§1 Subproper Forcing

We employ the notation and conventions of [SPCS] and [IT]. However, we change the definition of "subproper" slightly from the versions given in those papers. Our new definition reads:

Def Let $\mathbb{B}$ be a complete BA. Let $\delta = \delta(\mathbb{B}) =_{\mathrm{Df}}$ the smallest cardinality of a dense subset of $\mathbb{B}$. $\mathbb{B}$ is subproper as witnessed by $\theta > \omega_1$ iff $\mathbb{B} \in H_\theta$ and whenever $N = L_\tau^A =_{\mathrm{Df}} \langle L_\tau[A], \in, A\rangle$ is a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\theta < \tau$, then the following holds:

Let $\pi : \bar{N} \prec N$ s.t. $\pi \in N$, where $\bar{N}$ is countable, transitive and full. Let $s \in N$, $\pi(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$. Let $\bar{b} \in \bar{\mathbb{B}} \setminus \{0\}$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{B}, s$

(c) $C_\delta^N(\mathrm{rng}\,\sigma) = C_\delta^N(\mathrm{rng}\,\pi)$, where $C_\lambda^N(X) =_{\mathrm{Df}}$ the smallest $Y \prec N$ s.t. $\lambda \cup X \subset Y$.

(d) $\bar{G} = (\sigma^{-1})''G$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$ and $\bar{b} \in \bar{G}$.

Def $\mathbb{B}$ is subproper iff it is subproper as witnessed by some cardinal θ.

<u>Def</u> Ω <u>verifies</u> the subproperness of $\mathbb{B}$ iff every cardinal $\theta \geq \Omega$ witnesses the subproperness of $\mathbb{B}$.

<u>Note</u> This definition varies from the earlier definitions in [SPSC] and [IT] in that we no longer require τ to be regular, but only that $N = L^A_\tau$ be a ZFC$^-$ model. By a Löwenheim – Skolem argument we then need only consider N of cardinality $\bar{\bar{H}}_\theta = 2^\theta$ to determine whether $\mathbb{B}$ is subproper as verified by θ. Thus our new notion of subproperness is "locally based" in the sense that only $\mathcal{P}(H_\theta)$ is relevant to the question of whether θ witness sub-properness, regardless of what there may be further out in the universe. If θ is the least cardinal witnessing the subproperness of $\mathbb{B}$, it then follows easily that $(2^\theta)^+$ verifies subproperness.

It is also possible to employ a parameter in establishing the subproperness of $\mathbb{B}$:

Def $\mathbb{B}$ is subproper as witnessed by $\langle \theta, p \rangle$ iff $p \in H_\theta$ and whenever $N, \bar{N}, \pi$ are as in the earlier definition with $\pi(\bar{p}) = p$, then the same conclusion holds.

It is not hard to see that if the subproperness of $\mathbb{B}$ is witnessed by $\langle \theta, p \rangle$, then it is verified by $(2^\theta)^+$. We often make tacit use of this when establishing the subproperness of a given $\mathbb{B}$.

We note that our definition of subproperness differed from the earlier definition in two other respects:

(1) (c) replaces a weaker condition. We have thus made our definition harder to satisfy, which is in general not a good idea. However, in all examples of subproper forcing which have been found thus far, the weaker condition

was verified essentially by first verifying (c). (c) also seems to simplify the proofs of the iteration theorems. The weaker condition can be formulated as follows:

We suppose regular $\lambda_1, \ldots, \lambda_n$ to be given s.t. $\delta(\mathbb{B}) \leq \lambda_i < \theta$ and $\pi(\bar{\lambda}_i) = \lambda_i$ for $i = 1, \ldots, n$. Let $\bar{\lambda}_0 = On \cap \bar{N}$. In place of (c) we require:

(c') $\sup \pi''\bar{\lambda}_i = \sup \sigma''\bar{\lambda}_i$ $(i = 0, \ldots, n)$.

This can be derived from (c) as follows:

$\sup \pi''\bar{\lambda}_0 = \sup On \cap C^N_\delta(\text{rng} \quad) =$

$= \sup On \cap C^N_\delta(\text{rng}\,\sigma) = \sup \sigma''\bar{\lambda}_0$.

Replacing s by $\langle s, \lambda_1, \ldots, \lambda_n \rangle$, we can suppose σ so chosen that $\sigma(\bar{\lambda}_i) = \lambda_i$ $(i = 1, \ldots, n)$. But then:

$\sup \pi''\bar{\lambda}_i = \sup \lambda_i \cap C^N_\delta(\text{rng}\,\pi) =$

$= \sup \lambda_i \cap C^N_\delta(\text{rng}\,\sigma) = \sup \sigma''\bar{\lambda}_i$

for $i = 1, \ldots, n$.

A second minor difference is:

(2) We require $\pi \in N$. This is inessential and we have imposed the requirement simply to make our definition compatible with that of "α-subproper", given in the next chapter.

We shall now reprove the main iteration theorems for subproper forcing. The proofs differ somewhat from those given in [IT] (especially in Case 2 of Theorem 3). The changes are intended to facilitate the more complex iteration proofs in §2.

We now turn the proofs of the iteration theorems for subproper forcing. (These are proven in [IT], but we have modified some steps in preparation for the more difficult iterability proof in §2. The <u>two step iteration theorem</u> reads:

<u>Thm 1</u> Let $\mathbb{A} \subseteq \mathbb{B}$, where $\mathbb{A}, \mathbb{B}$ are complete BA's. Let $\mathbb{A}$ be subproper and $\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\dot{G}$ is subproper, where $\dot{G}$ is the canonical generic name. Then $\mathbb{B}$ is subproper.

prf.

Let θ be big enough that it verifies the subproperness of $\mathbb{A}$ and:
$\Vdash_{\mathbb{A}} (\check{\theta}$ verifies the subproperness of $\check{\mathbb{B}}/\dot{G})$

Let $N = L_\tau^A$ be a ZFC^- model s.t. $H_\theta \subset N$, $\theta < \tau$.

Let $\sigma : \bar{N} \prec N$ where $\bar{N}$ is countable and full. Let $\sigma(\bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{\theta}, \bar{s}) = \mathbb{A}, \mathbb{B}, \theta, s$. Let $\bar{b} \in \bar{\mathbb{B}} \setminus \{0\}$. We must find $b \in \mathbb{B} \setminus \{0\}$, $\dot{\sigma} \in V^{\mathbb{B}}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic and $\sigma = \dot{\sigma}^G$, then (a) – (d) in the definition of subproperness hold.

Let $a \in \bar{\mathbb{A}} \setminus \{0\}$, $\dot{\sigma}_0 \in V^{\mathbb{A}}$ s.t. whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\dot{\sigma}_0^{G_0} = \sigma$, then:

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{\theta}, \bar{s}) = \mathbb{A}, \mathbb{B}, \theta, s$.

(c) $C^N_{\delta_0}(\mathrm{rng}\,\sigma_0) = C^N_{\delta_0}(\mathrm{rng}\,\pi)$ where $\delta_0 = \delta(\mathbb{A})$

(d) $\bar{G}_0 = \sigma_0^{-1}{}''G_0$ is $\bar{\mathbb{A}}$-generic over $\bar{N}$ and $h_{\bar{\mathbb{A}}}(\bar{b}) \in \bar{G}_0$.

(Note As in [SPSC] we define:

$h_{\mathbb{A}}(b) = \bigwedge\{a \in \mathbb{A} \mid b \subset a\}$, when $\mathbb{A} \subseteq \mathbb{B} \ni b$.

It follows that $h_{\mathbb{A}}(b) = [\![\check{b}/\mathring{G} \neq 0]\!]_{\mathbb{A}}$.)

(Note $\delta_0 \leq \delta = \delta(\mathbb{B})$. Hence (c) implies $C^N_{\delta}(\mathrm{rng}\,\sigma_0) = C^N_{\delta}(\mathrm{rng}\,\pi)$.)

