§2 α-subproper forcing

α-subproper forcing generalizes the notion of α-proper forcing, invented by Shelah and lucidly exposited by Avraham in [PF]. In this forcing we deal not with a single embedding $\sigma : \bar{N} \prec N$ from a countable to an uncountable structure, but rather with an entire tower of embeddings.

Def Let $N = L_\tau^A$ be a ZFC⁻ model. Let $\Gamma = [\alpha, \beta]$ where $\alpha \leq \beta < \omega_1$. By an Γ-pretower for N we mean a

$\pi = \langle \pi^i \mid i \in \Gamma \rangle$ s.t.

- $\pi^i : N^i \prec N$ where N^i is countable and full
- $\text{rng}(\pi^i) \subset \text{rng}(\pi^j)$ and $\omega_1^{N^i} < \omega_1^{N^j}$ for $i < j$
- If λ is a limit point of Γ, then $\text{rng}(\pi^\lambda) = \bigcup_{\nu \in \Gamma \cap \lambda} \text{rng}(\pi^\nu)$.

We also set: $\pi^{ij} = (\pi^j)^{-1} \circ \pi^i$ for $i \leq j$, $i, j \in \Gamma$.

Since N^i is determined by N, π^i, we also denote it by N^{π^i}. We write: $\pi^{ij} = (\pi^j)^{-1} \circ \pi^i$.

By an α-pretower we mean a $[0,\alpha]$-pretower $\langle \pi^i \mid i \leq \alpha \rangle$. We shall generally state our definitions for α-towers, leaving it to the reader to work out the $[\alpha,\beta]$-version.

Def An α-pretower $\pi = \langle \pi^i \mid i \leq \alpha \rangle$ for N is a tower iff $\pi \in N$ and $\langle \pi^{h,i+1} \mid h \leq i \rangle \in N^{i+1}$ for $i < \alpha$.

Note If π is an α-tower, then $\pi^{i+1}(\pi^{h,i+1}) = \pi^h$ for $h \leq i < \alpha$, since $(\pi^{i+1}(\pi^{h,i+1}))(x) =$
$= (\pi^{i+1}(\pi^{h,i+1}))(\pi^{i+1}(x)) = \pi^{i+1}(\pi^{h,i+1}(x)) = \pi^h(x)$

Hence, if $\zeta \in N^h$, we have:

$$\pi^{i+1}(\sup \pi^{h,i+1}``\zeta) = \sup \pi^h``\zeta$$

Def Let π, σ be α-pretowers. σ is a revision of π iff

- $N^{\pi^i} = N^{\sigma^i}$ for $i \leq \alpha$
- $\bigcup \operatorname{rng} \pi^\alpha = \bigcup \operatorname{rng} \sigma^\alpha$
- $\bigcup \operatorname{rng} \pi^{i,i+1} = \bigcup \operatorname{rng} \sigma^{i,i+1}$ for $i < \alpha$.

Def Let π be an α-pretower for N. Let $x \in N$. π absorbs x iff $(\pi^i)^{-1}(x)$ exists for all $i \leq \alpha$.

Note If π absorbs x, we often write:
x^i for $(\pi^i)^{-1}(x)$.

Def Let σ be a revision of π. σ respects π at x iff σ absorbs x and $(\sigma^{i+1})^{-1}(x) = (\pi^i)^{-1}(x)$ for $i \leq \alpha$.

Thus x^i has the same meaning for σ, π if σ respects π at x.
We also say: σ is an x-revision of π to mean that σ respects π at x.

Def Let σ be a revision of π. Let $x \in N^\alpha$ (where π is an α-pretower). σ coincides with π at x iff

- $\sigma^\alpha(x) = \pi^\alpha(x)$
- Whenever $i < \alpha$ and $\pi^{i,\alpha}(\bar{x}) = x$, then $\sigma^{i,\alpha}(\bar{x}) = x$.

Def σ coincides with π on $u \subset N^\alpha$ iff σ coincides with π at each $x \in u$.
(In other words, $\pi^\alpha \upharpoonright u = \sigma^\alpha \upharpoonright u$ and $(\pi^i)^{-1} \upharpoonright u \subset (\sigma^i)^{-1} \upharpoonright u$ for $i < \alpha$.)

Def Let $\mathbb{B} \in N$ be a complete BA in N.
σ is an $x, \mathbb{B}$-revision of π iff

- σ is an x-revision of π and absorbs $\mathbb{B}$
- Af σ respects π at $\mathbb{B}$: and $\delta = \delta(\mathbb{B})$,
Then: $C^N_\delta(\text{rng}\,\sigma^\alpha) = C^N_\delta(\text{rng}\,\pi^\alpha)$ and
$C^{N^{i+1}}_{\delta^{i+1}}(\text{rng}\,\sigma^{i,i+1}) = C^{N^{i+1}}_{\delta^{i+1}}(\text{rng}\,\pi^{i,i+1})$
for all $i < \alpha$.

(Hence, being an $x, \mathbb{B}$-revision says little if σ does not respect π at $\mathbb{B}$.)

Def Let σ be an $x, \mathbb{B}$-revision of π. Let G be $\mathbb{B}$-generic over N. We say that x is an $x, \mathbb{B}, G$-revision of π iff

- $G^i =_{\text{Df}} (\sigma^i)^{-1}{}''G$ is $\mathbb{B}^i$-generic over N^i for $i \leq \alpha$, where $\mathbb{B}^i = (\sigma^i)^{-1}(\mathbb{B})$.
- $\sigma \in N[G]$
- $\langle \sigma^{h,i+1} \mid h \leq i \rangle \in N^{i+1}[G^{i+1}]$ for $i < \alpha$.

(Thus, being an $x, \mathbb{B}, G$-revision is also a much stronger statement if σ respects π at $\mathbb{B}$.)

Def Let σ be an $x, \mathbb{B}, G$ - revision of π.
The canonical completion $\tilde{\sigma} = \langle \tilde{\sigma}^i \mid i \leq \alpha \rangle$
of σ is defined by:
$\sigma^i : N^i[G^i] \prec N[G]$; $\sigma^i(G^i) = G$
for $i \leq \alpha$.
Then, setting $N^G = L_\tau^{A,G}$ where $N = L_\tau^A$,
we have: $\tilde{\sigma}$ is a tower for N^G.
Moreover $(N^G)^{\tilde{\sigma}^i} = L_{\tau_i}^{A^i, G^i}$, where
$N^i = L_{\tau_i}^{A^i}$.

This fact has many consequences for σ - e.g. $\sigma^{i+1}(\sup \sigma^{h,i+1} {}'' \xi) = \sup \sigma^h {}'' \xi$
for $\xi \in N^h$, $h \leq i < \alpha$.

Def Let $\mathbb{A}, \mathbb{B}$ be complete BA's in N with $\mathbb{A} \subseteq \mathbb{B}$. Let σ be an $x, \mathbb{A}, A$ - revision of π and σ' an $x, \mathbb{B}, B$ - revision of π. σ' coheres with σ wrt $\mathbb{A}$ iff
- σ' is an $\langle x, \mathbb{A} \rangle, \mathbb{B}, B$ - revision of σ'
- $A = B \cap \mathbb{A}$
- $A^i = B^i \cap \mathbb{A}^i$ for $i \leq \alpha$, where $A^i = (\sigma^i)^{-1} {}'' A$ and $B^i = (\sigma'^i)^{-1} {}'' B$ (and $\mathbb{A}^i = (\sigma^i)^{-1}(\mathbb{A})$).

