VI Large Cardinals in the "Ultimate" K^c

It is known that if Θ is a subtle cardinal and we build the model K^c up to Θ using only 1-small premice in the construction, then either K^c has a Woodin cardinal or Steel's "cheap covering lemma" holds in the form: The set Z of $\tau < \Theta$ s.t. $\tau^{+K^c} < \tau^+$ is not subtle.
(A set $X \subset \Theta$ is called <u>subtle</u> iff whenever $a_\alpha \subset V_\alpha$ for $\alpha \in X$ and $C \subset \Theta$ is cub in Θ, there are $\alpha, \beta \in X$ s.t. $\alpha < \beta$ and $a_\alpha = \alpha \cap a_\beta$. This is a "largeness" concept like stationarity and also satisfies Fodor's lemma. Θ is called a <u>subtle cardinal</u> iff Θ itself is a subtle set in Θ. If we suppose Θ to be ineffable (measurable), then the cheap covering lemma holds with "ineffable" ("of measure 1") in place of "subtle".)

Some constructions are known for obtaining a larger K^c by using a larger class of premice (e.g. 2-small). The failure of the cheap covering lemma then has correspondingly stronger consequences for K^c. We now consider the "ultimate K^c model", in which all premice in the sense of [NFS] are permitted in the construction.

We are, of course, very far from proving the existence of this structure. We show, however, that if this K^c exists and the cheap covering lemma fails, then K^c contains a subtle class of quite large cardinals.

Def A cardinal κ is quasi compact iff for each $A \subset H_{\kappa^+}$ there exist a cardinal λ, a set $A' \subset H_{\lambda^+}$ and an elementary map

$\pi : \langle H_{\kappa^+}, A \rangle \prec \langle H_{\lambda^+}, A' \rangle$ s.t. $\kappa = \mathrm{crit}(\pi)$.

(*Note* It follows easily that $\square_\kappa$ fails for quasicompact κ.)

Def κ is *strongly* quasi compact iff there exist λ, π s.t. whenever $A \subset H_{\kappa^+}$ there is $A' \subset H_{\lambda^+}$ with $\pi : \langle H_{\kappa^+}, A \rangle \prec \langle H_{\lambda^+}, A' \rangle$, $\kappa = \mathrm{crit}(\pi)$.

Assuming the existence of K^c up to a subtle cardinal θ we get:

Thm 1 If $\{\tau < \theta \mid \tau^{+K^c} < \tau^+\}$ is subtle, then so is the set of cardinals which are quasi compact in K^c.

Thm 2 If $\{\tau < \theta \mid \tau^{++K^c} < \tau^+\}$ is subtle, then so is the set of cardinals which are strongly quasi compact in K^c.

Thm 3 $\{\tau < \theta \mid \mathrm{cf}(\tau^{+K^c}) < \tau\}$ is not subtle.

(*Note* The corresponding versions hold if we suppose θ to be ineffable or measurable.)

The proofs of all three results are essentially the same, so we prove only Theorem 1. The construction we use for K^c is that of Steel in [5]. We define premice N_α, M_α by ind. on $\alpha < \theta$ as follows:

If N_α is defined we ask whether it is a weak mouse in the sense of I of these notes. If so, we set: $M_\alpha = \mathrm{core}(N_\alpha)$. If not, the construction stops. M_α is then undefined and N_β is undefined for $\beta > \alpha$.

Now suppose N_ξ, M_ξ to be defined for $\xi < \alpha$. We define N_α by cases as follows:

<u>Case 1</u> $\alpha = 0$. $N_0 =_{\mathrm{df}} \langle \emptyset, \emptyset \rangle$

<u>Case 2</u> $\alpha = \beta + 1$

We first define the notion of <u>background certificate</u>:

Let $N = \langle J_\nu^E, F\rangle$ be a premouse. $\langle Q, F^*\rangle$ is a <u>background certificate</u> for N iff the following holds: Let $\kappa = \mathrm{crit}(F)$, $\lambda = \mathrm{lh}(F)$. Then

- Q is a transitive ZF^- model
- $V_\kappa \in Q$
- F^* is an extender on Q with critical point κ *)
- $V_{\lambda+2} \subset \mathrm{Ult}(Q, F^*)$
- $F(X) = F^*(X) \cap \lambda$ for $X \in \mathbb{P}(\kappa) \cap Q \cap N$.

