

§3 Dee - Subproperness

Def A completeness system is a function  $\mathbb{D}$  defined on pairs  $\langle N, \mathbb{B} \rangle$  s.t.  $N = L_{\mathcal{L}}^A$  (or  $N = L_{\mathcal{L}}^{A_1, \dots, A_n}$ ) is a countable, full ZFC-model and  $\mathbb{B} \in N$  is a complete BA in  $N$  s.t. the following hold:

- (a) Each  $X \in \mathbb{D}(N, \mathbb{B})$  is a set of  $G$  which is  $\mathbb{B}$ -generic over  $N$ ;
- (b) If  $X \in \mathbb{D}(N, \mathbb{B})$  and  $a \in \mathbb{B} \setminus \{0\}$ , there is  $G \in X$  s.t.  $a \in G$
- (c) Let  $X_i \in \mathbb{D}(N, \mathbb{B})$  for  $i < \omega$ . Then  $\bigcap_i X_i \neq \emptyset$  and, in fact, for each  $a \in \mathbb{B} \setminus \{0\}$  there is  $G \in \bigcap_i X_i$  s.t.  $a \in G$
- (d) There is a formula  $\varphi$  and an  $\alpha \in H_{\omega_1}$  s.t.  $\mathbb{D}(N, \mathbb{B}) = \{A_u \mid u \in H_{\omega_1}\}$ , where  $A_u = \{G \mid G \text{ is } \mathbb{B}\text{-generic over } N \text{ and } H_{\omega_1} \models \varphi[N, \mathbb{B}, G, u, \alpha]\}$ .

Note (a), (b) express Shelah's notion of "completeness system". (c) expresses " $\omega$ -completeness" and (d) expresses "simplicity" in a sense close to that of Shelah. We shall only

make use of completeness systems  
satisfying (c), (d).

Def Let  $B$  be a complete BA.  $B$  is Deo-  
subproper as witnessed by  $\theta, \mathbb{D}$  iff  
 $B \in H_\theta$ ,  $\mathbb{D}$  is a completeness system,  
and the following holds:

Let  $N = L_\tau^A$  be a ZFC<sup>-</sup> model s.t.

$H_\theta \subset N$  and  $\theta < \tau$ . Let  $\pi: \bar{N} \prec N$ , where  
 $\bar{N}$  is countable and full, and  $\pi \in N$ .

Let  $\pi(\bar{\theta}, \bar{B}) = \theta, B$ . Then there is

$X \in \mathbb{D}(\bar{N}, \bar{B})$  s.t. whenever  $\bar{G} \in X$ , then

for all  $\bar{a} \in \bar{N}$ ,  $a = \pi(\bar{a})$ , there is

$b \in B \setminus \{0\}$  forcing that whenever

$G \ni b$  is  $B$ -generic, there is  $\sigma \in V[G]$

s.t.

- $\sigma: \bar{N} \prec N$

- $\sigma(\bar{\theta}, \bar{B}, \bar{a}) = \theta, B, a$

- $C_\delta^N(\text{rang } \sigma) = C_\delta^N(\text{rang } \pi)$

- $\sigma'' \bar{G} \subset G$ .

Def  $B$  is Dec-subproper as witnessed by  $\theta$  iff it is Dec-subproper as witnessed by  $\theta, \mathbb{D}$  for some  $\mathbb{D}$ .

Def  $B$  is Dec-subproper iff it is Dec-subproper as witnessed by some  $\theta$ .

Def  $B$  is Dec-subproper as verified by  $\theta$  iff it is Dec-subproper as witnessed by  $\theta'$  for all  $\theta' \geq \theta$ .

It again turns out that if  $B$  is Dec-subproper, then its Dec-subproperness is verified, since if  $\theta, \mathbb{D}$  witness the Dec-subproperness of  $B$ , then so do  $\theta', \mathbb{D}$  for all  $\theta' > 2^\theta$ .

We can, of course, modify our definition of "Dec-subproper as witnessed by  $\theta, \mathbb{D}$ " to require that  $p \in \text{rng}(\pi)$  for a fixed parameter  $p$ . We then say that  $B$  is "Dec-subproper as witnessed by  $\theta, \mathbb{D}, p$ ". The usual proof shows that every such  $B$  is fully subproper.

