recall the definition from [Sing]:

<u>Def</u> Let $\sigma : N \prec N'$ where N, N' are transitive ZFC$^-$ models. Let δ be a regular cardinal in N. σ is a δ-<u>cofinal</u> <u>map</u> of N to N' iff whenever $x \in N'$, then there is $u \in N$ with $\overline{\overline{u}} < \delta$ in N and $x \in \sigma(u)$.

<u>Fact 10</u> Let $\delta = \delta(\mathbb{B})$ in N. Then the map $e : N \prec \hat{N}$ is δ^{+N}-cofinal.

proof
Let $x = t^G \in \hat{N}$. Then there is $b \in G$ s.t. $b \Vdash t \in \check{V}$. Set: $u = \{z \mid \bigvee b \in \mathbb{B}\ b \Vdash t = \check{z}\}$, then $\overline{\overline{u}} \leq \delta$ in N and $b \Vdash t \in \check{u}$.
Hence $x = t^G \in \check{u}^G = e(u)$. QED

An immediate corollary is:

<u>Fact 11</u> Let δ be as above, $\delta \doteq e(\delta) = \delta(e(\mathbb{B}))$. Then $\hat{N} = C^{\hat{N}}_{\delta}(\text{rng } e)$.
(Here $C^N_\delta(u)$ = the smallest $X \prec N$ s.t. $u \cup \delta \subset X$.)

Now let $\mathbb{A} \subseteq \mathbb{B}$ be complete Boolean algebras in N, with $\mathbb{A}$ completely contained in $\mathbb{B}$. Let $B \subset \mathbb{B}$ be an ultrafilter on $\mathbb{B}$ and set: $A =: B \cap \mathbb{A}$.

<u>Assume</u>: A is $\mathbb{A}$-generic over N.

Set: $\tilde{\mathbb{B}} = \mathbb{B}/A$; $\tilde{B} = B/A =: \{b/A \mid b \in B\}$.
(Hence $\tilde{B}$ is an ultrafilter on $\tilde{\mathbb{B}}$, where $\tilde{\mathbb{B}}$ is a complete BA in $N[A]$.)

<u>Fact 12</u> Let $N, \mathbb{A}, \mathbb{B}, A, B$ etc. be as above. Then:
(a) $\tilde{\mathbb{B}}$ is well founded iff B is
(b) Let $\tilde{B}$ be well founded. Then $\tilde{B}$ is weakly generic iff B is.

proof:
We first prove $(\rightarrow)$, assuming to be well founded we form the completion $\langle e, N'[A'], \tilde{B}' \rangle$ of $N[A], \tilde{B}$. Then $e(\tilde{\mathbb{B}}) = e(\mathbb{B})/A'$ where A' is $e(\mathbb{A})$-generic over N' and $\tilde{B}'$ is $e(\tilde{\mathbb{B}})$-generic over $N'[A]$. Moreover $A' = \tilde{B}' \cap e(\mathbb{A})$. Set:
$B' = A' * \tilde{B}' =: \{b \in e(\mathbb{B}) \mid b/A \in \tilde{B}'\}$
Then B' is $\overset{*}{e}(\mathbb{B})$-generic over N' and $e''B \subset B'$, where

$e^* = e \upharpoonright N$, $e^* : N \prec N'$. But then B' is well founded over N by Fact 1. If $\tilde{B}$ is weakly generic over $N[A]$, then $\omega_1^N = \omega_1^{N'}$ and B' is weakly generic over N by Fact 1

QED ($\rightarrow$)