We are now ready to define the concept of α-subproperness:

<u>Def</u> Let $\mathbb{B}$ be a complete BA. Let $\alpha < \omega_1$. $\mathbb{B}$ is α-<u>subproper</u> <u>as</u> <u>witnessed</u> <u>by</u> the cardinal $\theta > \omega_1$ iff $\mathbb{B} \in H_\theta$ and the following holds:

Let $N = L_\tau^A$ be a ZFC^- model, where $H_\theta \subset N$ and $\theta < \tau$. Let $\beta \le \alpha$ and let π be a β-tower for N which absorbs $\theta, \mathbb{B}$. Let $u \subset N^\beta$ be finite. Let $\bar{b} \in \mathbb{B}^0 \setminus \{0\}$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, then there is $\sigma \in V[G]$ s.t. σ is a $\{\theta, \mathbb{B}\}, \mathbb{B}, G$-revision of π coinciding with π on u and s.t. $\bar{b} \in G^0$.

<u>Def</u> $\mathbb{B}$ is α-subproper iff it is α-subproper as verified by some θ.

<u>Def</u> $\mathbb{B}$ is ω_1-subproper iff it is α-subproper for all $\alpha < \omega_1$.

<u>Note</u> Clearly $\mathbb{B}$ is subproper iff it is 0-subproper.

Def θ verifies the α-subproperness of $\mathbb{B}$ iff every $\theta' \geq \theta$ witnesses the α-subproperness of $\mathbb{B}$.

Just as before, we can relativize the notion of α-subproperness to a fixed parameter p: We obtain the notion "$\mathbb{B}$ is α-subproper as witnessed by $\langle \theta, p \rangle$" by altering the above definition to require that $p \in H_\theta$, π absorbs p, and σ is a $\{p, \theta, \mathbb{B}\}$, $\mathbb{B}, G$-revision. It is again easily seen that this apparently weaker notion implies full α-subproperness — a fact that we shall often employ tacitly.

The two step iteration theorem reads:

<u>Thm 1</u> Let $\mathbb{A} \subseteq \mathbb{B}$ where $\mathbb{A}$ is α-subproper and $\Vdash_{\mathbb{A}} (\check{\mathbb{B}}/\dot{G}$ is α-subproper$)$. Then $\mathbb{B}$ is α-subproper.

proof

Let θ be big enough that it verifies the α-subproperness of $\mathbb{A}$ and $\Vdash_{\mathbb{A}} (\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\dot{G})$

Let $N = L^A_\tau$ be a ZFC^- model with $H_\theta \subset N$, $\theta < \tau$,

Let $\pi = \langle \pi_i \mid i \leq \alpha \rangle$ be an α-tower with $\pi^i : N^i \prec N$, $\pi^i(\theta^i, \mathbb{A}^i, \mathbb{B}^i) = \theta, \mathbb{A}, \mathbb{B}$ for $i \leq \alpha$.

Let $u \subset N^\alpha$ be finite. Let $e \in \mathbb{B}^0 \setminus \{0\}$.

Then there is $a \in \mathbb{A} \setminus \{0\}$ s.t. if $A \ni a$ is $\mathbb{A}$-generic, then there is $\sigma \in V[A]$ which is a $\{\theta, \mathbb{A}, \mathbb{B}\}, \mathbb{A}, A$ - revision of π coinciding with π on u and s.t.: $h_{\mathbb{A}^0}(e) \in A$. Let A, σ be given.

Let $\tilde{\sigma}$ be the canonical completion of σ, Set $\tilde{N} = L^{D,A}_\tau$, where $N = L^D_\tau$ and $\tilde{N}^i = L^{D^i, A^i}_{\tau^i}$ where $N^i = L^{D^i}_{\tau^i}$. Then $\tilde{\sigma}$ is a tower for $\tilde{N}$ with $\tilde{N}^i = \tilde{N}^{\tilde{\sigma}^i}$. Moreover, $\tilde{\sigma}$ absorbs $\theta, \mathbb{A}, \mathbb{B}, G$. Set: $\tilde{\mathbb{B}} = \mathbb{B}/A$. $\tilde{\mathbb{B}}$ is α-subproper in $V[A]$. Set:

$\tilde{e} = e/A$. Then $\tilde{e} \neq 0$ since $h_{A^0}(e) \in A^0$. Hence there is $\tilde{b} \in \tilde{\mathbb{B}} \setminus \{0\}$ which forces that, if $\tilde{B} \ni \tilde{b}$ is $\tilde{\mathbb{B}}$-generic, then there is $\sigma^* \in V[A][\tilde{B}]$ which is a $\{\theta, \mathbb{A}, \mathbb{B}, A\}, \tilde{\mathbb{B}}, \tilde{B}$-revision of $\tilde{\sigma}$ coinciding on u and s.t. $e \in \tilde{B}^0$. Let $\tilde{B}, \sigma^*$ be given and work in $V[A][\tilde{B}]$. Then $B = A * \tilde{B} =_{Df}$ $=_{Df} \{b \in \mathbb{B} \mid b/A \in \tilde{B}\}$ is $\mathbb{B}$-generic over V and $V[A][\tilde{B}] = V[B]$. Define:
$\sigma' = \langle \sigma'^i \mid i \leq \alpha \rangle$, where $\sigma'^i = \sigma^{*i} \upharpoonright N^i$.

<u>Claim</u> σ' is a $\{\theta, \mathbb{A}, \mathbb{B}\}, \mathbb{B}, B$-revision of π which coincides with π on u, and s.t. $e \in B^0$.

proof. Clearly:

(a) $e \in B^0$ since $e/A^0 \in \tilde{B}^0$.

(b) $\sigma'^h : N^h \prec N$, $h \leq i \leq \alpha \rightarrow rng(\sigma'^h) \subset rng(\sigma'^i)$, $rng(\sigma'^\lambda) = \bigcup_{i<\lambda} rng(\sigma'^i)$ for limit $\lambda \leq \alpha$

(c) $B^h = \sigma^{h}{}''B$ is $\mathbb{B}^h$-generic over N^h, since $B^h = A^h * \tilde{B}^h$.

(d) $\sigma' \in N[B]$, $\langle \sigma'^{h,i+1} \mid h \leq i \rangle \in N^{i+1}[B^{i+1}]$

(e) σ' coincides with σ (hence with π) on u,

(a)–(e) are immediate. It remains only to show:

(f) $C^N_\delta(\mathrm{rng}\,\sigma'^\alpha) = C^N_\delta(\mathrm{rng}\,\pi^\alpha)$ $(\delta = \delta(\mathbb{B}))$.

(g) $C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\sigma'^{i,i+1}) = C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\pi^{i,i+1})$

where $\delta^l = \delta(\mathbb{B}^l)$

We prove (f), the proof of (g) being virtually identical.

Since $\delta(\mathbb{B}) \geq \delta(\tilde{\mathbb{B}})$, we have:

(1) $C^{\tilde{N}}_\delta(\mathrm{rng}\,\tilde{\sigma}^\alpha) = C^{\tilde{N}}_\delta(\mathrm{rng}\,\sigma^{*\alpha})$

But:

(2) $N \cap C^{\tilde{N}}_\delta(\mathrm{rng}\,\sigma^{*\alpha}) = C^N_\delta(\mathrm{rng}\,\sigma'^\alpha)$

prf. $(\supset)$ is trivial. We show $(\subset)$.

Let $x \in N \cap C^{\tilde{N}}_\delta(\mathrm{rng}\,\sigma^{*\alpha})$. Then $x = \sigma^*(f)(\xi)$ for a $\xi < \delta$, where $f \in \tilde{N}^\alpha$ maps $\delta_\alpha = \delta(\mathbb{B}^\alpha)$ into N^α. Let $f = \mathring{f}^{A^\alpha}$, $\mathring{f} \in (\tilde{N}^\alpha)^{A^\alpha}$.

Then there is $a \in \mathbb{A}$ s.t. $a \Vdash_{\mathbb{A}} \check{x} = \sigma'(\mathring{f})(\check{\xi})$. But since $\delta \geq \delta(\mathbb{A})$, there is a dense set Δ in $\mathbb{A}$ s.t. $\bar{\bar{\Delta}} \leq \delta$. Hence there is such a $\Delta \in C = C^N_\delta(\mathrm{rng}\,\sigma'^\alpha)$ s.t. $\Delta \subset C$. We may assume $a \in \Delta$. Hence x is C-definable in $a, \sigma'(\mathring{f}), \xi$.