Clearly every Dec-subproper forcing is subproper. If, however, we restrict to forcings which are both  $\omega_1$ -subproper and Dec-subproper, we get a strong iteration theorem, generalizing the corresponding theorem which Shelah proved for proper forcings:

Thm 1 Let  $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$  be an RCS-iteration s.t.  $\mathbb{B}_0 = 2$  and:

(a)  $\mathbb{B}_{i+1} \neq \mathbb{B}_i$

(b)  $\mathbb{H}_i(\check{\mathbb{B}}_{i+1} / \dot{G}$  is  $\omega_1$ -subproper and Dec-subproper)

(c)  $\mathbb{H}_{i+1} \text{ card}(\delta(\check{\mathbb{B}}_i)) \leq \omega_1$ .

Then each  $\mathbb{B}_i$  adds no reals.

proof.

Let  $\theta$  be large enough that  $\mathbb{H}_i \mathbb{H}_\theta \check{\Sigma}_m \check{V}$  for all  $i < \alpha$ , where  $m$  is big enough that the formulae " $\forall x \in \check{V}_d$ " and

" $A$  is a complete BA and there are  $\langle \gamma, \mathbb{D} \rangle$  witnessing the Dec-subproperness of  $A$ " are  $\Sigma_m$ .

Then  $\Pi_i \cdot H_\theta^v = V_\theta^v$  and :

$\Pi_i \cdot \check{\theta}$  verifies the Dece-subproperness of  $B_{i+1}^v / G^v$  for all  $i < d$ .

Let  $N = L_{\check{\tau}}^A$  be a ZFC-model s.t.  $N$  is full,  $H_\theta \subset N$ ,  $\theta < \check{\tau}$  and  $cf(\check{\tau}) > \omega$ . (There are many such  $N$  of cardinality  $2^{\aleph_0}$ .) Let  $\pi^0 : N^0 \prec N$ , where  $N^0$  is countable and full and  $\pi^0(\theta^0, B^0, d^0) = \theta, B, d$ . By our assumption on  $N$  we can extend  $\pi^0$  to a tower  $\pi = \langle \pi^i \mid i \leq \beta \rangle$  of arbitrary countable height  $\beta$ . s.t.  $\pi^i : N^i \prec N$  and  $\pi^i(\theta^i, B^i, d^i) = \theta, B, d$ .

Hitherto we worked only with pretowers indexed by intervals  $[\nu, \beta]$ . Now, however, we shall also use sets of the form  $\Gamma_\nu = \{0\} \cup [\nu+1, \beta]$  to index pretowers. This necessitates some changes in our definition. The definition of " $\Gamma_\nu$ -tower" and " $\Gamma_\nu$ -pretower" is obvious. In particular,  $\pi \upharpoonright \Gamma_\nu$  is a tower,

Def By a  $\Gamma_\nu$ -reformation of  $\pi$   
 we mean a  $\Gamma_\nu$ -pretower  $\sigma$  s.t.  
 (a)  $\sigma \upharpoonright [\nu+1, \beta]$  is a revision of  
 $\pi \upharpoonright [\nu+1, \beta]$

(b)  $\text{rng}(\sigma^{\circ, \nu+1}) \subset \bigcup \text{rng}(\pi^{\nu, \nu+1})$ .

Note This is weaker than saying  
 that  $\sigma$  is a revision of  $\pi \upharpoonright \Gamma_\nu$ , since  
 that would require that  
 $\text{rng}(\sigma^{\circ, \nu+1}) = \bigcup \text{rng}(\pi^{\circ, \nu+1})$ .  
 Even if  $\sigma$  were a revision of  $\pi$ , it  
 does not follow that  $\sigma \upharpoonright \Gamma_\nu$  is a  
 revision of  $\pi \upharpoonright \Gamma_\nu$ , but  $\sigma \upharpoonright \Gamma_\nu$  is  
 a reformation of  $\pi$ .

Note  $\pi$  is a  $\Gamma_0$  reformation of itself.

We generally write "reformation" to mean  
 "reformation of  $\pi$ ".