<u>Case 2.1</u> $M_\beta = \langle J_\nu^E, \emptyset\rangle$ and there is F s.t. $\langle J_\nu^E, F\rangle$ is a premouse and for each countable $X \subset \mathbb{P}(\kappa) \cap J_\nu^E$ there is a background certificate $\langle Q, F^*\rangle$ s.t. $X \in Q$. Pick such F and set:

$N_\alpha = \langle J_\nu^E, F\rangle$.

*) We do not require $\mathbb{P}(\kappa) \subset Q$, since F^* need not be weakly amenable.

Case 2.2 Case 2.1 fails. Let $M_\beta = \langle J_\nu^E, E_\nu \rangle$.
Set: $N_\alpha = \langle J_{\nu+1}^E, \emptyset \rangle$.

Case 3 $Lim(\alpha)$.
For $\xi < \alpha$ set:

$\kappa_\xi = \kappa_{\xi,\alpha} = \inf\{\omega\rho^\omega_{N_i} \mid \xi \le i < \alpha\}$

$\mu_\xi = \mu_{\xi,\alpha} = \kappa_\alpha^+ =_{\text{Df}} \begin{cases} \kappa_\xi \text{ if } \kappa_\xi = On \wedge N_\xi; \\ \tau \text{ otherwise, where } \\ \tau \le On \wedge N_\xi \text{ is max} \\ \text{s.t. } \kappa \text{ is the largest} \\ \text{cardinal in } J_\tau^{E^{N_\xi}}. \end{cases}$

If we have:

(*) $J_{\mu_\xi}^{E^{N_\xi}} = J_{\mu_\xi}^{E^{N_i}}$ for all $\xi \le i < \alpha$,

we set: $N_\alpha = \langle \bigcup_{\xi<\alpha} J_{\mu_\xi}^{E^{N_\xi}}, \emptyset \rangle$.

If (*) fails, then N_α is undefined.

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It turns out that (*) can never fail in Case 3 and that in fact $\bigwedge i < \alpha \bigvee j < \alpha\ \mu_{i\alpha} < \mu_{j\alpha}$. (It is easily seen that $\mu_{i\alpha} \le \mu_{j\alpha}$ for $i \le j < \alpha$.)
Hence $N_\alpha = \langle J_\lambda^E, \emptyset \rangle$ for some λ.

From now on we make the assumption

(**) N_α is defined for all $\alpha < \theta$.

We can then define $\kappa_{\xi,\theta}$, $\mu_{\xi\theta}$ as above for $\xi < \theta$, again getting (*) at θ. We then define:

$$K^c =_{\mathrm{Df}} N_\theta = \bigcup_{\xi < \theta} J^{E^{N_\xi}}_{\mu_{\xi,\theta}}$$

It turns out that $K^c = J^E_\theta$ is a ZFC model. Each $K^c \| \nu = \langle J^E_\nu, E_{\omega\nu} \rangle$ is a weak mouse for $\nu < \theta$.

Recall that we assumed:

(***) θ is a subtle cardinal.

Before proving Thm 1, we need some preliminary facts which were stated and proven in [MOI] §1 as Fact 1 – Fact 9. These facts hold for all K^c models and we restate the salient conclusions here.