We now prove ($\leftarrow$). Let B be well founded over N. Form the completion $\langle e, N', B' \rangle$ of N, B. Then B' is $e(\mathbb{B})$-generic over N'. Set $A' = B' \cap e(\mathbb{A})$. Then A' is $e(\mathbb{A})$-generic over N' and $e``A \subset A'$. Hence e has a unique extension $e^* : N[A] \prec N'[A']$ s.t. $e^*(A) = A'$. Set $\tilde{\mathbb{B}}' = e(\mathbb{B})/A'$. Then $e^*(\tilde{\mathbb{B}}) = \tilde{\mathbb{B}}'$, (where $\tilde{\mathbb{B}} = \mathbb{B}/A$). Set:

$$\tilde{B}' = B'/A' =: \{b/_{A'} \mid b \in B\}.$$

Then $\tilde{B}'$ is $\tilde{\mathbb{B}}'$-generic over $N'[A']$. Since $e^*(b/_A) = e(b)/_{A'}$, it follows easily that $e^* `` \tilde{B} \subset \tilde{B}'$. Thus $\tilde{B}$ is well founded over N, by Fact 1. If B is weakly generic, then so is A and

$$\omega_1^{N[A]} = \omega_1^N = \omega_1^{N'} = \omega_1^{N'[A']}.$$

Hence $\tilde{B}$ is weakly generic by Fact 1,

QED (Fact 12)

Now let $\mathbb{A} \subseteq \mathbb{B}$ be complete BA's in N, and let $A \subset \mathbb{A}$ be a well founded ultrafilter on $\mathbb{A}$. Let $\langle \hat{e}, \hat{N}, \hat{A} \rangle$ be the completion of N, A. Then $\hat{A}$ is $\hat{e}(\mathbb{A})$-generic over $\hat{N}$. Let $\hat{B} \subset \hat{e}(\mathbb{B})$ be a well founded UF on $e(\mathbb{B})$ s.t. $\hat{A} \subset \hat{B}$. Let $\langle e', N', B' \rangle$ be the completion of of $\hat{N}, \hat{B}$. Set $e = e' \circ \hat{e}$. Then $B =: e^{-1}{}'' B'$ is a well founded ultrafilter on $\mathbb{B}$. Moreover, B is weakly generic if B is, since then $\mathbb{A}$ is weakly generic; hence $\omega_1^N = \omega_1^{\hat{N}}$ and $\hat{B} = \hat{e}^{-1}{}'' B'$ is weakly generic; hence $\omega_1^N = \omega_1^{\hat{N}} = \omega_1^{N'}$.

<u>Fact 13</u> In the above situation $\langle e, N', B' \rangle$ is the completion of N, B.

proof.

We apply Fact 6. Conditions (a)–(c) are immediate. We prove:

(d) $N' = \{ e(s)^{B'} \mid s \in N^{\mathbb{B}} \wedge \Vdash^N_{\mathbb{B}} s \in \check{V} \}$.

proof

Let $x \in N'$. Then $x = e'(t)^{B'}$, where $t \in \hat{N}^{\hat{e}(\mathbb{B})}$ and $\Vdash_{\hat{e}(\mathbb{B})} t \in \check{V}$.

But then $t = \hat{e}(s)^{\hat{A}}$, where $s \in N^{\mathbb{A}}$ and $\Vdash_{\mathbb{A}} s \in \check{V}$. Hence:

$$x = e'(\hat{e}(s)^{\hat{A}})^{B'} = (\acute{e}(s)^{e'(\hat{A})})^{B'}$$

where $e'(\hat{A}) = B' \cap e(\mathbb{A})$. Hence

$$x = (e(s)^{B' \cap e(\mathbb{A})})^{B'}.$$ But there is an $r \in N^{\mathbb{B}}$ s.t. $r^B = (s^{B \cap \mathbb{A}})^B$ whenever B is $\mathbb{B}$-generic over N and $s^{B \cap \mathbb{A}} \in N^{\mathbb{B}}$. But there is an $r \in N^{\mathbb{B}}$ s.t.

$$\Vdash^{N}_{\mathbb{B}} r = (\check{s}^{\mathring{B} \wedge \check{\mathbb{A}}}),$$

where $\mathring{B}$ is the canonical $\mathbb{B}$-generic name (i.e. $[\![\check{b} \in \mathring{B}]\!] = b$ for $b \in \mathbb{B}$).

But then whenever B is $\mathbb{B}$-generic over N, we have:

$$r^{B} = (s^{B \cap \mathbb{A}})^{B},$$

since $\check{s}^{B} = s$, $\check{\mathbb{A}}^{B} = \mathbb{A}$, and $\mathring{B}^{B} = B$.