Hence $x \in C$.

Since $\sigma'^{\alpha} = \tilde{\sigma}^{\alpha} \upharpoonright N$, the same proof shows:

(3) $N \cap C_{\sigma}^{\tilde{N}}(\operatorname{rng} \tilde{\sigma}^{\alpha}) = C_{\sigma}^{N}(\operatorname{rng} \sigma^{\alpha})$.

Hence $C_{\sigma}^{N}(\operatorname{rng} \sigma'^{\alpha}) = C_{\sigma}^{N}(\operatorname{rng} \sigma^{\alpha}) = C_{\sigma}^{N}(\operatorname{rng} \pi^{\alpha})$.

QED (Thm 1)

The proof of Thm 1 contains much more information than we have stated. We can drop the assumption that $\mathbb{A}$ is α-subproper, merely assuming:

$\Vdash_{\mathbb{A}}$ ($\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\mathring{A}$),

$\mathring{A}$ being the canonical $\mathbb{A}$-generic name.

We assume that $a \in \mathbb{A} \setminus \{0\}$ forces the existence of a $\theta, \mathbb{A}, \mathring{A}$-revision of π (but not necessarily a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle, \mathbb{A}, \mathring{A}$ revision). But then there is $\dot{\sigma} \in V^{\mathbb{A}}$ s.t. a forces $\dot{\sigma}^{A}$ to be a $\theta, \mathbb{A}, \mathring{A}$-revision of π, when $A \ni a$ is $\mathbb{A}$-generic. We can replace our fixed $u \subset N^{\alpha}$ by $\dot{u}^{A}$, where $a \Vdash_{\mathbb{A}} (\dot{u} \subset \check{N}^{\alpha}$ is finite). Similarly we can replace $e \in \mathbb{B}^{0} \setminus \{0\}$ by $\dot{e}^{A}$, where $a \Vdash_{\mathbb{A}} \dot{e} \in (\dot{\sigma}^{0})^{-1}(\check{\mathbb{B}})$ and

$a \Vdash_{\mathbb{A}} h_{\mathbb{A}}^{\check{}}(\dot{e}) \in \mathring{A}^{0}$ ($\mathring{A}^{0}$ being an abbreviation for $(\dot{\sigma}^{0})^{-1}$ "$\mathring{A}$.) We then let $\tilde{b}$ force the existence of σ^{*}'s which is a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle, \tilde{\mathbb{B}}, \tilde{B}$-revision of $\tilde{\sigma}$, coinciding with $\tilde{\sigma}$ on $u = \check{u}^A$ and s.t. $\tilde{e} = e/A \in \tilde{B}^\circ$, where $e = \check{e}^A$. Since for every $A \ni a$ there is such a $\tilde{b}$, we may assume $\tilde{b} = \mathring{b}^A$, where a forces $\mathring{b}$ to have these properties. We may also assume w.l.o.g. that $\Vdash_{/A} \mathring{b} \in \check{\mathbb{B}}/\mathring{A}$ and $[\![\mathring{b} \neq 0]\!] = a$. But then there is $b \in \mathbb{B}$ s.t. $\Vdash_{\mathbb{A}} \check{b}/\mathring{A} = \mathring{b}$. Hence $h_{\mathbb{A}}(b) = [\![\check{b}/\mathring{A} \neq 0]\!] = a$. Letting $\sigma' i =_{Df} \sigma^* i \upharpoonright N^i$ as before, it is forced by b that σ' is a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle, \mathbb{B}, B$ - revision of σ for generic $B \ni b$.

(We must replace (f) by: $C_\sigma^N(\operatorname{rng} \sigma'\alpha) = C_\sigma^N(\operatorname{rng} \sigma^\alpha)$; similarly for (g), since σ' will not necessarily be a $\langle \theta, \mathbb{A}, \mathbb{B} \rangle$ revision of π.)

σ' then coincides with σ on $u = \check{u}^A$ and is s.t. $e = \check{e}^A \in B^\circ$. Since $\sigma^*(A^i) = A$ for $i \leq \alpha$, it follows easily that σ' coheres with σ wrt. $\mathbb{A}$.

Putting all of this together, we get:

<u>Lemma 2</u> Let $\mathbb{A} \subseteq \mathbb{B}$ be complete BA's. Let $\Vdash_{\mathbb{A}}$ ($\check{\theta}$ verifies the $\check{\alpha}$-subproperness of $\check{\mathbb{B}}/\mathring{A}$), where $\mathring{A}$ is the canonical $\mathbb{A}$-generic name.

Assume also:

Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let π be a tower for N with $\pi : N' \prec N$, $\pi'(\theta') = \theta$. Let $a \in \mathbb{A} \setminus \{0\}$ force that, whenever $A \ni a$ is $\mathbb{A}$-generic, then $\dot{\sigma}^A$ is an $\langle x, \theta \rangle, \mathbb{A}, A$-revision of π, $\dot{u}^A \subset N^\alpha$ is finite, $\dot{e}^A \in \mathbb{B}^0 = (\sigma^0)^{-1}(\mathbb{B})$ and $h_{\mathbb{A}}(\dot{e}^A) \in A$.

Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. $a = h_{\mathbb{A}}(b)$ and whenever $B \ni b$ is $\mathbb{B}$-generic, $A = B \cap \mathbb{A}$, $\sigma = \dot{\sigma}^A$, $u = \dot{u}^A$, $e = \dot{e}^A$, then there is $\sigma' \in V[B]$ which is an $\langle x, \theta, \mathbb{A}, \mathbb{B} \rangle, \mathbb{B}, B$-revision of σ cohering with σ w.r.t. $\mathbb{A}$ and coinciding with σ on u. Moreover, $e \in B^0$.

<u>Note</u> To show that $\mathbb{B}$ does not collapse ω_1, we must assume that $\mathbb{A}$ does not do so. To show that $\mathbb{B}$ is α-subproper we must assume that $\mathbb{A}$ has the property.

Thm 3 Let $\mathbb{B} = \langle \mathbb{B}_\nu \mid \nu < \gamma \rangle$ be an RCS-iteration. Let $\alpha < \omega_1$. Assume that for all $i+1 < \alpha$:

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i (\check{\mathbb{B}}_{i+1} / \dot{G})$ is $\check{\alpha}$-subproper

(c) $\Vdash_{i+1} (\check{\delta(\mathbb{B}_i)}$ has cardinality $\leq \omega_1)$.

Then every $\mathbb{B}_i$ is α-subproper

proof

Set $\delta_i = \delta(\mathbb{B}_i)$. As before we get:

(1) $\delta_i \leq \delta_j$ for $i \leq j < \gamma$

(2) $\bar{\bar{\nu}} \leq \delta_\nu$ for $\nu < \gamma$

By induction on $i < \gamma$ we prove:

Claim Let $h \leq i$, Let G_h be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i / G_h$ is α-subproper in $V[G_h]$.

The cases $i = 0$, $i = h$, $i = j+1$ follow exactly as before in §1 Thm 3, using the two step theorem.

There remains the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis $\mathbb{B}_j / G_h$ is α-subproper in $V[G_h]$ for $h \leq j < \lambda$. We consider two cases, as before:

Case 1 $\mathrm{cf}(\lambda) \leq \delta_i$ for an $i < \lambda$.

It again suffices to prove the claim for $h \geq i$, since it will then hold for smaller ordinals by the two step thm. But then $cf(\lambda) \le \omega_1$ in $V[G_n]$. We display the proof in the special case:

$$cf(\lambda) \le \omega_1 \text{ in } V,$$

showing that $\mathbb{B}_\lambda$ is α-subproper in V, since we can then repeat the proof in $V[G_n]$ to show that $\mathbb{B}_\lambda / G_n$ is α-subproper. (The induction hypothesis holds in $V[G_n]$ just as before.)