Now let $G_0 \ni a$ be as above, $\sigma_0 = \dot{\sigma}_0^G$. Let σ_0^* be the unique extension of σ_0 s.t.

$\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G]$, $\sigma_0^*(\bar{G}) = G$. Set:

$\bar{N}^* = L^{\bar{A}, \bar{G}}_{\bar{\tau}}$, $N^* = L^{A, G}_{\tau}$, where

$\bar{N} = L^{\bar{A}}_{\bar{\tau}}$, $N = L^{A}_{\tau}$. Set $\mathbb{B}^* = \mathbb{B}/G_0$.

Then $\mathbb{B}^*$ is subproper as verified by θ in $V[G]$. Set $H^*_\theta = H_\theta^{V[G_0]} = H_\theta[G_0]$.

Then $\mathbb{B}^* \in H^*_\theta$, $H^*_\theta \subset N^*$, $\theta < \tau$, where N^* is a ZFC$^-$ model. But

$\sigma_0^* : \bar{N}^* \prec N$, $\sigma_0^*(\bar{\theta}, \bar{\mathbb{B}}^*, \bar{\imath}) = \theta, \mathbb{B}^*, \imath$, where $\bar{\mathbb{B}}^* = \bar{\mathbb{B}}/\bar{G}_0$. Moreover $\bar{b}^* \in \bar{\mathbb{B}}^* \setminus \{0\}$ where $\bar{b}^* = \bar{b}/\bar{G}_0$. But then, by the subproperness of $\mathbb{B}^*$, there is $b^* \in \mathbb{B}^* \setminus \{0\}$ s.t. whenever $G^* \ni b^*$ is $\mathbb{B}^*$-generic over $V[G_0]$, then there is $\sigma^* \in V[G_0][G^*]$ satisfying (a)–(d) with $\bar{N}^*, N^*, \sigma_0^*$ in place of $\bar{N}, N, \pi$ – i.e.

(a*) $\sigma^* : \bar{N}^* \prec N^*$

(b*) $\sigma^*(\bar{\theta}, \bar{G}_0, \bar{\mathbb{B}}, \bar{\imath}) = \theta, G_0, \mathbb{B}, \imath$ (hence $\sigma^*(\bar{\mathbb{B}}^*) = \mathbb{B}^*$)

(c*) $C_{\delta^*}^{N^*}(\mathrm{rng}\,\sigma^*) = C_{\delta^*}^{N^*}(\mathrm{rng}\,\sigma_0^*)$,
where $\delta^* = \delta(\mathbb{B}^*)$ (hence $\delta^* \le \delta(\mathbb{B})^V$),

(d*) $\bar{G}^* = (\sigma^*)^{-1}{}''G^*$ is $\bar{\mathbb{B}}^*$-generic over $\bar{N}^*$ and $\bar{b}^* \in \bar{G}^*$.

Since this holds whenever $G_0 \ni a$, we may assume $b^* = \dot{b}^{G_0}$, where a forces $\dot{b}^{\dot{G}_0}$ to have these properties whenever $G_0 \ni a$ is generic. We may also assume w.l.o.g.

$$\Vdash_{\mathbb{A}} \dot{b} \in \check{\mathbb{B}}/\dot{G}, \quad [\![\dot{b} \neq 0]\!]_{\mathbb{A}} = a.$$

But then there is a unique $b \in \mathbb{B}$ s.t.

$$\Vdash_{\mathbb{A}} \check{b}/\dot{G} = \dot{b}. \text{ Hence } h_{\mathbb{A}}(b) = [\![\dot{b} \neq 0]\!] = a.$$

Now let $G \ni b$ be $\mathbb{B}$-generic. Set $G_0 = G \cap \mathbb{A}$, $G^* = G/G^0 =_{\mathrm{df}} \{c/G^0 \mid c \in G\}$.

Set: $b^* = b/G_0 = \dot{b}^{G_0}$. Then $G^* \ni b^*$ is $\mathbb{B}^* = \mathbb{B}/G_0$ - generic over $V[G_0]$ and $b^* \in G^*$. Let $\bar{\mathbb{B}}^*, N^*, \bar{N}^*, \sigma_0^*$ be defined as above. Let $\sigma^* \in V[G] = V[G_0][G^*]$ satisfy (a*) — (d*), with $\sigma_0^* = \dot{\sigma}_0^{G_0}$.

Set: $\sigma = \sigma^* \upharpoonright N$. <u>We claim</u>:

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\mathbb{B}}, \bar{\theta}, \bar{r}) = \mathbb{B}, \theta, r$

(c) $C_\delta^N(\mathrm{rng}\,\sigma) = C_\delta^N(\mathrm{rng}\,\pi)$ for $\delta = \delta(\mathbb{B})$

(d) $\bar{G} = (\sigma^{-1})''G$ is $\bar{\mathbb{B}}$ - generic over $\bar{N}$ and $\bar{b} \in \bar{G}$.

(a), (b) are immediate; (d) follows by?

$\bar{G} = \bar{G}_0 * \bar{G}^* = \{c \in \bar{\mathbb{B}} \mid c/\bar{G}_0 \in \bar{G}^*\}$, where $\bar{G}^* = (\sigma_0^*)^{-1}{}''G^*$ is $\bar{\mathbb{B}}^* = \bar{\mathbb{B}}/\bar{G}_0$ - generic over $\bar{N}[\bar{G}_0]$ and $\bar{G}_0 = (\sigma_0^{*-1})(G_0)$ is $\bar{\mathbb{A}}$ - - generic over $\bar{N}$.

We prove (c).

Since $\delta \geq \delta(\mathbb{B}^*)$, we have:

$$C_\delta^{N^*}(\mathrm{rng}\,\sigma^*) = C_\delta^{N^*}(\mathrm{rng}\,\sigma_0^*).$$

<u>Claim</u> $C_\delta^N(\mathrm{rng}\,\sigma) = N \cap C_\delta^{N^*}(\mathrm{rng}\,\sigma^*)$

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($\subset$) is trivial. We prove ($\supset$).

Let $x \in N \cap C_\delta^{N^*}(\mathrm{rng}\,\sigma^*)$. Then x is N^*-definable in $\xi < \omega_1$, $\sigma^*(w)$, where $w \in \bar{N}^*$.

$x =$ that x s.t. $N^* \models \varphi[x, \xi, \sigma^*(w)]$

$w = \mathring{w}^{\bar{G}_0}$, $\mathring{w} \in \bar{N}$, where $\bar{G}_0 = (\sigma_0^{*-1})(G_0) = \sigma_0^{-1}{}''G_0$.

Since $\sigma : \bar{N} \prec N$, $\sigma(\bar{\mathbb{B}}) = \mathbb{B}$, we have $\sigma(\bar{\delta}) = \delta$ where $\bar{\delta} = \delta(\bar{\mathbb{B}})$. Hence there is $f \in \bar{N}$ mapping $\bar{\delta}$ onto a dense subset of $\bar{\mathbb{A}}$. But then there is $\nu < \delta$ s.t. $\sigma(f)(\nu)$ forces $\varphi(\check{x}, \check{\xi}, \sigma(\mathring{w}))$, since $\sigma^*(w) = \sigma(\mathring{w})^{G_0}$. Hence:

$$x = \text{that } x \text{ s.t. } \sigma(f)(\nu) \Vdash^N_{\mathbb{A}} \varphi(\check{x}, \check{\xi}, \sigma(\mathring{w})) \in C^N_\delta(\mathrm{rng}\,\sigma)$$

Similarly, $N \cap C^N_\delta(\mathrm{rng}\,\sigma_0^*) = C^N_\delta(\mathrm{rng}\,\sigma_0)$.