Since $e : N \prec N'$ we have:

$$\Vdash^{N'}_{e(\mathbb{B})} e(r) = (e(s)^{\mathring{B} \cap e(\check{\mathbb{A}})}).$$

Since B' is $e(\mathbb{B})$-generic over N', we conclude:

$$e(r)^{B'} = (e(s)^{B' \cap e(\mathbb{A})})^{B'} = x.$$

QED (Fact 13)

Fact 14 Let $\mathbb{A} \subseteq \mathbb{B}$ be complete BA's in N. Let $B \subset \mathbb{B}$ be an ultrafilter on $\mathbb{B}$ which is well founded wrt. N. Set $A = B \cap \mathbb{A}$. Let:

$\langle e, \hat{N}, \hat{A} \rangle$ be the completion of N, A

$\langle e', N', A' \rangle$ be the completion of N, B.

There is a unique $\sigma : \hat{N} \prec N'$ s.t. $\sigma e = e'$ and $\sigma `` \hat{A} \subset B' \cap e'(\mathbb{A})$.

proof.

$e' : N \prec N'$ and $\sigma' `` A \subset A'$ where $A' = B' \cap e'(\mathbb{A})$

Hence $\sigma = \mathrm{int}(e', A')$ is the unique such function. QED

Def We denote σ by $\mathrm{int}(A, B)$ (or $\mathrm{int}(\langle \mathbb{A}, A \rangle, \langle \mathbb{B}, B \rangle)$ (or $\mathrm{int}(N, \langle \mathbb{A}, A \rangle, \langle \mathbb{B}, B \rangle)$, if we wish to mention all relevant parameters).

Fact 15 Let $\mathbb{A}, \mathbb{B}, A, B, \sigma = \mathrm{int}(A, B)$ be as above. Then $\langle \sigma, N', A' \rangle$ is the completion of $\hat{N}, \hat{B}$ where $\hat{B} = \sigma^{-1} `` B'$.

proof.

$\hat{B}$ is obviously well founded wrt. $\hat{N}$. We apply Fact 6. Conditions (a)–(c) are trivially satisfied. We prove (d). Let $x \in N'$. Then $x = e'(t)^{B'}$ where $t \in N^{\mathbb{B}}$ and $\Vdash_{\mathbb{B}} t \in \check{V}$. But then $x = \sigma(\because(t))^{B'}$ where $(t) \in \hat{N}^{\hat{\mathbb{B}}}$ and and $\Vdash_{\hat{\mathbb{B}}} e(t) \in V'$

QED

If $\mathbb{A} \subseteq \mathbb{B}$ is completely contained in $\mathbb{B}$, we shall often have recourse to a function $h : \mathbb{B} \to \mathbb{A}$ defined by:

Def $h(b) = h_{\mathbb{A}}(b) = h_{\mathbb{A},\mathbb{B}}(b) =: \bigcap \{a \in \mathbb{A} \mid b \subset a\}$.

It follows easily that:

$$h(\bigcup_i b_i) = \bigcup_i h(b_i)$$

$$a \cap h(b) = h(a \cap b) \text{ if } a \in \mathbb{A}.$$

If is $\mathbb{A}$-generic, we of course have:

$$b/A = 0 \text{ in } \mathbb{B}/A \longleftrightarrow \forall a \in A \; a \cap b = 0$$

for $b \in \mathbb{B}$. It follows easily that:

$$b/A = 0 \longleftrightarrow h(b) \in A.$$

Thus $h(b) = [\![\check{b}/\mathring{A} = 0]\!]_{\mathbb{A}}$, where $\mathring{A}$ is the canonical name for an $\mathbb{A}$-generic set.