Def Let  $\sigma$  be a  $\Gamma_i$  reformation and  
 $\sigma'$  a  $\Gamma_j$  reformation, where  $i \leq j < \alpha$

$\sigma'$  respects  $\sigma$  at  $x$  iff  $\sigma'$  respects

$\sigma \upharpoonright \Gamma_i$  at  $x$ .

(We then also say:  $\sigma'$  is an  $x$ -  
 -reformation of  $\sigma$ .)

Def Let  $\sigma, \sigma', i, j$  be as above. Let  $u \in N^\beta$ .  
 $\sigma'$  coincides with  $\sigma$  on  $u$  iff  
 $\sigma'$  coincides with  $\sigma \upharpoonright \Gamma_i$  on  $u$ .

Def Let  $\sigma, \sigma', i, j$  be as above.

$\sigma'$  is strong over  $\sigma$  iff

(a)  $\sigma'$  respects  $\sigma$  at  $\langle \theta, B, \sigma^o(i), \sigma^o(j) \rangle$

(b)  $\sigma' \upharpoonright [i+1, \beta]$  is a  $\langle \theta, B, \sigma^o(i), \sigma^o(j) \rangle, B_{\sigma^o(i)}$ -

- revision of  $\sigma \upharpoonright [i+1, \beta]$

(c)  $C_\sigma^N(\text{rng } \sigma^o) = C_\sigma^N(\text{rng } \sigma'^o)$ , where

$$\sigma = \sigma(B_{\sigma^o(j)})$$

(d)  $C_{\delta_n}^{N^n}(\text{rng } \sigma^{on}) = C_{\delta_n}^{N^n}(\text{rng } \sigma'^{on})$ , where

$$\delta_n = \sigma(B_{\sigma^{on}(j)}^n) \text{ for } n \in \Gamma_i$$

Note This is stronger than saying that

$\sigma'$  is a  $\langle \theta, B, \sigma^o(i), \sigma^o(j) \rangle, B_{\sigma^o(j)}$ -revision

of  $\sigma \upharpoonright \Gamma_i$ .

Def  $\langle \sigma, G \rangle$  is a  $\Gamma_i$ -witness iff  $\sigma$  is a  $\Gamma_i$ -reformation,  $G$  is  $B_{\sigma^o(i)}$ -generic over  $N$  and:

(a)  $G^h = (\sigma^h)^{-1} \circ G$  is  $B_{\sigma^{on}(i)}^h$ -generic over  $N^h$  for  $h \in \Gamma_i$ .

(b)  $\sigma \in N[G]$  and  $\langle \sigma^{n,i+1} \mid h \in \Gamma_i^i \rangle \in N^{i+1}[G^{i+1}]$  for  $i \leq i' < \beta$ , where  $\Gamma_i^i = \Gamma_i \cap (i+1)$ .

Note  $\Gamma_i^i = \{0\}$ ;  $\Gamma_i^j = \{0\} \cup [i+1, j]$  for  $i > 0$

Def Let  $\bar{G} \in \mathbb{B}_i^0$ .  $\langle \sigma, G \rangle$  is a witness for  $\bar{G}$

iff  $\langle \sigma, G \rangle$  is a witness and  $\bar{G} = G^0$ .

Def Let  $\langle \sigma, G \rangle$  be an  $\Gamma_i$ -witness and  $\langle \sigma', G' \rangle$  an  $\Gamma_j$ -witness, where  $i \leq j$ .

$\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$  iff

(a)  $\sigma'$  is strong over  $\sigma$

(b)  $G = G' \cap \mathbb{B}_{\sigma^0(i)}$  and

$G^h = G'^h \cap \mathbb{B}_{\sigma^0(h(i))}$  for  $h \in \Gamma_i$

where  $G^h = (\sigma^h)^{-1} \circ G$ ,  $G'^h = (\sigma'^h)^{-1} \circ G'$

(In other words,  $\langle \sigma', G' \rangle$  coheres with  $\langle \sigma, \Gamma_i, G \rangle$  wrt.  $\mathbb{B}_{\sigma^0(i)}$ .)

Note Since  $\mathbb{B}_0 = \mathbb{Z} = \mathbb{B}_0^h$  for all  $h$ , we know that  $\langle \pi, \{1\} \rangle$  is a  $\Gamma_0$ -witness for  $\{1\}$ .