But then:

$$C^N_\delta(\mathrm{rng}\,\sigma) = C^N_\delta(\mathrm{rng}\,\sigma_0) = C^N_\delta(\mathrm{rng}\,\pi)$$

by (c*). QED (Thm 1)

This proof shows more than we have stated. We can drop the assumption that $\mathbb{A}$ is subproper, assuming instead that $\mathbb{A}$ does not collapse ω_1 and that $a, \mathring{\sigma}_0$ have the stated properties. We then get a $b \in \mathbb{B} \setminus \{0\}$ s.t. $h_{\mathbb{A}}(b) = a$ and b has the stated properties. We can also dispense with the map π, assuming simply that $\sigma_0(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{A}, \mathbb{B}, s$ whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \mathring{\sigma}_0^{G_0}$, $\sigma(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}, \bar{s}) = \theta, \mathbb{A}, \mathbb{B}, s$ and $C^N_\delta(\mathrm{rng}\,\sigma) = C^N_\delta(\mathrm{rng}\,\sigma_0)$ whenever

$G \ni b$ is $\mathbb{B}$-generic, $G_0 = G \cap \mathbb{A}$, and $\sigma_0 = \mathring{\sigma}_0^{\,G}$.

Rather than taking $\bar{x}$ as a fixed element of $\bar{N}$ we could take: $\bar{x} = \mathring{x}^{G_0}$, where $\mathring{x} \in V^{\mathbb{A}}$ and all $\mathring{x} \in \check{\bar{N}}$, thus getting $\sigma(\mathring{x}^{G_0}) = \sigma_0(\mathring{x}^{G_0})$. We also note that σ "$\bar{G}_0 \subset G_0$ (where, again, $\bar{G}_0 = (\sigma_0^{-1})$"$G_0$), since $\sigma^*(\bar{G}_0) = G_0$. Instead of taking $\bar{b}$ as a fixed element we can take $\bar{b} = \mathring{b}^{G_0}$, where $\mathring{b} \in V^{\mathbb{A}}$, all $\mathring{b} \in \check{\bar{\mathbb{B}}}$ and all $h_{\check{\mathbb{A}}}(\mathring{b}) \in \mathring{\bar{G}}$, where $\mathring{\bar{G}}^{G_0} = \bar{G}_0$. (Equivalently, all $\mathring{b}/\mathring{\bar{G}} \neq 0$.)

Putting all of this together, we get:

Lemma 2 Let $\mathbb{A} \subseteq \mathbb{B}$ s.t. $\mathbb{A}$ does not collapse ω_1 and $\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\dot{G}$ is subproper ($\dot{G}$ being the canonical $\mathbb{A}$-generic name). Let θ be big enough that:

$\Vdash_{\mathbb{A}} \check{\theta}$ verifies the subproperness of $\check{\mathbb{B}}/\dot{G}$.

Let $N = L_{\tau}^{A}$ be a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\theta < \tau$, Let $\bar{N}$ be countable and full. Let $a \in \mathbb{A} \setminus \{0\}$ and $\dot{\sigma}_0, \dot{s}, \dot{b} \in V^{\mathbb{A}}$ s.t. whenever $G_0 \ni a$ is $\mathbb{A}$-generic, $\sigma_0 = \dot{\sigma}^{G_0}$, $\bar{s} = \dot{s}^{G_0}$ and $\bar{b} = \dot{b}^{G_0}$, then:

(i) $\sigma_0 : \bar{N} \prec N$ and $\sigma_0(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}) = \theta, \mathbb{A}, \mathbb{B}$

(ii) $\bar{G}_0 = (\sigma_0^{-1})'' G_0$ is $\bar{\mathbb{A}}$-generic over $\bar{N}$

(iii) $\bar{s} \in \bar{N}$, $\bar{b} \in \bar{\mathbb{B}}$ and $\bar{b}/\bar{G}_0 \neq 0$.

Then there is $b \in \mathbb{B}$ s.t. $a = h_{\mathbb{A}}(b)$ and whenever $G \ni b$ is $\mathbb{B}$-generic, $G_0 = G \cap \mathbb{A}$, $\bar{s} = \dot{s}^{G_0}$; $\bar{b} = \dot{b}^{G_0}$, and $\sigma_0 = \dot{\sigma}_0^{G_0}$, then there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$ and $\sigma(\bar{\theta}, \bar{\mathbb{A}}, \bar{\mathbb{B}}) = \theta, \mathbb{A}, \mathbb{B}$

(b). $C_{\delta}^{N}(\operatorname{rng}\sigma) = C_{\delta}^{N}(\operatorname{rng}\sigma_0)$, where $\delta = \delta(\mathbb{B})$

(c) $\bar{G} = (\sigma^{-1})'' G$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$

(d) $\bar{b} \in \bar{G}$ and $\bar{G}_0 \subset \bar{G}$

(e) $\sigma(\bar{s}) = \sigma_0(\bar{s})$.

The proof is exactly like that of Thm 1.

By an *iteration* we mean a sequence $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ s.t. $\mathbb{B}_0 = 2$; $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \le j$ (i.e. the complete BA $\mathbb{B}_i$ is completely included in the complete BA $\mathbb{B}_j$); and whenever $\lambda < \alpha$ is a limit ordinal, then $\mathbb{B}_\lambda$ is completely generated by $\bigcup_{i<\lambda} \mathbb{B}_i$. We may also allow indices of the form $[\beta, \alpha)$ instead of an ordinal. (We then require that $\mathbb{B}_\lambda$ be completely generated by $\bigcup_{\beta \le i < \lambda} \mathbb{B}_i$ whenever λ is a limit point of $[\beta, \alpha)$.) ~~At~~ If $\langle \mathbb{B}_i \mid i < \alpha \rangle$ is an iteration, $\nu < \alpha$ and G is $\mathbb{B}_\nu$-generic, then $\mathbb{B}/G$ is the iteration $\langle \mathbb{B}_i/G \mid \nu \le i < \alpha \rangle$ in $V[G]$. By a *thread* in $\langle \mathbb{B}_i \mid i < \alpha \rangle$ we mean $\langle b_i \mid i < \eta \rangle$ for some $\eta < \alpha$ s.t. $b_0 = 1$, $b_i \in \mathbb{B}_i$, and $b_i = h_{\mathbb{B}_i}(b_j)$ for $i \le j < \eta$. Now let η be a limit point we call a thread $\langle b_i \mid i < \eta \rangle$ *good* iff there is $i < \eta$ s.t either $b_j = b_i$ for $i \le j < \eta$, or else $b_i \Vdash_{\mathbb{B}_i} \mathrm{cf}(\check{\eta}) = \omega$. Following Donder, we call $\mathbb{B}$ a *revised countable support iteration* iff the following holds:

If λ is a limit and $\langle b_i \mid i<\lambda \rangle$ is a good thread; then $\bigcap_i b_i \neq 0$ in $\mathbb{B}_\lambda$; moreover, the set of all such $\bigcap_i b_i$ is dense in $\mathbb{B}_\lambda$.

The salient properties of RSC iterations are listed in [IT] §1.

The main iteration theorem for subproper forcing reads:

<u>Thm 3</u> Let $\mathbb{B} = \langle \mathbb{B}_i \mid i<\alpha \rangle$ be an RCS-iteration s.t. for all $i+1<\alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1}/\mathring{G}$ is subproper)

(c) $\Vdash_{i+1} (\delta(\check{\mathbb{B}}_i)$ has cardinality $\leq \omega_1$)

Then every $\mathbb{B}_i$ is subproper.

proof.

Set: $\delta_i = \delta(\mathbb{B}_i)$. Then

(1) $\delta_i \leq \delta_j$ for $i \leq j < \alpha$, since if X is dense in $\mathbb{B}_j$, then $\{h_i(a) \mid a \in X\}$ is dense in $\mathbb{B}_i$. (Here we write h_i for $h_{\mathbb{B}_i}$.)