(<u>Note</u> Fact 2 in [MOI] §1 was misstated and should read: Let $\kappa = \omega\rho^\omega_{N_\ell} = \kappa_{\ell i}$. Then $\ell < h < i \to \mu_{\ell i} < \mu_{hi}$)

Set: $\mu_\xi = \mu_{\xi,\theta}$, $\kappa_\xi = \kappa_{\xi,\theta}$.
Let $\omega < \lambda < \theta$ s.t. λ is a limit ordinal and is cardinally absolute in K^c (i.e. if $\tau < \lambda$ is a cardinal in J^E_λ, then in $J^E_\theta = K^c$). Set:

$$\delta = \delta(\lambda) = \text{lub}\{\xi \mid \mu_\xi < \lambda\}.$$

Since $\mu_\xi \le \mu_{\xi'}$ for $\xi \le \xi' < \theta$ and $\sup_{\xi<\theta} \mu_\xi = \theta$, we know that $\delta < \theta$.
In [MOI] §1 we prove:

(1) δ is a limit ordinal

(2) $\mu_i = \mu_{i,\delta}$ for $i < \delta$

(3) $N_\delta = J^E_\lambda$

(4) $M_\delta = N_\delta$ and $\mu_\delta = \lambda$

(5) If λ is a cardinal in K^c, then $\mu_\delta = \kappa_\delta = \lambda$.

(These facts were also stated in [NFS] but some of the proofs given there were confused).

We now prove Thm 1. Suppose not. Then there is a subtle set Z s.t. each $\tau \in Z$ is a cardinal with $\tau + \kappa^c < \tau^+$ and no $\tau \in Z$ is quasicompact. For $\tau \in Z$ let $A_\tau \subset (H_{\tau^+})^{\kappa^c}$ be a counterexample to quasi compactness in κ^c. Set $\tilde{\tau} = \tau + \kappa^c$, $\bar{K}_\tau = J^E_{\tilde{\tau}}$. (Hence $\bar{K}_\tau = (H_{\tau^+})^{\kappa^c}$). Select $Q = Q_\tau \in H_{\tau^+}$ s.t.

- Q is a transitive ZF^- model
- $V_\tau, \bar{K}_\tau, A_\tau \in Q$

Let $f^\tau : \tau \leftrightarrow Q_\tau$

Let a_τ be the set of all tuples $\langle \varphi, \langle \xi_1, \ldots, \xi_n \rangle, \langle \zeta_1, \ldots, \zeta_m \rangle \rangle$ s.t. φ is a 1-st order fmla in the language of Q_τ, $\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m < \tau$ and $Q_\tau \models \varphi[f_\tau(\vec{\xi}), \vec{\zeta}, \bar{K}_\tau, A_\tau]$.

For $\lambda \in Z$ set: $Z_\lambda = \{\kappa \in Z \cap \lambda \mid a_\kappa = V_\kappa \cap a_\lambda\}$

Then $Z^* = \{\lambda \in Z \mid \sup Z_\lambda = \lambda\}$ is subtle. (If not, then $Z \setminus Z^*$

subtle. For $\lambda \in Z \setminus Z^*$ pick η_λ s.t. $a_\alpha \neq V_\alpha \cap a_\lambda$ for $\alpha \in (\eta_\lambda, \lambda) \cap (Z \setminus Z^*)$. By Fodor there is η s.t. $\{\lambda \mid \eta_\lambda = \eta\}$ is subtle. Pick α, β s.t. $\alpha < \beta$, $\eta_\alpha = \eta_\beta = \eta$, $a_\alpha = \alpha \cap a_\beta$. Then $\alpha < \eta < \alpha$. Contr!.) From now

~~mit.~~ on let $\gamma \in Z^*$, $\kappa \in Z_\gamma$. There is obviously a map $\pi = \pi_{\kappa\gamma} : Q_\kappa \prec Q_\gamma$ defined by: $\pi(f_\kappa(\xi)) = f_\gamma(\xi)$ for $\xi < \kappa$. Moreover $\pi \restriction \kappa = \mathrm{id}$ and $\pi(\kappa) = \gamma$. Clearly $\pi(\bar{K}_\kappa) = \bar{K}_\gamma$ and $\pi(A_\kappa) = A_\gamma$. Set $F^* = \pi \restriction \mathbb{P}(\kappa)$. Let $\tilde{\pi} : Q \longrightarrow_{F^*} \tilde{Q}$. $\tilde{Q}$ is