Now let $N = L^A_\tau$ be a ZFC^- model s.t. $H_\theta \subset N$, $\theta < \tau$. We shall prove:

<u>Main Claim</u> Let $\beta \le \alpha$. Let π be a β-tower for N which absorbs $\langle \theta, \mathbb{B}, \lambda \rangle$. Let $e \in \mathbb{B}^0 \setminus \{0\}$. Then let $u \subset N^\beta$ be finite. Let there is $b \in \mathbb{B}_\lambda \setminus \{0\}$ which forces that if $G \ni b$ is $\mathbb{B}_\lambda$-generic, then in $V[G]$ there is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_\lambda, G$-revision of $\overline{\pi}$ which which coincides with π on u and s.t. $e \in G^0$.

The proof will be by induction on β, but we shall need a stronger induction hypothesis. We split into two subcases.

<u>Case 1.1</u> $cf(\lambda) = \omega$

Let $f = \langle \xi_i \mid i<\omega \rangle$ be the N-least ω-sequence which is monotone and cofinal in λ and s.t. $\xi_0 = 0$. Set $f^h = \langle \xi_i^h \mid i<\omega \rangle = (\pi^h)^{-1}(f)$ for $h \leq \beta$.

Clearly, if σ is any $\langle \theta, \mathbb{B}, \lambda \rangle$-revision of π, then $\sigma^h(f^h) = f$ and $\sigma^h(\xi_i^h) = \xi_i$ for $i<\omega$.

We now refer back to the definitions of $\mathbb{B}_{<\lambda}$, $\mathbb{B}^+_{<\lambda}$ that we gave at the end of §1. Recall that $G \subset \mathbb{B}_{<\lambda}$ is called $\mathbb{B}_{<\lambda}$-<u>generic</u> iff $G \cap \mathbb{B}_\nu$ is $\mathbb{B}_\nu$-generic for all $\nu<\lambda$. G is then called $\mathbb{B}^+_{<\lambda}$-generic iff the set G^+ of $b \in \mathbb{B}^+_{<\lambda}$ s.t. $h_\nu(b) \in G$ for $\nu<\lambda$ meets every dense V-subset of $\mathbb{B}^+_{<\lambda}$. Setting $G^{++} = \{ b \in \mathbb{B}_\lambda \mid \forall a \in \mathbb{B}^+_\lambda \; a \subset b \}$, this is equivalent to saying that G^{++} is $\mathbb{B}_\lambda$-generic.

We now define:

Def Let $G \in \mathbb{B}_{<\lambda}$, where $\xi \leq \lambda$ is a limit ordinal. σ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - *revision of* π iff

- σ is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_\xi$ - revision of π
- G^i is $(\mathbb{B}^i_{<\xi^i})^+$ - generic over N^i for $i \leq \beta$, where $G^i =_{\text{df}} (\sigma^i)^{-1} {}'' G$
- $\sigma \in N[G]$ and $\langle \sigma^{h, i+1} \mid h \leq i \rangle \in N^{i+1}[G^{i+1}]$ for $i < \beta$.

Note It follows that, if σ respects π at $\mathbb{B}_\xi$, then $C^N(\text{rng}\, \sigma^\alpha) = C^N_\delta(\text{rng}\, \pi^\alpha)$ and $C^{N^{i+1}}_{\delta^i}(\text{rng}\, \sigma^{i, i+1}) = C^{N^{i+1}}_{\delta^i}(\text{rng}\, \pi^{i, i+1})$, where $\delta = \delta(\mathbb{B}_\xi)$ and $\delta^i = \delta(\mathbb{B}^i_{\xi^i})$.

Note that this definition makes no assumption about the genericity of G.

Clearly it suffices to show:

If $\beta \leq \alpha$ and π, u, e are as in the Main Claim, then there is $b \in (\mathbb{B}_{<\lambda})^+$ s.t. whenever G is $\mathbb{B}_{<\lambda}$ generic and $b \in G^+$, then there is $\sigma \in V[G]$ which is a $\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{<\lambda}, G$ - revision of π which coincides with π on u and is s.t. $e \in (G^0)^+$.

<u>Def</u> Let $G \subset \mathbb{B}_{<\lambda}$ be $\mathbb{B}_{<\lambda}$ - generic. Set: $G_i = G \cap \mathbb{B}_{\xi_i}$. By a <u>good</u> <u>matrix</u> <u>for</u> G <u>w.r.t.</u> π we mean a sequence $\langle \sigma_i \mid i<\omega \rangle$ s.t.

(a) $\sigma_i = \langle \sigma_i^h \mid h < \beta \rangle$ is a $\langle\langle \theta, \mathbb{B}, \lambda \rangle, \mathbb{B}_{\xi_i}, G \rangle$ - revision of π which coheres with σ_l w.r.t. $\mathbb{B}_{\xi_l}$ for $l \leq i$

(b) $G^i = \bigcup_{l<\omega} G_l^i$ is $\mathbb{B}^+_{<\lambda^l}$ - generic over N^i for $i \leq \beta$.

(c) If $x \in N^i$, there is $j < \omega$ s.t. $\sigma_l^i(x) = \sigma_j^i(x)$ for $l \geq j$.

(d) If $x \in N^\lambda$, x being a limit ordinal, then there are $h < \lambda$, $j < \omega$, $\bar{x} \in N^h$ s.t. $x = \sigma_l^{h\lambda}(\bar{x})$ for all $l \geq j$

(e) If $x \in N^h$, there are $j < \omega$, $w \in N^h$ s.t. $\bar{\bar{w}} \leq \delta^h = \sup_i \delta(\mathbb{B}^h_{\xi_i h})$ in N^h and:

- $\pi^{h,h+1}(x) \in \sigma_j^{h,h+1}(w) = \sigma_l^{h,h+1}(w)$ for $l \geq j$ if $h < \beta$
- $\pi^\beta(x) \in \sigma_j^\beta(w) = \sigma_l^\beta(w)$ for $l \geq j$ if $h = \beta$

(f) $\langle \sigma_i \mid i < \omega \rangle \in N[G]$ and $\langle \sigma_i^{h,j+1} \mid h \leq j, i < \omega \rangle \in N^{j+1}[G^{j+1}]$ for $j < \beta$.

<u>Note</u> By (c), if $i \leq h \leq \beta$, there is $j < \omega$ s.t. $\sigma_l^{ih} = \sigma_j^{ih}$ for all $l \geq j$.

We can then define: $\sigma = \sigma_\omega$ by:

$\sigma^h(x) = \sigma_j^h(x)$ if $\sigma_l^h(x) = \sigma_j^h(x)$ for all $l \geq j$.

Clearly $\sigma^h : N^h \prec N$. If we set $\sigma^{hi} = (\sigma^i)^{-1} \cdot \sigma^h$ for $h \leq i \leq \beta$, we get:

$\sigma^{hi}(x) = \sigma_i^{hi}(x)$ if $\sigma_l^{hi}(x) = \sigma_i^{hi}(x)$ for $l \geq i$.

<u>Sublemma 3.1</u> σ is a $\langle \theta, \mathbb{B}, \lambda \rangle$, $\mathbb{B}_{<\lambda}$, G - revision of π which coheres with σ_i wrt $\mathbb{B}_{\xi_i}$ for $i < \omega$.

proof.

(1) σ is a pretower.

We must show: $\operatorname{rng}(\sigma^\eta) = \bigcup_{i<\eta} \operatorname{rng}(\sigma^i)$ for limit $\eta \leq \beta$. This follows from (d) which says that each $x \in N^\eta$ has the form $\sigma^{i\eta}(\bar{x})$ for an $\bar{x} \in N^i$, $i < \eta$. QED(1)

(2) $C_\delta^N(\operatorname{rng} \sigma^\beta) = C_\delta^N(\operatorname{rng} \pi^\beta)$, where $\delta = \sup_{i<\lambda} \delta(\mathbb{B}_{\xi_i}) \leq \delta(\mathbb{B}_\lambda)$.

proof.