Def Let  $i < \alpha^0$ .  $\bar{G} \in \mathbb{B}_i^0$  is good for  $i$  iff

for all  $l \leq i$ ,  $G_l = \bar{G} \cap \mathbb{B}_l^0$  is  $\mathbb{B}_l^0$ -generic

over  $N^0$  and  $\bar{G}_l \in N^{l+1}$ .



Def Let  $i < \aleph^0$ .  $\bar{G} \in \mathbb{B}_i^0$  is superb for  $i$  iff  $\bar{G}$  is good for  $i$  and whenever  $l \leq i \leq i'$ ,  $\langle \sigma, \bar{G} \rangle$  witnesses  $\bar{G}_l$  and  $u \in N^B$  is finite, then there is  $\langle \sigma', \bar{G}' \rangle$  witnessing  $\bar{G}_{i'}$  s.t.  $\langle \sigma', \bar{G}' \rangle$  is strong over  $\langle \sigma, \bar{G} \rangle$  and  $\sigma'$  coincides with  $\sigma$  on  $u$ .

Note This definition appears to be non-sensical, since it makes a statement about generic objects which do not exist in  $V$ . A correct formulation would say that the statement holds in  $\text{Coll}(\aleph, \omega)$  for  $\aleph \geq \bar{N}$ .

Note Taking  $l=0$  in the above definition, it follows that  $\bar{G}_i$  has a witness for each  $i' \leq i$ .

Def  $i$  is superb iff for all  $a \in \mathbb{B}_i^0 \setminus \{0\}$  there is  $\bar{G} \ni a$  which is superb for  $i$ .

Lemma 1.1 If  $i < \aleph^0$  is superb, then  $\mathbb{B}_i^0$  adds no reals to  $N^0$ .

Proof

Suppose not. There is then  $a \in \mathbb{B}_i^0 \setminus \{0\}$ ,  $t \in N^0 \setminus \mathbb{B}_i^0$  s.t. all  $\langle \bar{G} \ni a \rangle$   $t \notin \bar{V}$ . Let  $\bar{G} \ni a$  be superb for  $i$ .

Let  $\langle \sigma, \bar{G} \rangle$  witness  $\bar{G}$ . Let  $\sigma^* \supset \sigma$  s.t.  $\sigma^* \upharpoonright N[\bar{G}] \in N[\bar{G}]$ ;  $\sigma^*(\bar{G}) = a$ . Then  $t^{\bar{G}} \in N$ , since  $\bar{G} \in N^{i+1} \subset N$ . But since  $t^{\bar{G}} \notin N^0$ , we have  $t^{\bar{G}} = \sigma^*(t^{\bar{G}}) \notin N$ . Contr!

QED (1.1)

Hence it suffices to show that each  $i < d^0$  is superb, since then:

$N^0 \models N^0 \models \mathbb{B}_i^0$  adds new reals,  
hence  $N \models N^0 \models \mathbb{B}_i$  adds new reals,  
since  $\pi^0: N^0 < N$ ,  $\pi^0(\mathbb{B}_i^0) = \mathbb{B}_i$ .

We shall actually prove:

Main Claim Let  $l \leq i < d^0$ . Let  $\bar{G}$  be superb for  $l$ . Let  $a \in \mathbb{B}_i^0$  s.t.  $h_l(a) \in \bar{G}$ . Then there is  $\bar{G}' \ni a$  s.t.  $\bar{G} \subset \bar{G}'$  and  $\bar{G}'$  is superb for  $i$ .

Remark Taking  $l=0$ , it follows that  $i$  is superb. Hence the main claim proves the Theorem. We shall prove the Main Claim by induction on  $i$ . First, however, we prove some preliminary lemmas on superbtiness.