(2) $\bar{\bar{\nu}} \leq \delta_\nu$ for $\nu < \alpha$

proof of (2). Suppose not.
Let ν be the least counterexample. Then $\nu > 0$ is a cardinal. If $\nu < \omega$, then $\delta_\nu < \omega$ and hence $\mathbb{B}_\nu$ is atomic with $\delta_\nu =$ the number of atoms. Let $\nu = n+1$. Then $\delta_n < \delta_\nu < n+1$ by (a). Hence $\delta_n < n$. Contradiction! Hence $\nu \geq \omega$ is a cardinal. If ν is a limit cardinal, then $\delta_\nu \geq \sup_{i<\nu} \delta_i \geq \nu$. Contradiction! Thus ν is a successor cardinal. Let $X \in \mathbb{B}_\nu$ be dense in $\mathbb{B}_\nu$ with $\overline{\overline{X}} = \delta_\nu < \nu$. Then $X \subset \mathbb{B}_\eta$ for an $\eta < \nu$ by the regularity of ν. Hence $\mathbb{B}_\eta = \mathbb{B}_\nu$, contradicting (a). QED (2)

By induction on $i < \alpha$ we prove:

<u>Claim</u> Let G_h be $\mathbb{B}_h$ - generic, where $h \leq i$. Then $\mathbb{B}_i / G_h$ is subproper in $V[G_h]$.

(Hence $\mathbb{B}_i \simeq \mathbb{B}_i / \{1\}$ is subproper in V, taking $h = 0$, $\mathbb{B}_h = 2$.)

The case $h = i$ is trivial, since then $\mathbb{B}_i / G_h \simeq 2$. Hence $i = 0$ is trivial.

Now let $i = j+1$.

Then $\mathbb{B}_i/G_h \subset \mathbb{B}_i/G_h$. Let $\tilde{G}$ be $\mathbb{B}_i/G_h$-generic over $V[G_h]$. Then $G' = G_h * \tilde{G} =_{\mathrm{Df}} \{b \in \mathbb{B}_j \mid b/G_h \in \tilde{G}\}$ is $\mathbb{B}_i$-generic over V. But then $(\mathbb{B}_i/G_h)/\tilde{G} \simeq \mathbb{B}_i/G'$ is subproper in $V[G'] = V[G_h][\tilde{G}]$ by (b). Thus we have shown:

$$\Vdash_{\mathbb{B}_i/G_h} (\check{\mathbb{B}}_i/G_h)/\mathring{G} \text{ is subproper.}$$

But $\mathbb{B}_j/G_h$ is subproper in $V[G_h]$ by the induction hypothesis, so it follows by the two step theorem that $\mathbb{B}_i/G_h$ is subproper in $V[G_h]$.

. . . . . .

There remains the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis $\mathbb{B}_i/G_h$ is subproper in $V[G_h]$ for $h \leq i < \lambda$.

<u>Case 1</u> $cf(\lambda) \le \delta_i$ for an $i < \lambda$.

Then $cf(\lambda) \le \omega_1$ in $V[G]$ for $i < h < \lambda$, whenever G_h is $\mathbb{B}_h$-generic. It suffices to prove the claim for such h, since if $h \le i < j$ and G_h is $\mathbb{B}_h$-generic, we can use the two step theorem to show - exactly as in the successor case - that $\mathbb{B}_\lambda / G_h$ is subproper in $V[G_h]$

But then it suffices to prove:

<u>Claim</u> Assume $cf(\lambda) \le \omega_1$ in V, Then $\mathbb{B}_\lambda$ is subproper,

since the same proof can then be carried out in $V[G_h]$ to show that $\mathbb{B}_\lambda / G_h$ is subproper.

(<u>Note</u> To do this we need, of course, to know that $\mathbb{B}/G_h$ satisfies the induction hypothesis in $V[G_h]$ - i.e if $h \le i \le j < \lambda$ and $\tilde{G}$ is $\mathbb{B}_i / G_h$ - generic over $V[G_h]$, then $(\mathbb{B}_j / G_h) / \tilde{G}$ is subproper in $V[G_h][\tilde{G}]$. This is clear, however, since, setting $G = G_h * \tilde{G} =_{\mathrm{df}} \{b \in \mathbb{B}_i \mid b/G_h \in \tilde{G}\}$, we have: G is $\mathbb{B}_i$ - generic over V and $\mathbb{B}_i/G \simeq (\mathbb{B}_i/G_h)/\tilde{G}$ is subproper in $V[G] = V[G_h][\tilde{G}]$. |

Now let θ be big enough that:

$\Vdash_{\mathbb{B}_i} \check{\theta}$ verifies the subproperness of $\check{\mathbb{B}}_{i+1}/\dot{G}$

for $i < \lambda$, $\dot{G}$ being the canonical generic name.

Let $N = L_\tau^A$ be a ZFC - model s.t. $H_\theta \subset N$, $\theta < \tau$.

Let $\pi : \bar{N} \prec N$, where $\bar{N}$ is countable and full.

Let $\pi(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$. Let $\bar{b} \in \bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$.

Claim There is $b \in \mathbb{B}_\lambda$ s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$ - generic, there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $C_\delta^N(\mathrm{rng}\,\sigma) = C_\delta^N(\mathrm{rng}\,\pi)$, where $\delta = \sup_{i<\lambda} \delta(\mathbb{B}_i)$. (Hence $\delta \le \delta(\mathbb{B}_\lambda)$.)

(d) $\bar{G} = (\sigma^{-1})''G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{N}$ and $\bar{b} \in \bar{G}$.

Let f be N-least s.t. $f : \omega_1 \to \lambda$ cofinally. Then $f = \pi(\bar{f})$, where $\bar{f}$ has the corresponding definition in $\bar{N}$.

Let $\langle \bar{\xi}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\lambda}$ s.t. $\bar{\xi}_0 = 0$. Set: $\xi_i = \pi(\bar{\xi}_i)$.

It follows easily that for any $\sigma : \bar{N} \prec N$ with $\sigma(\bar{\lambda}) = \lambda$ we have: $\sigma(\bar{f}) = f$, hence $\sigma(\bar{\xi}_i) = \xi_i$ for $i < \omega$.

Set: $\tilde{\lambda} =_{\text{df}} \sup \pi{}''\bar{\lambda} = \sup\limits_{i<\omega} \xi_i$.

In order that (d) of the above Claim holds, we first construct a "master sequence" $\langle \bar{b}_i \mid i < \omega \rangle$: Call an ultrafilter $\bar{G}$ on $\bar{\mathbb{B}}_\lambda$ good iff whenever $b \in \bar{\mathbb{B}}_\lambda$ s.t. $b = \bigcap\limits_{\nu<\lambda} h_\nu(b)$ and $h_\nu(b) \in \bar{G}$ for $\nu < \lambda$, then $b \in \bar{G}$. a <u>master sequence</u> is a sequence $\langle \bar{b}_i \mid i < \omega \rangle$ s.t.

(a) $\bar{b}_i \in \bar{\mathbb{B}}_{\bar{\xi}_i} \setminus \{0\}$; $\bar{b}_i = h_{\bar{\xi}_i}(\bar{b}_j)$ for $i \leq j < \omega$.