($\subset$) $\sigma^\beta(x) = \sigma_i^\beta(x) \in C_\delta^N(\operatorname{rng} \sigma_i^\beta) = C_\delta^N(\operatorname{rng} \pi^\beta)$

($\supset$) Let $x \in C_\delta^N(\operatorname{rng} \pi^\beta)$. Then $x = \pi(f)(\xi)$ where $\xi < \delta$, $f \in N^\beta$. But $\pi(f) \in \sigma(w)$ for a $w \in N^\beta$, $\bar{\bar{w}} \leq \delta$.

Hence $\pi(f) = \sigma(g)(\varsigma)$ where $\varsigma < \delta$. H

Hence $x = (\sigma(g)(\varsigma))(\xi) \in C_\delta^N(\operatorname{rng} \sigma)$,

QED(2)

Similarly:

(3) $C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\sigma^{i,i+1}) = C^{N^{i+1}}_{\delta^{i+1}}(\mathrm{rng}\,\pi^{i,i+1})$

for $i < \beta$, where $\delta^i = \sup_{h<\omega} \sigma(\mathbb{B}^i_{\xi^i_n})$.

By (f) we trivially have:

(4) $\sigma \in N[G]$ and $\langle \sigma^{i,h+1} \mid i \leq h \rangle \in N^{h+1}[G^{h+1}]$ for $h < \beta$

Finally:

(5) σ coheres with σ_i wrt $\mathbb{B}_{\xi_i}$ for $i < \omega$.

proof.

We must show: $\sigma^h{}''G^h_i \subset G_i$ for $h \leq \beta$, $i < \omega$.

Let $b \in G^h_i$. Then $\sigma^h(b) = \sigma^h_j(b) \in G_j \cap \mathbb{B}_{\xi_i} = G_i$ for some $j \geq i$.

QED (Sublemma 3.1)

We note that σ also has the properties:

- Let $u \subset N^\beta$ be finite s.t. each σ_i coincides with π on u. Then σ coincides with π on u.
- Let $b \in (\mathbb{B}^i_{<\lambda^i})^+$ s.t. $h_{\xi_h}(b) \in G^i_h$ for $h < \omega$. Then $b \in G^{i+}$.

Thus it suffices to show for $\beta \leq \alpha$:

(*) Let π be a β-tower for N which absorbs $\Theta, \mathbb{B}, \lambda$. Let $u \subset N^\beta$ be finite.

Let $e \in (\mathbb{B}^0_{<\lambda^0})^+ \setminus \{0\}$. Then there is $b \in \mathbb{B}^+_{<\lambda}$ and a sequence $\langle \dot\sigma_i \mid i<\omega \rangle \in N$ s.t. $\dot\sigma_i \in N^{\mathbb{B}_{\xi_i}}$ for $i<\omega$ and whenever G is $\mathbb{B}_{<\lambda}$-generic, $b \in G^+$, $G_i = G \cap \mathbb{B}_{\xi_i}$ $(i<\omega)$ and $\sigma_i = \dot\sigma_i^{G_i}$ for $i<\omega$, then $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix which coincides with π on u and is s.t. $e \in (G^0)^+$, where: $G^h_i = (\sigma_i h)^{-1} {}'' G_i$, $G^h = \bigcup_i G^h_i$.

We prove this by induction on β. However, as induction hypothesis we need the even stronger statement:

(**) Let $i<\omega$ and let G_i be $\mathbb{B}_{\xi_i}$-generic. Then (*) holds in $V[G_i]$ with $\mathbb{B}/G_i$ in place of $\mathbb{B}$ and $N^{G_i} = L^{A, G_i}_\tau$ in place of N (where $N = L^A_\tau$) and $\langle \xi_j \mid i \leq j < \omega \rangle$ in place of $\langle \xi_j \mid j<\omega \rangle$.

(Recall that $\mathbb{B}/G_i = \langle \mathbb{B}_\nu / G_i \mid \nu \geq i \rangle$.)

It will suffice at each stage of the induction to display the proof of (*), since the same proof can then be repeated in $V[G_i]$.

<u>Case 1.1.1</u> $\beta = 0$. The construction of b, $\langle \dot{\sigma}_i \mid i<\omega \rangle$ and the verification that $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix for G whenever G is $\mathbb{B}_{<\lambda}$ - generic, $b \in G^+$, and $\sigma_i = \dot{\sigma}_i^{G_i}$ $(i<\omega)$ is given in Case 1 of the proof of §1 Lemma 3. To see that we can have $\langle \dot{\sigma}_i \mid i<\omega \rangle \in N$, note that $S \in N$, where $S = \{\langle b, i, y, x \rangle \mid b \Vdash_{\mathbb{B}_{\xi_i}} \dot{\sigma}(\check{x}) = \check{y}\}$.
To see this note that $S \subset \mathbb{B}_\lambda \times C$, where $C = C_\delta^N(\mathrm{rng}\,\pi)$, $\delta = \sup_i \delta(\mathbb{B}_{\xi_i})$.
Let $\tilde{\tau} = \sup \pi^\beta{}'' On_{N^\beta}$, $\tilde{N} = L^A_{\tilde{\tau}}$, where $N = L^A_\tau$.
Then $\tilde{N} \prec N$ and $\pi^\beta : N^\beta \prec \tilde{N}$ cofinally.
But then $C \subset \tilde{N}$; hence $C = C_\delta^{\tilde{N}}(\mathrm{rng}\,\pi) \in N$ and $\bar{\bar{C}} = \delta < \theta$ in N. Let $f \in N$ map a $\gamma < \theta$ onto $\mathbb{B}_\lambda \times C$. Let $\bar{S} = f^{-1}{}''S$. Then $\bar{S} \in \mathbb{P}(\gamma) \subset H_\theta \subset N$. Hence $S = f''\bar{S} \in N$

QED (Case 1.1.1)

Case 1.1.2 $\beta = \nu + 1$

Then (*) holds at ν. We use:

Fact The statement "(*) holds at ν" is uniformly expressible over N in parameters from rng (π^β).

proof.

(*) says that if $u \subset N^\nu$ is finite and $e \in \mathbb{B}^{\circ +}_{<\lambda^\circ} \setminus \{0\}$, then there are $b \in \mathbb{B}^+_{<\lambda} \setminus \{0\}$, $\langle \dot\sigma_i \mid i<\omega\rangle \in N$ s.t. if G is $\mathbb{B}_{<\lambda}$ - generic and $\sigma_i = \dot\sigma_i^{G_i}$ ($i<\omega$), then $\langle\sigma_i \mid i<\omega\rangle$ is a good matrix with certain properties. (The quantification over the nonexistent G can be replaced by the statement that the above holds in $N^{\mathrm{coll}(\omega, \overline{\overline{\mathbb{B}_\lambda}})}$.) Hence it suffices to show that, if G is $\mathbb{B}_{<\lambda}$ - generic and $\langle\sigma_i \mid i<\omega\rangle \in N[G]$, then the statement:

"$\langle\sigma_i \mid i<\omega\rangle$ is a good matrix for G"

is uniformly expressible over $N[G]$ in parameters from rng π^β;

...

This at first glance seems dubious, since the statement involves clauses of the form:

$\pi^h : N^h \prec N$ $(h \le \nu)$ and

$C^N_\sigma(\text{rng}\,\sigma^\nu) = \text{rng}\, C^N_\sigma(\text{rng}\,\pi^\nu)$.

However $\pi^\beta(\langle \pi^h \beta \mid h \le \nu \rangle) = \pi \restriction \beta$ and, letting $\tilde{\tau} = \sup \pi^\nu{}''\,On_{N^\nu}$, we have: $\pi^\beta(\tilde{\tau}') = \tilde{\tau}$, where $\tilde{\tau}' = \sup \pi^{\nu\beta}{}''\,On_{N^\nu}$.

