Dee-Subproper Forcing

The concepts of α-proper and Dee-proper forcing were developed by Shelah in [PIF] and masterfully exposited by Uri Abraham in [PF]. In these notes we generalize the concepts to subproper forcing and then use them to solve an iteration problem left over from [PIF].

In §2 we introduce α-subproper forcing and prove that a revised countable support iteration of α-subproper forcings, subject to the usual restraints, yields an α-subproper forcing. α-proper forcings are trivially α-subproper. In addition, we show that every subcomplete forcing is α-subproper. In §1 we lay the groundwork for §2 by revisiting the notion of subproper forcing. We slightly revise the definition and redo the proof of the iteration theorem, in preparation for the more difficult proof in § 2.

In § 3 we introduce Dee-subproper forcing. This generalizes the notion of simple Dee-proper forcing, as defined in [PF]. (A generalization of the broader concept is readily available but, alas, we were unable to prove a reasonable iteration theorem for it.) Subcomplete forcings are trivially Dee-subproper. We prove that an RCS iteration of forcings which are ω_1-subproper and Dee-subproper, subject to the usual restraints, will not add new reals.

In § 4 we then apply our methods to a variant $\mathbb{N}'$ of Namba forcing. $\mathbb{N}'$ was treated extensively in [PIF] (where it is called Nm'). The conditions are Namba trees which have a special form:

a single finite stem followed by ω_2 many branchings at each node thereafter. In [PIF] it was shown that, assuming CH, $\mathbb{N}'$ adds no reals and is essentially different from Namba forcing $\mathbb{N}$. It was also shown that if $\mathbb{N}'$ is semiproper, then a strong form of Chang's conjecture holds. Shelah developed a theory of 'I-condition' forcings, which enabled him to iterate Namba forcing without adding new reals. Since, however, $\mathbb{N}'$ does not appear to satisfy an I-condition, the question, whether $\mathbb{N}'$ can be iterated without collapsing ω_1 was left open in [PIF]. In [SPSC] we showed, assuming CH + $2^{\omega_1} = \omega_2$, that $\mathbb{N}'$ is subproper, hence can be iterated without collapsing ω_1. In §4 we show, assuming only CH, that $\mathbb{N}'$ is ω_1-subproper and Dee-subproper, Hence it can be iterated without adding new reals.

In an appendix we prove the companion theorem for Namba forcing $\mathbb{N}$: If CH holds, then $\mathbb{N}$ is subcomplete. (In [LF] we had shown this under the additional assumption $2^{\omega_1} = \omega_2$.) In the course of the proof we verify the following Lemma (without CH): If G is Namba-generic and $c \in V[G]$ is any ω-sequence which is monotone and cofinal in ω_2^V, then c is a Namba-generic sequence.

Bibliography

[PF] Uri Abraham Proper Forcing
in Handbook of Set Theory Springer 2010

[LF] Jensen $\mathcal{L}$-Forcing (handwritten) *)

[SPSC] Jensen Subproper and Sub-complete Forcing (handwritten) *)

[EN] Jensen The Extended Namba Problem (handwritten) *)

[IT] Jensen Iteration Theorems for Subcomplete and Related Forcings (handwritten) *)

[PIF] Shelah Proper and Improper Forcing Springer Perspectives in Math. Logic 1991

*) These notes are on my website. (To find it, enter "Ronald B. Jensen" in Google.)