(b) If $\bar{G}$ is a good ultrafilter on $\bar{\mathbb{B}}_{\bar{\lambda}}$ s.t. $\bar{b}_i \in \bar{G}_i =_{df} \bar{G} \cap \bar{\mathbb{B}}_{\bar{\xi}_i}$ for $i < \omega$, then $\bar{G}$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{N}$. Moreover, $\bar{b} \in \bar{G}$

We define this as follows: Let Δ be the set of $b \in \bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$ s.t. $b = \bigcap_{\nu < \bar{\lambda}} h_\nu(b)$. Then Δ is dense in $\bar{\mathbb{B}}_{\bar{\lambda}}$, since we are doing an RSC - iteration. Assume w.l.o.g. that $\bar{b} \in \Delta$. Let $\langle \Delta_i \mid i < \omega \rangle$ enumerate the $\Delta_i \subset \Delta$ s.t. $\Delta_i \in \bar{N}$ and Δ_i is dense in Δ. We successively pick $c_j \in \Delta_j$ s.t. $h_{\bar{\xi}_i}(c_j) = h_{\bar{\xi}_i}(c_i)$ for $i \leq j$. Set $c_0 = \bar{b}$, and set: $\bar{b}_j = h_{\bar{\xi}_j}(c_j)$. Given c_i, let X_i be a maximal antichain in $\{h_{\bar{\xi}_i}(d) \mid d \subset c_i \wedge d \in \Delta_i\}$. Then $\bigcup X_i = b_i = h_{\bar{\xi}_i}(c_i)$. For $a \in X_i$ pick $d_a \subset c_i$ s.t. $d_a \in \Delta_i$, $h_{\bar{\xi}_i}(d_a) = a$. Set: $c_{i+1} = \bigcup_{a \in X_i} d_a$. Then $h_{\bar{\xi}_i}(c_{i+1}) = \bigcup X_i = \bar{b}_i$

This completes the construction. If $G \subset \bar{\mathbb{B}}_{\bar{\lambda}}$ is a good ultrafilter and $G_i = G \cap \bar{\mathbb{B}}_{\bar{\xi}_i}$ is $\bar{\mathbb{B}}_{\bar{\xi}_i}$ - generic over $\bar{N}_i$ with $\bar{b}_i \in \bar{G}_i$ for $i \leq \omega$, then there is $a \in X_i$ s.t. $a \in \bar{G}_i$.

But then $a \cap c_{i+1} \subset d_a \in \Delta_i$. But $b_l \subset h_{\bar{\xi}_l}(c_{i+1})$ for $l > i$. Hence

$$c_{i+1} = \bigcap_{l>i} h_{\bar{\xi}_i}(c_{i+1}) \in G. \text{ QED}$$

Using Lemma 2 we successively construct $b_i \in \mathbb{B}_i \setminus \{0\}$, $\dot{\sigma}_i \in V^{\mathbb{B}_i}$ s.t.

I $b_i = h_{\xi_i}(b_j)$ for $i \leq j < \omega$

II Let $G \ni b_i$ be $\mathbb{B}_i$-generic. Set:

$G_l = G \cap \mathbb{B}_{\xi_l}$, $\sigma_l = \dot{\sigma}_l^{G_l}$, $\bar{G}_l = (\sigma_l)^{-1}{}''G_l$ $(l \leq i)$

Let $\langle x_i \mid i < \omega \rangle$ be a fixed enumeration of $\bar{N}$. Then:

(a) $\sigma_i : \bar{N} \prec N$

(b) $\sigma_i(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $C^N_{\delta_i}(\text{rng}\, \sigma_i) = C^N_{\delta_i}(\text{rng}\, \pi)$ $(\delta_i = \delta(\mathbb{B}_i))$

(d) $\bar{G}_i \ni \bar{b}_i$ is $\bar{\mathbb{B}}_{\bar{\xi}_i}$-generic over $\bar{N}$ and $\bar{G}_j \subset \bar{G}_i$ for $j \leq i$ (hence $\sigma_i{}''\bar{G}_j \subset G_j$)

(e) $\sigma_i(x_j, w_j) = \sigma_j(x_j, w_j)$ for $j \leq i$, where w_j = the $\bar{N}$-least w s.t. $\bar{\bar{w}} = \bar{\delta}$ in $\bar{N}$ and $\pi(x_j) \in \sigma_j(w)$, where $\bar{\delta}_j = \delta(\bar{\mathbb{B}}_{\bar{\xi}_j})$, $\delta_j = \delta(\mathbb{B}_{\xi_j})$

We construct b_i, $\dot{\sigma}_i$ as follows:

Case 1 $i=0$. $b_0 = 1$, $\dot{\sigma}_0 = \check{\pi}$

Case 2 $i=j+1$. By Lemma 2 there is $b \in \mathbb{B}_{\xi_i}$, $\dot{\sigma} \in V^{\mathbb{B}_{\xi_i}}$ s.t. if $G \ni b$ is $\mathbb{B}_{\xi_i}$-generic, $G_j = G \cap \mathbb{B}_{\xi_j}$, $\sigma = \dot{\sigma}^G$, $\sigma_j = \dot{\sigma}_j^{G_j}$, Then

(i) $\sigma : \bar{N} \prec N$ and $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{\imath}) = \theta, \mathbb{B}, \lambda, \imath$

(ii) $C^N_{\delta_i}(\operatorname{rng} \sigma) = C^N_{\delta_i}(\operatorname{rng} \sigma_j)$

(iii) $\bar{G} \ni \bar{b}_i$ is $\bar{\mathbb{B}}_{\bar{\xi}_i}$-generic over $\bar{N}$ and $\bar{G}_j \subset \bar{G}$

(since $h_{\bar{\xi}_j}(\bar{b}_i) = \bar{b}_j \in \bar{G}_j$)

(iv) $\sigma(x_l, w_l) = \sigma_j(x_l, w_l)$ for $l \leq j$.

(since $x_l = \check{x}_l^{G_j}$, $w_l = \dot{w}_l^{G_j}$ for a $\dot{w}_l \in V^{\mathbb{B}_{\xi_j}}$ for $l \leq j$.)

set: $\dot{\sigma}_i = \dot{\sigma}$, $b_i = b$.

The verifications are trivial.

Now set: $b = \bigcap_{i<\omega} b_i$. Then $b \in \mathbb{B}_{\tilde{\lambda}} \subset \mathbb{B}_\lambda$. Let $G \ni b$ be $\mathbb{B}_\lambda$-generic. Set: $G_i = G \cap \mathbb{B}_{\xi_i}$, $\sigma_i = \dot{\sigma}_i^{G_i}$, $\bar{G}_i = \sigma_i^{-1}{}''G_i$. Then $\bar{G}_i \subset \bar{G}_j$ for $i \leq j$.

Since $\sigma_i(x_i) = \sigma_j(x_i)$ for $j \geq i$ we can define $\sigma : \bar{N} \to N$ by:

$\sigma(x) = \sigma_i(x)$ where $\sigma_j(x) = \sigma_i(x)$ for $j \geq i$.

It follows that:

(a) $\sigma : \bar{N} \prec N$ and $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$.

(b) $C^N_\delta(\operatorname{rng} \sigma) = C^N_\delta(\operatorname{rng} \pi)$,

where $\delta = \sup_i \delta_i \leq \delta(\mathbb{B}_\lambda)$

(c) $\bar{G} = \sigma^{-1}{}''G$ is $\bar{\mathbb{B}}_\lambda$ - generic over $\bar{N}$; $\bar{b} \in \bar{G}$.

(a) is immediate. We prove (b).

($\subset$) $\sigma(x) = \sigma_i(x) \subset C^N_{\delta_i}(\operatorname{rng} \pi) \subset C^N_\delta(\operatorname{rng} \pi)$ for some i. Hence $\operatorname{rng} \sigma \subset C^N_\delta(\operatorname{rng} \pi)$ and $C^N_\delta(\operatorname{rng} \sigma) \subset C^N_\delta(\operatorname{rng} \pi)$.

($\supset$) Let $z \in C^N_\delta(\operatorname{rng} \pi)$. Then $z = \pi(f)(\nu)$ where $\nu < \delta$ and $f \in \bar{N}$, $f : \bar{\delta} \to \bar{N}$. Let Let i: $\pi(f) \in \sigma_i(w_i) = \sigma(w_i)$ Let $g \in \bar{N}$, $g : \delta_i \xrightarrow{\text{onto}} w_i$. Then $\pi(f) = \sigma_i(g)(\xi)$ where $\xi < \delta_i$. Hence $z = (\sigma_i(g)(\xi))(\nu) \in C^N_\delta(\operatorname{rng} \sigma)$.

QED (b)

We now prove (c).