At $\tilde{N} = L^{\tilde{A}}_{\tilde{\tau}}$ (where $N = L^A_\tau$), then $\tilde{N} \in \text{rng}(\pi^\beta)$, $\text{rng}(\pi^\nu) \subset \tilde{N}$, and $\tilde{N} \prec N$. Thus we can replace the questionable clauses by:

$\pi^h : N^h \prec \tilde{N}$ $(h \le \nu)$ and

$C^{\tilde{N}}_\sigma(\text{rng}\,\sigma^\nu) = \text{rng}\, C^{\tilde{N}}_\sigma(\text{rng}\,\pi^\nu)$.

QED (Fact)

But since $\sigma^\beta : N^\beta \prec N$, the corresponding statement holds over N^β. Thus, letting $\bar{u} = (\sigma^{\nu\beta})^{-1}{}''u$ (where $u \subset N^\beta$ is finite), there are $\bar{b} \in (\mathbb{B}^\beta_{<\lambda^\beta})^+$, $\langle \dot{\bar{\sigma}}_i \mid i < \omega \rangle \in N^\beta$ s.t. if $\bar{G}$ is $\mathbb{B}^\beta_{<\lambda^\beta}$-generic, $\bar{b} \in \bar{G}^+$, and $\bar{\sigma}_i = \dot{\bar{\sigma}}_i^{\bar{G}}$ $(i < \omega)$, then $\langle \bar{\sigma}_i \mid i < \omega \rangle$ is a good matrix for $\bar{G}$ wrt. $\bar{\pi} = \langle \pi^{h\beta} \mid h \le \nu \rangle$ which coincides with π on $\bar{u}$ and is s.t. $e \in (\bar{G}^0)^+$.

Set $\bar{b}_i = h_{\xi_i \beta}(\bar{b})$. Note that if $\bar{G}_i \ni \bar{b}_i$ is any $\mathbb{B}^{\beta}_{\xi_i \beta}$ - generic set, then $\dot{\bar{\sigma}}_i^{\bar{G}_i}$ is a revision of $\bar{\pi}$. By Case 1.1.1, however, there are $b \in (\mathbb{B}_{<\omega})^+ \setminus \{0\}$ and $\langle \dot{\sigma}_i' \mid i<\omega \rangle \in N$ s.t. $\sigma_i' \in N^{\mathbb{B}_{\xi_i}}$ and whenever G is $\mathbb{B}_{<\lambda}$ - generic and $\sigma_i' = \dot{\sigma}_i'^G$ $(i<\omega)$, then $\langle \sigma_i' \mid i<\omega \rangle$ is a good matrix for G wrt. the tower $\pi \upharpoonright \{\beta\}$ of length o s.t. σ_i' coincides with $\pi \upharpoonright \{\beta\}$ on u and $e_i \in G_i^{\beta} = \sigma_i'{}'' G_i$ for $i<\omega$, (where $e_i = h_{\xi_i \beta}(e)$), just as above, letting $b_i = h_{\xi_i}(b)$, we have:

$b_i \Vdash_{\xi_i} (\dot{\sigma}_i'$ is a revision of $\pi \upharpoonright \{\beta\}$).

But then there is obviously a term $\dot{\sigma}_i \in N^{\mathbb{B}_{\xi_i}}$ s.t. if $G_i \ni b_i$ is $\mathbb{B}_{\xi_i}$ - generic, then $\sigma_i = \dot{\sigma}_i^{G_i}$ is the revision of $\bar{\pi}$ defined by: $\sigma_i^{\beta} = \sigma_i'^{\beta}$ and $\sigma_i^h = \sigma_i^{\beta} \circ \bar{\sigma}_i^h$ for $h < \beta$. Then $\langle \dot{\sigma}_i \mid i<\omega \rangle$ has the desired properties. QED (Case 1.1.2)

<u>Case 1.1.3</u> β is a limit ordinal

Fix a sequence $\langle \beta_i \mid i<\omega \rangle$ which is monotone and cofinal in β with $\beta_0 = 0$ and β_i a successor ordinal for $i>0$. We also write: $\tilde{\beta}_i = (\beta_{i+1} - 1)$. Set $\langle x_i^h \mid i<\omega \rangle$ = the N-least enumeration of of N^h for $h \leq \beta$. Then $\langle x_i^h \mid h \leq j \rangle \in N^{j+1}$ for $j < \beta$. In order to simplify our notation we also write: $\hat{\mathbb{B}}_i^h = \mathbb{B}^h_{\xi_i h}$.

We must produce a good matrix $\langle \sigma_i \mid i < \omega \rangle$. To do this we essentially define σ_i by induction on i. To make sure this works, however, we also anticipate the matrix "from below", simultaneously constructing $\mu(i)$ s.t. $\mu(i)$ is a good matrix for $N^{\beta_{i+1}}$ with $\mu(i)^h_l = \sigma_l^{h,\beta_{i+1}}$ for $l \leq i$.

Of course, we are working in V, and will not directly construct $\langle \sigma_i \mid i<\omega \rangle$ but rather an $a \in \mathbb{B}^+_{<\lambda}$ and $\langle \dot{\sigma}_i \mid i<\omega \rangle$ s.t. $\dot{\sigma}_i \in N^{\mathbb{B}_{\xi_i}}$ and whenever G is $\mathbb{B}_{<\lambda}$ - generic and $a \in G^+$, then, letting $\dot{\sigma}_i^{G_i} = \sigma_i$, $\langle \sigma_i \mid i<\omega \rangle$ will be the desired good matrix. Thus, we inductively construct $a_i = h_{\xi_i}(a)$ and $\dot{\sigma}_i$.

We construct $a_i, \dot{\sigma}_i, \dot{u}_i$ s.t.

(I) (a) $a_i \in \mathbb{B}_{\xi_i}$

(b) $\dot{\sigma}_i, \dot{u}_i \in N^{\mathbb{B}_{\xi_i}}$

(c) $a_\ell = h_{\xi_\ell}(a_i)$ for $\ell \leq i$.

(II) Let $G \ni a_i$ be $\mathbb{B}_{\xi_i}$ - generic. Set:

$G_\ell = G \cap \mathbb{B}_{\xi_\ell}$, $\sigma_\ell = \dot{\sigma}_\ell^{G_\ell}$, $u_\ell = \dot{u}_\ell^{G_\ell}$ $(\ell \leq i)$.

Then:

(a) σ_i is a $\langle\langle \theta, \mathbb{B}, \lambda\rangle, \mathbb{B}_{\xi_i}, G_i\rangle$ revision of π. Moreover $\sigma_0 = \pi$

(b) σ_i coheres with σ_ℓ w.r.t. $\mathbb{B}_{\xi_\ell}$ for $\ell \leq i$

(c) $u_i \subset N^\beta$ is finite

(d) $\sigma_i \upharpoonright [\beta_\ell, \beta]$ coincides with $\sigma_\ell \upharpoonright [\beta_\ell, \beta]$ on u_ℓ for $\ell \leq i$

(e) $u \cup \bigcup_{\ell < i} u_\ell \subset u_i$.

(f) $x_h^\beta, w_h \in u_i$ for $h < i$, where w = the N^β- least w s.t. $\bar{\bar{w}} \leq \delta^\beta$ in N^β and $\pi(x_i^\beta) \in \sigma_i^\beta(w)$ (where $\delta^\beta = \sup_{h<\omega} \delta(\mathbb{B}^\beta_{\xi_h^\beta})$).

(g) $\sigma_i^{\beta_h}(x_i^{\beta_h}) \in u_i$ for $h < i$.

Simultaneously we construct $\dot{b}^i$, $\ddot{\mu}(i) \in N^{\mathbb{B}_{\xi_i}}$ s.t.

III Let G be as in II. Set: $b^i = (\dot{b}^i)^G$ and $\dot{\mu}(i) = \ddot{\mu}(i)^G$. Then:

(a) $b^i \in (\mathbb{B}^{\beta_{i+1}}_{<\lambda^{\beta_{i+1}}})^+$. Set $b^i_\ell = h_{\xi_\ell \beta_{i+1}}(b^i)$.