$Q = Q_\kappa$ well founded since there is σ s.t. $\sigma : \tilde{Q} \longrightarrow_{\Sigma_0} Q_\gamma$ defined by: $\sigma(\tilde{\pi}(f)(\alpha)) = \pi(f)(\alpha)$ for $\alpha < \gamma$, $f : \kappa \to Q_\kappa$, $f \in Q_\kappa$. It follows easily that $\sigma \restriction (\gamma + 1) = \mathrm{id}$ and $V_\gamma^{\tilde{Q}} = V_\gamma^{Q_\gamma} = V_\gamma$. Now let $\lambda \in Z_\gamma$ s.t. $\lambda > \kappa$. Set:

$F(X) = F^*(X) \cap \lambda$ for $X \in \mathbb{P}(\kappa) \cap \bar{K}_\kappa$.

Then F is an extender of length λ on $\bar{K}_\kappa$.

(1) F is weakly amenable.

prf. Let $X = \langle X_i \mid i < \kappa \rangle \in \bar{K}_\kappa$. Then $\pi(X) = \langle \pi(X)_i \mid i < \gamma \rangle \in \bar{K}_\gamma$. Hence for $\alpha < \lambda$ we have: $\{i \mid X_i \in F_\alpha\} = \{i < \kappa \mid \alpha \in \pi(X)_i\} \in \mathbb{P}(\kappa) \cap K^c \subset \bar{K}_\kappa$

QED (1)

Now let $\bar{\pi} : \bar{K}_\kappa \xrightarrow[F]{} \bar{K}$. There is $\bar{\sigma} : \bar{K} \xrightarrow[\Sigma_0]{} \bar{K}_\gamma$ defined by: $\bar{\sigma}(\bar{\pi}(f)(\alpha)) = \pi(f)(\alpha)$. Thus $\bar{K}$ is well founded.

(2) $\bar{K} = J^E_\nu$, where $\nu = \mathrm{ht}(\bar{K})$.

proof.

Pick $\xi \in (\lambda, \nu)$ s.t. $\omega\rho^\omega_{\bar{K}\|\xi} = \lambda$. There are arbitrarily large such ξ, so it suffices to show: $\bar{K}\|\xi = K^c\|\xi$. Let $K' = \bar{\sigma}(\bar{K}\|\xi)$. Then $K' = K^c\|\sigma(\xi)$. But $\bar{\sigma}\restriction(\bar{K}\|\xi) : \bar{K}\|\xi \xrightarrow[\Sigma_\omega]{} K'$ and $\lambda = \mathrm{crit}(\bar{\sigma})$. Since λ is a limit cardinal in $\bar{K}\|\xi$, it follows from §8 Lemma 4 of [NFS] that either $\bar{K}\|\xi = \mathrm{core}(K')$ or $\bar{K}\|\xi$ is a proper segment of K'. But the first alternative is impossible, since $\omega\rho^{\omega}_{\bar{K}\|\xi} < \omega\rho^{\omega}_{K'}$. QED(2)

(3) $\langle J^E_\nu, F\rangle$ is a premouse.

prf.
All conditions except the initial segment condition are trivial. We verify the initial segment condition as stated in I of these notes.
Define $C = C_{\langle J^E_\nu, F\rangle}$ as in I. If not, then there is a least $\lambda' \in C$ s.t. $F|\lambda' \notin J^E_\nu$ (hence $F|\lambda' \notin J^{E}_{\nu}$), Set $F' = F|\lambda'$ and let $\pi': J_\kappa \xrightarrow[F]{} J^{E'}_{\nu'}$. It follows that $\langle J^{E'}_{\nu'}, F'\rangle$ is a premouse with background certificate $\langle Q, F^*\rangle$. Exactly as above, however, we get $J^{E'}_{\nu'} = J^E_{\nu'}$.
Now let $\delta = \delta(\lambda')$ in the sense of the above definition.