$h(b) \in A$ is a necessary and sufficient condition for the existence of an UF B on $\mathbb{B}$ s.t. $b \in B$ and $A \subset B$. (To see sufficiency, let $\tilde{B}$ be an UF on $\mathbb{B}/A$ s.t. $b/A \in \tilde{B}$. Then set:

$$B = A * \tilde{B} =: \{b \in \mathbb{B} \mid b/A \in \tilde{B}\}.)$$

The following fact will be useful:

<u>Fact 16</u> Let $\mathbb{A} \subseteq \mathbb{B}$ be complete BA's in N.
Let $A \subset \mathbb{A}$ be a well founded UF on $\mathbb{A}$.
Let $\langle e, \hat{N}, \hat{A} \rangle$ be the completion of N, A.
Let $b \in e(\mathbb{B})$ s.t. $h_{e(\mathbb{A})}(b) \in \hat{A}$. Then there is $d \in \mathbb{B}$ with the properties:

- $h_{\mathbb{A}}(d) \in A$
- Let B be any well founded UF on $\mathbb{B}$ s.t. $A \cup \{d\} \subset B$. Let $\langle e', N', B' \rangle$ be the completion of N, B. Let $\sigma = \text{int}(A, B)$. Then $\sigma(b) \in B'$.

proof.
Let $b = e(t)^{\hat{A}}$, where $\Vdash_{\mathbb{A}} t \in \check{V}$. Let $s \in N^{\mathbb{B}}$ s.t. $\Vdash_{\mathbb{B}} s = \check{t}^{(\dot{B} \cap \check{\mathbb{A}})}$, where $\dot{B}$ is the canonical $\mathbb{B}$-generic name.

Set: $d = [\![s \in \dot{B}]\!]_{\mathbb{B}}$.

Then:

$$(1)\ e(d) = [\![e(s) \in \dot{B}]\!]_{e(\mathbb{B})}$$
$$= [\![\check{e(t)}^{(\dot{B} \cap \check{e(\mathbb{A})})} \in \dot{B}]\!]_{e(\mathbb{B})}$$

Since $\hat{A}$ is $e(\mathbb{A})$-generic over $\hat{N}$ and $h_{e(\mathbb{A})}(b) \in \hat{A}$, we can find a $\hat{B} \supset \check{A}$ s.t. $\hat{B}$ is $e(\mathbb{B})$-generic over $\hat{N}$ and $\hat{A} \cup \{b\} \subset \hat{B}$.

(If $\hat{N}$ is uncountable we may have to work in the generic collapse of some cardinal in order to find $\hat{B}$.)

But then

(2) $b = \dot{e}(t)^{\check{A}} = e(t)^{\hat{B} \cap e(A)} \in \hat{B}$.

Hence by (1):

(3) $e(d) \in \hat{B}$.

Hence $e(d)/\hat{A} \neq 0$ and

(4) $h_{/\hat{A}}(e(d)) \in \hat{A}$.

But $e(h_{/A}(d)) = h_{/A}(e(d))$ and $A = e^{-1}{}''\hat{A}$. Hence:

(5) $h_{/A}(d) \in A$,

Now let $B \supset A \cup \{d\}$ be a well founded ultrafilter on N. Let $\langle e', N', B' \rangle$ be the completion of N, B. Then

$e'(d) = [\![e'(s) \in \dot{B}]\!]_{e'(B)} \in B'$, where

B' is $e'(B)$-generic over N'. Hence

$e'(s)^{B'} = e'(t)^{A'} \in B'$, where $A' = B' \cap e'(A)$.

Let $\sigma = \mathrm{int}(A, B)$. Then

$\sigma : \hat{N} \prec N'$ with $\sigma''\hat{A} \subset A'$ where $\hat{A}$ is $e(\mathbb{A})$-generic over $\hat{N}$ and A' is $e'(\mathbb{A}) = \sigma(e(\mathbb{A}))$-generic over N'.

Hence σ extends uniquely to a $\sigma^* : \hat{N}[\hat{A}] \prec N'[A']$ with $\sigma^*(\hat{A}) = A'$.

Hence $e'(t)^{A'} = \sigma e(t)^{A'} = \sigma e(t)^{\sigma^*(\hat{A})} =$
$= \sigma^*(e(t)^{\hat{A}}) = \sigma(b) \in B$.

QED (Fact 16)