(b) $\dot{\mu}(i) = \langle \dot{\mu}(i)_\ell \mid \ell < \omega \rangle$, where

$\dot{\mu}(i)_\ell \in (N^{\beta_{i+1}})^{\hat{\mathbb{B}}_\ell^{\beta_{i+1}}}$ for $\ell < \omega$

(c) $b^i_i = 1$; $b^k_i \in G_i^{\beta_{k+1}}$ for $k < i$

IV Let $G, b^i, \dot{\mu}(i)$ be as in III. Let $H \supset G_i^{\beta_{i+1}}$ be $\mathbb{B}^{\beta_{i+1}}_{\lambda^{\beta_{i+1}}}$ – generic over $N^{\beta_{i+1}}$ s.t. $b^i \in H^+$.

Set: $H_\ell = H \cap \hat{\mathbb{B}}_\ell^{\beta_{i+1}}$ for $\ell < \omega$. (Hence $H_\ell = G_\ell^{\beta_{i+1}}$ for $\ell \le i$.) Set:

$\mu(i)_\ell = (\dot{\mu}(i)_\ell)^{H_\ell}$ $(\ell < \omega)$. Then:

(a) $\mu(i) = \langle \mu(i)_\ell \mid \ell < \omega \rangle$ is a good matrix for $N^{\beta_{i+1}}$ w.r.t. H

(b) $\mu(i)_\ell = \langle \sigma_\ell^{h, \beta_{i+1}} \mid h \le \tilde{\beta}_i \rangle$ for $\ell \le i$.

(c) $\mu(i)_\ell \upharpoonright [\beta_h, \tilde{\beta}_i]$ coincides with $\mu(i)_h \upharpoonright [\beta_h, \tilde{\beta}_i]$ on $(\sigma^{\beta_h})^{-1}$ " u_h for $h \le i$, $h \le \ell < \omega$.

We of course set: $H_l^{j} = (\mu(i)_l^{j})^{-1} \text{"}\underline{H}_l$ for $l \leq \omega$, $j \leq \tilde{\beta}_i$. But then $H^{j} = \bigcup_l H_l^{j}$ in $(\mathbb{B}^{d}_{<\lambda^{j}})^{+}$ - generic over N' for $j \leq \tilde{\beta}_i$. Hence, for $h < i$, we can form $b^h = (\dot{b}^h)^{H^{\beta_{h+1}}}$, $\mu(h) = \dot{\mu}(h)^{H^{\beta_{h+1}}}$.

We shall ensure that:

(d) $b^h \in (H^{\beta_{h+1}})^{+}$ for $h < i$; Moreover $e \in (H^0)^{+}$.

But then $\mu(h)_l = \dot{\mu}(h)_l^{H^{\beta_{h+1}}}$ is defined for $l < \omega$ and satisfies the above conditions.

We ensure:

(e) $\mu(h)_l^{j} = \mu(i)_l^{j, \beta_{h+1}}$ for $j \leq \tilde{\beta}_h$, $l < \omega$.

In this context it is useful to write:

$\mu(i)_l^{j, \beta_{i+1}} =_{Df} \mu(i)_l^{j}$ for $j \leq \tilde{\beta}_i$, $l < \omega$,

With this convention we have:

$\mu(h)_l^{j,k} = \mu(i)_l^{j,k}$ for $l < \omega$, $j \leq k \leq \beta_{h+1}$ for $h \leq i$, and we can, without confusion, write $\mu_l^{j,k}$.

We note that IV (a) – (e) hold "locally" – i.e. from IV we can derive:

V Let G, b^i, $\mu(i)$ be as in III. Let $j \geq i$ and let $H \ni b^i_j$ be $\hat{\mathbb{B}}^{\beta_i+1}_j$ – generic over N^{β_i+1} s.t. $H \supset G^{\beta_i+1}_i$. Set: $H_l = H \cap \hat{\mathbb{B}}^{\beta_i+1}_l$ for $l \leq j$. Set: $\mu(i)_l = (\mu(i)_l)^{H_l}$ $(l \leq j)$. Then

(a) $\mu(i)_j$ is a $\langle\langle \theta^{\beta_i+1}, \mathbb{B}^{\beta_i+1}, \lambda^{\beta_i+1}\rangle, \hat{\mathbb{B}}^{\beta_i+1}_j, H\rangle$ – revision of π cohering with $\mu(i)_l$ w.r.t. $\hat{\mathbb{B}}^{\beta_i+1}_l$ for $l \leq j$

(b) $\mu(i)_l = \langle \sigma^{h, \beta_i+1}_l \mid h \leq \tilde{\beta}_i \rangle$ for $l \leq i$

(c) $\mu(i)_j \upharpoonright [\beta_h, \tilde{\beta}_i]$ coincides with $\mu(i)_h \upharpoonright [\beta_h, \tilde{\beta}_i]$ on $(\sigma^{\tilde{\beta}_h})^{-1}$ " u_h for $h \leq i$

(d) $b^h_j \in H^{\beta_h+1}_j$ for $h < i$

(e) $\mu(h)^k_l = \mu(i)^{k, \beta_h+1}_l$ for $k \leq \tilde{\beta}_h$, $l \leq j$.

This is because H can be extended to a $\mathbb{B}^{\beta_i+1}_{<\lambda^{\beta_i+1}}$ – generic H' s.t. $b^i \in H'^+$, since $b^i_j = h_{\xi^{\beta_i+1}_j}(b^i) \in H$.

We are now ready to prove (*). Let $a = \bigcap_i a_i$. Let G be $\mathbb{B}_{<\lambda}$-generic, where $a \in G^+$. Set: $G_i = G \cap \mathbb{B}_{\xi_i}$; $\sigma_i = \dot{\sigma}_i^{G_i}$.

<u>Claim</u>: $\langle \sigma_i \mid i<\omega \rangle$ is a good matrix.

Set: $G_i^h = (\sigma_i^h)^{-1}{}''G_i$. Then $G_i^h \subset G_j^h$ for $i \leq j$, by coherence. Set $G^h = \bigcup_{i<\omega} G_i^h$. Then

(1) G^h is $\mathbb{B}^h_{<\lambda^h}$-generic over N^h.

(2) $b^i \in (G^{\beta_{i+1}})^+$ for $i<\omega$, where $b^i = (\dot{b}^i)^{G_i}$

proof
$b^i_h \in G_h^{\beta_{i+1}}$ for $i<h$ by III (c). But
$b^i = \bigcap_{i<h} b^i_h$. QED(2)

Thus, letting $\mu(i) = \dot{\mu}(i)^{G_i}$, we have:

(3) $\mu(i) = \dot{\mu}(i)^{G^{\beta_{i+1}}}$ exists and satisfies IV (a)–(e) with $H = G^{\beta_{i+1}}$.

Thus we can write $\mu_l^{hj} = \mu(i)_l^{hj}$ for $l<\omega$, $h \leq j \leq \beta_{i+1}$, the choice of $i<\omega$ being irrelevant. By IV (b):

(4) $\mu_l^{hj} = \sigma_l^{hj}$ for $l<\omega$, $h \leq j < \beta$.

We now verify (a)–(f) in the definition of good matrix:

(a) is immediate

We prove (b): (b) is proven for $i < \beta$, so let $i = \beta$. Let $\Delta \in N^\beta$ be strongly dense in $\mathbb{B}^\beta_{\lambda^\beta}$. We must find $c \in (G^\beta)^+$ s.t. $c \in \Delta$. Let $\Delta \in U_{i_0}$. Let $j > i_0$ s.t. $\sigma_j^{\beta_j, \beta}(\bar{\Delta}) = \Delta$. Then $\sigma_l^{\beta_l, \beta}(\bar{\Delta}) = \Delta$ for all $l \geq j$. $\bar{\Delta}$ is strongly dense in $\mathbb{B}^{\beta_j}_{\lambda^{\beta_j}}$. Hence there is $\bar{c} \in (G^{\beta_j})^+$ s.t. $\bar{c} \in \bar{\Delta}$. Let $\bar{c} = x_k^{\beta_j}$. Assume w.l.o.g. that $j > k$. Let $c = \sigma_j^{\beta_j \beta}(\bar{c})$. Then $c = \sigma_l^{\beta_j \beta}(\bar{c})$ for all $l \geq j$. Hence, since $h_{\xi_l^{\beta_j}}(\bar{c}) \in G_l^{\beta_j}$, we have $\sigma_l^{\beta_j \beta}(h_{\xi_l^{\beta_j}}(\bar{c})) = h_{\xi_l^\beta}(c) \in$ $\in G_l^\beta$. Hence $c = \bigcap_{l \geq j} h_{\xi_l^\beta}(c) \in$ $\in (G^\beta)^+$. But $c = \sigma_j^{\beta_j, \beta}(\bar{c}) \in$ $\in \sigma_j^{\beta_j \beta}(\bar{\Delta}) = \Delta$. QED (b).