(1) $\bigcup_{i<\omega} \bar{G}_i \subset \bar{G}$, since if $\bar{b} \in \bar{G}_i$, then $\sigma(\bar{b}) = \sigma_j(\bar{b}) \in \sigma_j{}''\bar{G}_j \subset G$ for a $j \geq i$.

Clearly $\bar{b}_i \in \bar{G}$ and $\bar{G}$ is an ultrafilter. Hence $\bar{G}$ is $\mathbb{B}_\lambda$-generic over $\bar{N}$, since $\langle \bar{b}_i \mid i<\omega \rangle$ is a master sequence. Moreover $\bar{b} \in G$, since $\bar{b}_i \subset h_{\bar{\xi}_i}(\bar{b})$.

QED (Case 1)

<u>Case 2</u> Case 1 fails.

In this case our proof will diverge from that given in [IT], both because we find the new proof more conceptual and because it adapts more readily to the rather complex construction which will be used in §2.

λ is regular and $\lambda > \delta_i$ for $i<\lambda$. This implies that $\mathbb{B}_\lambda$ satisfies the λ-chain condition, since at λ we took a direct limit and there are stationarily many $\tau < \lambda$ of cofinality ω_1, at which we also took a direct limit. (Let $X \subset \bigcup_{i<\lambda} \mathbb{B}_i$ be a maximal antichain in $\mathbb{B}_\lambda$. Then there is $\tau < \lambda$ s.t. $\bar{X} = X \cap \bigcup_{i<\tau} \mathbb{B}_i$ is a maximal antichain in $\mathbb{B}_\tau$, hence in $\mathbb{B}_\lambda$. Hence $\bar{X} = X$.)

We again let $N = L^{A_\tau}$ be a ZFC^--model s.t. $H_\theta \subset N$ and $\theta < \tau$.

We again let $\sigma : \bar{N} \prec N$ where $\bar{N}$ is countable and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$. Let $\bar{b} \in \bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$. We claim that there is $b \in \mathbb{B}_\lambda \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$ - generic, then there is $\sigma \in V[G]$ with:

(a) $\sigma : \bar{N} \prec N$

(b) $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $C^N_\delta(\operatorname{rng} \sigma) = C^N_\delta(\operatorname{rng} \pi)$, where

$$\delta = \sup_{i<\lambda} \delta(\mathbb{B}_i)$$

(d) $\bar{G} = \sigma^{-1}{}'' G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{N}$ and $\bar{b} \in \bar{G}$.

Since $\bar{\mathbb{B}}_{\bar{\lambda}}$ satisfies the $\bar{\lambda}$ - chain condition in $\bar{N}$ it follows that an ultrafilter $\bar{G}$ on $\bar{\mathbb{B}}_{\bar{\lambda}}$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{N}$ iff every $\bar{G}_\nu = \bar{G} \cap \bar{\mathbb{B}}_\nu$ $(\nu < \bar{\lambda})$ is $\bar{\mathbb{B}}_\nu$ - generic over $\bar{N}$. Hence we do not need a master sequence.

As before, we let $\langle \bar{\xi}_i \mid i < \omega \rangle$ be monotone and cofinal in $\bar{\lambda}$ s.t. $\bar{\xi}_0 = 0$.

However, we do not have the function $\bar{f}$ used in Case 1. Hence we will not be able to ensure that $\sigma(\bar{\xi}_i) = \pi(\bar{\xi}_i)$ for $\sigma : \bar{N} \prec N$ s.t. $\sigma(\bar{\lambda}) = \lambda$. However, letting $\tilde{\lambda} = \sup \pi'' \bar{\lambda}$, we will be able to choose our σ s.t. $\sup \sigma'' \bar{\lambda} = \tilde{\lambda}$. This will have to suffice.

Let $\langle \eta_i \mid i < \omega \rangle$ be cofinal in $\tilde{\lambda}$ s.t. $\eta_0 = 0$. (If we wished, we could take $\eta_i = \pi(\bar{\xi}_i)$.) We construct $\langle c_i \mid i < \omega \rangle$, $\langle \dot{\pi}_i \mid i < \omega \rangle$ s.t.

(I) (a) $c_i \in \mathbb{B}_{\eta_i}$, $h_{\eta_l}(c_i) = c_l$ for $l \leq i$.

(b) $\dot{\pi}_i \in V^{\mathbb{B}_{\eta_i}}$

(II) Let $G \ni c_i$ be $\mathbb{B}_{\eta_i}$ - generic. Set: $G_\nu = \mathbb{B}_\nu \cap G$ $(\nu \leq \eta_i)$, $\pi_h = \dot{\pi}_h^{G_{\eta_h}}$ $(h \leq i)$.

(a) $\pi_i : \bar{N} \prec N$

(b) $\pi_i(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $\pi_i(\bar{\xi}_l) = \pi_h(\bar{\xi}_l)$ for $h \le i$, $l \le m_h$, where:

$m_h =_{\text{Df}}$ the least m s.t. $\pi_h(\bar{\xi}_m) > \eta_{h+1}$

(d) $C^N_{\tilde{\lambda}_i}(\text{rng } \pi_i) = C^N_{\tilde{\lambda}}(\text{rng } \pi)$

(hence $\tilde{\lambda} = \sup \pi_i{}''\bar{\lambda}$).

Simultaneously we define $\dot{\Gamma}_i \in V^{\mathbb{B}_{\eta_i}}$ s.t.

<u>III</u> Let $G \ni c_i$ be as in II. Set $\Gamma_l = \dot{\Gamma}_l^{G_l}$ ($l \le i$). Then $\Gamma_i = \langle \langle \xi_j, \dot{\sigma}_j, b_j \rangle \mid j \le m_i \rangle$ where:

(a) $\Gamma_l \subset \Gamma_i$ for $l \le i$

(b) $\xi_j = \pi(\bar{\xi}_j)$ for $j \le m_i$

(c) $b_j \in \mathbb{B}_{\xi_j}$, $b_l = h_{\xi_l}(b_j)$ for $l \le j \le m_i$

(d) $\dot{\sigma}_j \in V^{\mathbb{B}_{\xi_j}}$

<u>IV</u> Let G, Γ_i be as in III. Let $G' \supset G$ be $\mathbb{B}_{\xi_{m_i}}$ - generic s.t. $b_{m_i} \in G$. Set:

$G_\nu = G' \cap \mathbb{B}_\nu$ ($\nu \le \xi_{m_i}$), $\sigma_h = \dot{\sigma}_h^{G'}$ ($h \le m_i$).

Then for $j \le m_i$:

(a) $\sigma_j : \bar{N} \prec N$

(b) $\bar{G}_{\bar{\xi}_j} = \sigma_j^{-1}{}''G'_{\xi_j}$ is $\bar{\mathbb{B}}_{\bar{\xi}_j}$ - generic over $\bar{N}$. Moreover, and $\bar{G}_{\bar{\xi}_l} \subset \bar{G}_{\bar{\xi}_j}$ for $l \le j$.

$h_{\bar{\xi}_j}(\bar{b}) \in \bar{G}_{\bar{\xi}_j}$.

(c) $\sigma_i(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{\imath}) = \theta, \mathbb{B}, \lambda, \imath$

(d) $\sigma_i(x_\ell) = \sigma_\ell(x_\ell)$ for $\ell \leq i$ (where $\langle x_\ell \mid \ell < \omega \rangle$ is a fixed enumeration of $\bar{N}$)

(e) $\sigma_i(w_\ell) = \sigma_\ell(w_\ell)$ for $\ell \leq i$, where w_ℓ = the least $w \in \bar{N}$ s.t. $\bar{\bar{w}} \leq \bar{\lambda}$ and $\pi(x_\ell) \in \sigma_\ell(w)$,

(f) $\sigma_i(\bar{\xi}_\ell) = \xi_\ell$ if $\ell \leq m_h$ and $h \leq i$ s.t. $h = 0$ or $m_{h-1} \leq j'$.