Then $N_\delta = J^E_{\lambda'}$ and $\mu_\delta = \kappa_\delta = \lambda'$. Hence $\kappa_\xi \geq \lambda'$ for all $\xi \geq \delta$. Now let $\delta' = \delta(\nu')$. Then $N_{\delta'} = J^E_{\nu'}$. Since $\langle Q; F^* \rangle$ is a sufficient background certificate for $\langle J^E_{\nu'}, F' \rangle$, we have: $N_{\delta'+1} = \langle J^E_{\nu'}, F' \rangle$. But $\mu_\xi \geq \mu_{\delta'} = \nu'$ for $\delta' \leq \xi$. ~~Hence~~ $F' = E_{\nu'}$. Hence $F' \in K^c$. Contr!

QED (3).

But then $\langle Q, F^* \rangle$ is a background certificate for $\langle J^E_\nu, F \rangle$ and it follows as before that $F = E_\nu \in$ $\in K^c$. Hence $\bar{\pi} \in K^c$. Note, however, that if $\pi_{\kappa\lambda}$ is defined like $\pi = \pi_{\kappa\lambda}$, then $\pi_{\lambda\gamma}\pi_{\kappa\lambda} = \pi$, $\pi_{\lambda\gamma} \restriction \lambda = id$, $F = \pi_{\kappa\lambda} \restriction \mathbb{P}(\kappa)$. It follows easily that $\bar{\pi} = \pi_{\kappa\lambda} \restriction \bar{K}_\kappa$, where $\pi_{\kappa\lambda}(Q_\kappa) = Q_\lambda$, $\pi_{\kappa\lambda}(A_\kappa) = A_\lambda$ and $\pi_{\kappa\lambda}(\bar{K}_\kappa) = \bar{K}_\lambda = (H_{\lambda^+})^{K^c}$.

But then $\tilde{\pi} : \langle \tilde{K}_\kappa, A_\kappa \rangle \prec \langle \tilde{K}_\lambda, A_\lambda \rangle$. Hence A_κ is not a counterexample to the quasi-compactness of κ in K^c. Contr! QED (Thm 1)

<u>Note</u> A stronger background condition in Case 2.1 of the def. of K^c would be: For each $B \subset \kappa$ (in V) there is a background certificate $\langle Q, F^* \rangle$ s.t. $B \in Q$. If the ultimate K^c were defined in this way we would still get the above results, replacing "subtle" by "2-subtle". ($X \subset \Theta$ is 2 subtle iff whenever $a_\alpha \subset \alpha$ for all $\alpha \in X$ and $a_{\alpha\beta} \subset \alpha$ for $\alpha < \beta$ in X, then whenever $C \subset \Theta$ is cub in Θ, there are $\alpha, \beta, \gamma \in C \cap X$ s.t. $\alpha < \beta < \gamma$, $a_\alpha = \alpha \cap a_\beta$, $a_\beta = \beta \cap a_\gamma$, $a_{\alpha\beta} = a_{\alpha\gamma} = \alpha \cap a_{\beta\gamma}$

<u>Note</u> A connection between quasi- and supercompactness is given by: Let κ be $(2^\kappa)^+$-supercompact. Let $\sigma : V \prec W$, ${}^{(2^\kappa)}W \subset W$, $\kappa = \mathrm{crit}(\sigma)$, $\lambda = \sigma(\kappa)$. Then λ, $\sigma \upharpoonright H_{\kappa^+}$ verify that κ is strongly quasicompact in W.