We now prove (c).
Let $i \leq \beta_n$ and let $j_0 < \omega$ s.t. $\sigma_l^{i \beta_n}(x) = \sigma_{j_0}^{i \beta_0}(x)$ for $l \geq j$.

(j_0 exists because $\sigma_l^{i,\beta_h} = \mu_l^{i,\beta_h}$ and $\mu(h)$ is a good matrix for N^{β_h+1}.)

Let $\sigma_{j_0}^{i,\beta_h}(x) = x_k^{\beta_h}$. Let $j_1 > j_0, k$.

For $l \geq j_1$ we have:

$\sigma_l^{i}(x) = \sigma_l^{\beta_h}(\sigma_l^{i,\beta_h}(x)) =$

$= \sigma_{j_1}^{\beta_h}(\sigma_{j_1}^{i,\beta_h}(x)) = \sigma_{j_1}^{i}(x)$. QED (c)

(d) holds for $\lambda < \beta$, since $\mu(i)$ is a good matrix, where $\beta_i > \lambda$. (d) holds at β by II (d).

(e) holds at $h < \beta$ because $\mu(i)$ is a good matrix, where $h < \beta_i$. (e) holds at β by II (f).

We prove (f). We can assume w.l.o.g. that: $\langle \sigma_i \mid i < \omega \rangle \in N$.

This follows by the fact that:

$S = \{\langle b, i, y, x \rangle \mid i < \omega \wedge b \in \mathbb{B}_{\xi_i} \wedge b \Vdash \sigma^{i}(\check{x}) = \check{y}\} \in$

$\in N$.

(To see this, note that $S \subset \mathbb{B}_\lambda \times C$, where $C = C_\delta^N(\operatorname{rng} \pi)$, where $\delta = \sup_i \delta(\mathbb{B}_{\xi_i})$. But, just as in the proof of Case 1.1.2, we have $C \in N$, $\bar{\bar{C}} < \theta$ in N. Thus, setting $\bar{S} = f^{-1}{}''S$, where $f \in N$ maps a $\delta < \theta$ onto $\mathbb{B}_\lambda \times C$, we have $\bar{S} \subset \mathcal{P}(\delta) \subset H_\theta \subset N$ + hence $S = f''\bar{S} \in N$.)

But then $\langle \sigma_i \mid i<\omega \rangle = \langle \dot{\sigma}_i^{G_i} \mid i<\omega \rangle \in N[G]$.

For $j < \beta$, $\langle \sigma_i^{h, j+1} \mid h \leq j, i<\omega \rangle \in N^{j+1}$ because $\mu(k)$ is a good matrix, taking $j \leq \beta_k$. QED (Claim)

It remains only to note:

- σ_i coincides with π on u by (d), since $\sigma_0 = \pi$ and $u \subset u_0$
- $e \in (G^0)^+$ by IV (d)

This completes the proof of (*).

All that remains is to define a_i, σ_i, u_i, b^i and $\mu(i)$ and verify I - IV.

We proceed by induction on i:

Case 1 $i=0$. Set $a_0 = 1$, $\dot\sigma_0 = \check\pi$. By the induction hypothesis there are b, $\langle \mu_i \mid i<\omega\rangle$ satisfying (*) at $\tilde\beta_0$. But, just as in the proof of Case 1.1.2, this fact is expressible over N in parameters from $\operatorname{rng}\pi^{\beta_1}$. Since $\pi^{\beta_1}: N^{\beta_1} \prec N$, the corresponding statements hold in N^{β_1}. This gives us $b^0 \in (\mathbb{B}_{<\lambda^0})^+$, $\dot\mu(0) = \langle \dot\mu(0)_i \mid i<\omega\rangle \in N^{\beta_1}$ satisfying (*) for N^{β_1} wrt $\langle \pi^{h\beta_1} \mid h \le \tilde\beta_0\rangle$.
Set: $\dot b^0 = \check b^0$, $\ddot\mu(0) = \check{\dot\mu}(0)$.
The verifications are straightforward.

Case 2 $i = k+1$. We first construct a_i, σ_i.
By Lemma 2, there are a, $\dot\sigma$ s.t. $a \in \mathbb{B}_{\xi_i}$ and $a_k = h_{\xi_k}(a)$, $\dot\sigma \in V^{\mathbb{B}_{\xi_i}}$ s.t. whenever $G \ni a$ is $\mathbb{B}_{\xi_i}$-generic, then $\sigma = \dot\sigma^G$ is a $\langle\langle \theta, \mathbb{B}, \lambda\rangle, \mathbb{B}_{\xi_i}, G\rangle$-revision of $\pi \restriction [\beta_i, \beta]$ coinciding with $\dot\sigma_k^{G_{k+1}} \restriction [\beta_i, \beta]$ on $u_k = \dot u_k^{G_k}$, and s.t. $b_i^k \in G^{\beta_i}$. (Here $b_i^k = \dot b^{G_k}$, where $a_k \Vdash_{\xi_k} \dot b = h_{\hat{\mathbb{B}}_i^{\beta_i}}(\dot b^k)$. Thus $h_{\xi_k \beta_i}(b_i^k) = b_k^k = 1$.) Set: $a_i = a$.

Since $b_i^k \in G^{\beta_i}$ and $G_k^{\beta_i} \subset G^{\beta_i}$, we can form $\mu(k)_i = (\dot\mu(k)_i)^{G^{\beta_i}}$, where $\dot\mu(k) = \ddot\mu(k)^{G_k}$. Set:

$$\sigma_i^h = \begin{cases} \sigma^h & \text{if } \beta_i \le h \le \beta \\ \sigma^{\beta_i} \circ \mu(k)_i^h & \text{if } h < \beta_i \end{cases}$$

Let $\dot\sigma_i$ be the $\dot\sigma_i \in V^{\mathbb{B}_{\xi_i}}$ s.t. $\sigma_i = \dot\sigma_i^G$ satisfies the above definition for all $\mathbb{B}_{\xi_i}$-generic $G \ni a$. We can assume w.l.o.g. that $\dot\sigma_i \in N$, since, arguing as in Case 1.1.2, we have $S \in N$, where:

$$S = \{\langle b, h, y, x\rangle \mid h \le \beta \wedge b \Vdash_{\xi_i} \dot\sigma_i(\check x) = \check y\},$$

Noting that $\mu(k)_i \upharpoonright [\beta_l, \tilde\beta_k]$ coincides with $\mu(k)_l \upharpoonright [\beta_l, \tilde\beta_k] = \langle \sigma_l^{h, \beta_i} \mid \beta_l \le h \le \tilde\beta_k \rangle$ on $(\sigma_l^{\beta_i})^{-1}{}'' u_l$ for $l \le k$, we get:

$\sigma_i \upharpoonright [\beta_l, \tilde\beta_k]$ coincides with $\sigma_l \upharpoonright [\beta_l, \tilde\beta_k]$ on u_l for $l \le k$.

Finally set:

$$u = \dot u^G = u_k \cup \{x_i^\beta, w_i\} \cup \{\sigma_i^{\beta_h}(x_i^{\beta_h}) \mid h < \omega\}.$$

The verification of I, II is straightforward.