(g) Let $j = m_i$. Let h be least s.t. $\xi_i \leq \eta_h$ (hence $h > i+1$). Set m_i^+ = the least m s.t. $\sigma_{i-1}(\bar{\xi}_m) > \eta_h$. Then $\sigma_i(\bar{\xi}_\ell) = \sigma_{i-1}(\bar{\xi}_\ell)$ for $\ell \leq m_i^+$.

(h) $C^N_{\delta_i}(\operatorname{rng} \sigma_i) = C^N_{\delta_i}(\operatorname{rng} \pi)$, where $\delta_i = \delta(\mathbb{B}_{\xi_i})$.

<u>Note</u> If G' is $\mathbb{B}_{\xi_i}$-generic with $b_i \in G'$ and $G \cap \mathbb{B}_{\xi_i} \subset G'$, it follows that (a)–(h) hold, since we can then extend G' to G'' s.t. $G \subset G''$, G'' is $\mathbb{B}_{m_i}$-generic, and $b_{m_j} \in G''$.

We also have:

V Let G be as in II, where $i = k+1$. Then:

(a) $h_{\eta_i}(b_{m_k}) \in G$

(b) $\pi_i = (\dot{\sigma}_{m_k - 1})^G$

Note $\dot{\sigma}_{m_k-1} \in V^{\mathbb{B}_{\eta_i}}$, since $\xi_{m_k-1} \le \eta_i < \xi_{m_k}$.

Moreover, $b_{m_k-1} = h_{\xi_{m_k-1}}(b_{m_k})$. Hence $b_{m_k-1} \in G$, which guarantees that IV (a)-(h) hold at $j = m_k - 1$ with $G' = G \cap \mathbb{B}_{\xi_{m_k-1}}$.

We now construct $c_i, \dot{\sigma}_i, \dot{\Gamma}_i$ and verify I – V by induction on i.

Case 2.1 $i = 0$.

Set $c_0 = 1$, $\dot{\pi}_0 = \check{\pi}$. By repeated use of Lemma 1.1, just as in Case 1, we construct $\Gamma = \langle\langle \xi_i, \dot{\sigma}_i, b_i \rangle \mid i \leq m_0 \rangle$ satisfying III (a) – (d) and IV (a) – (h). Set $\dot{\Gamma}_0 = \check{\Gamma}$. V holds vacuously. The other verifications are trivial.

Case 2.2 $i = k+1$.

We first define c_i.

$\dot{\Gamma}_k$ gives us names $\dot{\xi}, \ddot{\sigma}, \dot{b}, \dot{m}_k$ s.t.

$c_k \Vdash_{\eta_k} (\dot{m}_k < \omega \wedge \dot{\xi}, \ddot{\sigma}, \dot{b}$ are functions on $\dot{m}_k + 1)$,

$c_k \Vdash_{\eta_k} \dot{\Gamma}_k = \langle\langle \dot{\xi}(l), \ddot{\sigma}(l), \dot{b}(l) \rangle \mid l \leq \dot{m}_k \rangle$,

where $c_k \Vdash_{\eta_k} (\dot{\xi}(l) \leq \check{\eta}_i < \dot{\xi}(\dot{m}_k)$ for $l < \dot{m}_k)$.

Let $\dot{c} \in V^{\mathbb{B}_{\eta_k}}$ s.t.

$\Vdash_{\eta_k} (\check{c}_k \in \dot{G} \wedge \dot{c} = h_{\check{\eta}_i}(\dot{b}(\dot{m}_k)/\dot{G}) \vee (\check{c}_k \notin \dot{G} \wedge \dot{c} = 0))$,

$\dot{G}$ being the canonical generic name.

Then $\Vdash_{\eta_k} \dot{c} \in \breve{\mathbb{B}}_{\eta_i}/\dot{G}$.

Set: c_i = the unique $c \in \mathbb{B}_{\gamma_i}$ s.t.
$\Vdash_{\gamma_i} \check{c}/\mathring{G} = \dot{c}$. Then $c_i \in \mathbb{B}_{\gamma_i}$ and
$h_{\gamma_h}(c_i) = [\![\check{c}_i / \mathring{G} \neq 0]\!] = c_k$.
Thus I(a) holds. $\overline{\text{V}}$(a) is immediate.
We now define $\dot{\pi}_i$. We know that
$\dot{\sigma}_{m_h - 1} \in V^{\mathbb{B}_{\gamma_i}}$, where:
$$\dot{\sigma}_{m_h-1} = \ddot{\sigma}^G(m_k - 1), \text{ where } G \ni c_k \text{ is } \mathbb{B}_{\gamma_h}\text{-}$$
-generic *)

(*) Note If $\mathbb{A} \subseteq \mathbb{B}$, we suppose $V^{\mathbb{A}}$, $V^{\mathbb{B}}$ to be so defined that $V^{\mathbb{A}} \subseteq V^{\mathbb{B}}$ — i.e.
If $t \in V^{\mathbb{A}}$, then $t \in V^{\mathbb{B}}$; moreover
$t^{G_0} = t^G$, where G is $\mathbb{B}$-generic and
$G_0 = G \cap \mathbb{A}$.)

But then there is $\dot{\pi} \in V^{\mathbb{B}_{\gamma_i}}$ s.t.
$$c_k \Vdash_{\gamma_i} \dot{\pi} = \ddot{\sigma}^{\dot{G}}(\dot{m}_h - 1)$$
Set $\dot{\pi}_i = \dot{\pi}$, Then
$$\dot{\pi}_i^G = (\ddot{\sigma}^G(m_k - 1))^G = (\dot{\sigma}_{m_h - 1})^G$$
whenever $G \ni c_i$ is $\mathbb{B}_{\gamma_i}$- generic. Hence
$\overline{\text{V}}$(b) holds, I(b) is trivial.

$\underline{II}$ is then straightforward, using that $\underline{IV}$ holds at k.

We now define $\dot{\Gamma}_i$ and verify $\underline{III}$, $\underline{IV}$.

We first assume $G \ni c_i$ to be $\mathbb{B}_{\eta_i}$ - generic and show that there is a $\Gamma_i \in V[G]$ satisfying $\underline{III}$, $\underline{IV}$ (with Γ instead of $\Gamma_i = \dot{\Gamma}_i^G$).

$\Gamma_k = \dot{\Gamma}_k^G = \langle\langle \xi_\ell, \dot{\sigma}_\ell, b_\ell \rangle \mid \ell \leq m_k \rangle$ is given.

For $l \leq m_i$ set: $\xi_\ell = \pi_i(\bar{\xi}_\ell)$. This extends the sequence $\langle \xi_\ell \mid \ell \leq m_k \rangle$ given by Γ_k since $\pi_i(\bar{\xi}_\ell) = \pi_k(\bar{\xi}_\ell)$ for $\ell \leq m_k$.

(Note that $m_k = m_i$ with $\xi_k > \eta_{i+1}$ is possible! In this case we shall have: $\Gamma_k = \Gamma_i$.) Set:

$$a := c_i \wedge [\![\dot{\Gamma}_k = \check{\Gamma}_k \wedge \bigwedge_{\ell \leq m_i} \dot{\pi}_i(\check{\bar{\xi}}_\ell) = \check{\xi}_\ell]\!]$$

Then $a \in G$ where $h_{\eta_i}(b_{m_k}) \in G$. Hence $a \wedge b_{m_k} \neq 0$. By repeated use of Lemma 2, just as in Case 1, we then construct $\bar{b}_\ell^s \in \mathbb{B}_{\xi_\ell}$, $\dot{\sigma}_\ell^s \in V^{\mathbb{B}_{\xi_\ell}}$ $(m_k \leq \ell \leq m_i)$ s.t.

$\bar{b}_{m_k} = a \wedge b_{m_k}$, $\bar{b}_\ell = h_{\eta'_\ell}(\bar{b}_{\ell+1})$ for $m_k \le \ell < m_i$, and whenever $G' \ni \bar{b}_{m_i}$ in $\mathbb{B}_{\xi_{m_i}}$ - generic with $G \subset G'$ and $\sigma_\ell = \dot{\sigma}_\ell^{G'}$ for $\ell \le m_i$, then IV (a)-(h) are satisfied.

Finally set: $b_\ell = \bar{b}_\ell \cup (b_{m_k} \setminus a)$ $(m_k \le \ell \le m_i)$.

The sequence $\Gamma = \langle\langle \xi_\ell, \dot{\sigma}_\ell, b_\ell\rangle \mid \ell \le m_i\rangle$ clearly extends Γ_k and satisfies III, IV (with Γ in place of $\Gamma_i = \dot{\Gamma}_i^G$).

Note, however, that Γ really depends only on a, rather than G. Following this up, we consider the set S of sequences $s = \langle s_0, \xi_0, \dots, \xi_n\rangle$ s.t.

$$\xi_0 < \dots < \xi_{n-1} \le \eta_{i+1} < \xi_n \text{ and } a_s \ne 0,$$

where $a_s = c_i \cap [\![\dot{\Gamma}_k = \check{s}_0 \wedge \bigwedge_{\ell \le n} \dot{\pi}_i(\check{\xi}_\ell) = \check{\xi}_\ell]\!]_{\eta_i}$.

Note that $a_s \cap a_{s'} = 0$ if $s \ne s'$.

For $s = \langle s_0, \xi_0, \dots, \xi_n\rangle \in S$ we then define $\Gamma^s = \langle\langle \xi_\ell, \dot{\sigma}_\ell, b_\ell\rangle \mid \ell \le n\rangle$

exactly as before, with $\dot{\sigma}^s_\ell, b^s_\ell$ $(\ell \le m^s_k)$ given by $\Gamma^s_k =_{\text{Df}} s_0$, and $\dot{\sigma}^s_\ell, b^s_\ell$ $(m^s_k \le \ell \le m)$ defined as before, with $\bar{b}_{m_k} = a_s \wedge b_{m^s_k}$, $\bar{b}_\ell$ $(m^s_k \le \ell \le m)$ as before and $b_\ell = \bar{b}_s \vee (b_{m^s_k} \setminus a_s)$ $(m^s_k \le \ell \le m)$.

If $G \ni a_s$ is $\mathbb{B}_{\eta_i}$-generic, it follows as before that $\Gamma^s_k = \Gamma_k = \dot{\Gamma}^G_k$ and Γ^s satisfies III, IV (with Γ^s in place of $\Gamma_i = \dot{\Gamma}^G_i$).

But $c_i = \bigcup_{s \in S} a_s$. Hence there is $\dot{\Gamma}_i \in V^{\mathbb{B}}$ s.t. $c_i = \bigcup_{s \in S} [\![\dot{\Gamma}_i = \check{\Gamma}^s]\!]_{\eta_i}$.

Γ^i clearly satisfies III, IV.

This completes the construction.

We now complete the proof of (*).

Let $c = \bigcap_{i<\omega} c_i$. Then $c \in \mathbb{B}_{\tilde\lambda} \subset \mathbb{B}_\lambda$.

Let $G \ni c$ be $\mathbb{B}_\lambda$-generic. Set:

$G_\nu = G \cap \mathbb{B}_\nu$ for $\nu < \tilde\lambda$. Set $\Gamma_i = \dot\Gamma^{G_{\eta_i}}$.

Then $\Gamma_i \subset \Gamma_j$ for $i \leq j < \omega$. Set:

$$\Gamma = \bigcup_i \Gamma_i = \langle\langle \xi_j, \dot\sigma_j, b_j\rangle \mid j < \omega\rangle.$$

(It is clear that $\sup_i m_i = \omega$, since if $k > i$ s.t. $\eta_k < \pi_i(\bar\xi_{m_i}) < \eta_{k+1}$, then $\pi_k(\bar\xi_{m_i}) = \pi_i(\bar\xi_{m_i})$ and hence $m_k > m_i$.)

If $k > i$ s.t. $m_k > m_i$, then $G_{\eta_k} \supset G_{\eta_i}$ is $\mathbb{B}_{\eta_k}$-generic with $b_{m_i} \in G_{\eta_k}$, since $b_{m_i} \subset h_{\eta_k}(b_{m_k}) \in G_{\eta_k}$. Hence $b_j \in G$ for all $j < \omega$. Setting $\sigma_i = \dot\sigma_i^G = \dot\sigma_i^{G_{\xi_i}}$ for $i < \omega$, we see that III (a)-(d), IV (a)-(h) hold (with $G = G'$ in IV). The rest of the proof is just like Case 1.

QED (Thm 3)

We note for later reference that the proof of Thm 3 shows slightly more than was stated.

Def For limit $\lambda \leq \alpha$ set:

$$\mathbb{B}_{<\lambda} = \bigcup_{i<\lambda} \mathbb{B}_i,$$

$\mathbb{B}^+_{<\lambda}$ = the set of $b \in \mathbb{B}_\lambda$ s.t.

$$b = \bigcap_{i<\lambda} h_i(b).$$

Then $\mathbb{B}^+_\lambda$ is dense in $\mathbb{B}_\lambda$, since $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ is an RSC iteration.

Def Let $G \subset \mathbb{B}_{<\lambda}$.

G is $\mathbb{B}_{<\lambda}$-generic over V iff

$G_i = G \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic for $i < \lambda$.

G is $\mathbb{B}^+_{<\lambda}$-generic over V iff the set:

$$G^+ = \{ b \in \mathbb{B}^+_{<\lambda} \mid h_i(b) \in G \text{ for } i < \lambda \}$$

meets every strongly dense set in $\mathbb{B}_\lambda$. (In other words,

$$G^{++} = \{ b \in \mathbb{B}_\lambda \mid \bigvee a \in G^+ \; a \subset b \}$$

is fully $\mathbb{B}_\lambda$-generic.)

<u>Lemma 3.1</u> Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RCS iteration satisfying (a)–(c) in the statement of Thm 3. Let $\lambda < \alpha$ be a limit ordinal s.t. $\mathrm{cf}(\lambda) \leq \omega_1$ or λ is regular in V with $\lambda > \delta(\mathbb{B}_i)$ for $i < \lambda$. Let θ be big enough that:

$\Vdash_{\mathbb{B}_i} (\check{\theta}$ verifies the subproperness of $\check{\mathbb{B}}_{i+1}/\dot{G})$ for $i < \alpha$

Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$, $\theta < \tau$.

Let $\pi : \overline{N} \prec N$, where $\overline{N}$ is countable and full.

Let $\pi(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{s}) = \theta, \mathbb{B}, \lambda, s$. Let $\tilde{\lambda} = \sup \pi``\overline{\lambda}$.

Let $\overline{b} \in (\overline{\mathbb{B}}^+_{<\overline{\lambda}}) \setminus \{0\}$. There is $b \in \mathbb{B}^+_{<\tilde{\lambda}}$ s.t. whenever $G \subset \mathbb{B}_{<\tilde{\lambda}}$ is $\mathbb{B}_{<\tilde{\lambda}}$-generic and $h_i(b) \in G$ for $i < \tilde{\lambda}$, then there is $\sigma \in V[G]$ s.t.

(a) $\sigma : \overline{N} \prec N$

(b) $\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $C_\delta^N(\mathrm{rng}\,\sigma) = C_\delta^N(\mathrm{rng}\,\pi)$,

where $\delta = \sup\{\delta(\mathbb{B}_i) \mid i < \tilde{\lambda}\}$

(d) $\overline{G} = (\sigma^{-1})``G$ is $(\overline{\mathbb{B}}^+_{<\overline{\lambda}})$-generic and $\overline{b} \in \overline{G}^+$

Moreover, $\tilde{\lambda} = \sup \sigma``\overline{\lambda}$.