Lemma 1.2 Let  $i < d^0$ . The following are equivalent:

(a)  $\bar{G}$  is superb for  $i$

(b) Let  $l \leq i \leq i$ . Let  $l^* \leq j^* < d$  and let  $a \in \mathbb{B}_{l^*}$ ,  $\sigma, u \in V^{\mathbb{B}_{l^*}}$  s.t. whenever  $G \ni a$  is  $\mathbb{B}_{l^*}$ -generic and  $\sigma = \sigma^G$ ,  $u = u^G$ , then  $\langle \sigma, G \rangle$  witnesses  $\bar{G}_l$  and  $u \in N^B$  is finite and  $\sigma^0(j) = j^*$ . There  $G' \subset \mathbb{B}_{j^*}$ ,  $\sigma' \in V[G']$  s.t.  $a \in G'$ ,  $\langle \sigma', G' \rangle$  witnesses  $\bar{G}_j$ , and is strong over  $\langle \sigma, G \rangle$ , coinciding with  $\sigma$  on  $u$ , where  $\sigma = \sigma^G$  and  $G = G' \cap \mathbb{B}_{l^*}$ .

(c) Let  $l, j, l^*, j^*$ ,  $\sigma, u$  be as above. There is  $b \in \mathbb{B}_{j^*}$  s.t.  $h_{l^*}(b) = a$  and whenever  $G \ni b$  is  $\mathbb{B}_{j^*}$ -generic, there is  $\sigma' \in V[G']$  as in (b)

(d) Let  $l \leq i \leq i$ ,  $l^* \leq j^* < d$  and let  $G \subset \mathbb{B}_{l^*}$ ,  $\sigma \in V[G]$  be s.t.  $\langle \sigma, G \rangle$  witnesses  $\bar{G}_l$ . Let  $u \in N^B$  be finite. Let  $\sigma^0(j) = j^*$ . There is  $b' \in \mathbb{B}_{j^*}$  s.t.  $h_{l^*}(b') = 1$  and whenever  $G' \ni b'$  is  $\mathbb{B}_{j^*}$ -generic with  $G' \supset G$ , then there is  $\sigma' \in V[G']$  s.t.  $\langle \sigma', G' \rangle$  witnesses  $\bar{G}_j$ ,  $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$  and  $\sigma'$  coincides with  $\sigma$  on  $u$ .

proof of 1.2

(a)  $\rightarrow$  (b) Let  $G \ni a$  be  $\mathbb{B}_{p^*}$ -generic. Let  $\sigma = \sigma^{\circ} G$ ,  $u = u^{\circ} G$ . By (a) there exist  $\langle \sigma', G' \rangle$  with the desired properties.

(b)  $\rightarrow$  (c)

For any such  $a$  there is  $\langle \sigma', G' \rangle$  with the stated properties, by (b). Hence there is  $b_a \in G'$  which forces the existence of  $\sigma'$  with these properties. Since  $b_a, a \in G'$ , we can assume w.l.o.g.  $b_a \subset a$ . Now let  $A$  be a maximal antichain in:

$$\{ h_{p^*}(b_{a'}) \mid a' \subset a, a' \neq \emptyset \}$$

For  $c \in A$  pick  $b(c) = b_{a'}$  s.t.  $c = h_p(b_{a'})$ .

$$\text{Set: } b = \bigcup \{ b(c) \mid c \in A \}.$$

Then  $h_{p^*}(b) = \bigcup A = a$  and

$b$  has the desired property. QED (b)  $\rightarrow$  (c)

(c)  $\rightarrow$  (d) Let  $\langle \sigma, G \rangle$  be given. There

is then  $a \in \mathbb{B}_{p^*}$ ,  $\sigma^{\circ} \in V^{\mathbb{B}_{p^*}}$ ,  $u^{\circ} \in V^{\mathbb{B}_{p^*}}$

s.t.  $\sigma = \sigma^{\circ} G$ ,  $u = u^{\circ} G$ , and  $a$

forces  $\langle \sigma, \cdot \rangle$  to witness  $\bar{G}_e$  and

$u$  to be finite and  $\sigma(i) = j^*$ . Let

$b$  be as in (c). Let  $b' = b \cup \tau a$ . Then

$b'$  has the desired property.

QED (c)  $\rightarrow$  (d)

(d)  $\rightarrow$  (a) Let  $\langle G, \sigma \rangle$  witness  $\bar{G}_\beta$  with  $\sigma^\circ(\ell) = \ell^\nu$  and  $\sigma^\circ(j) = j^*$ . Let  $b'$  be as in (d). Extend  $G$  to a  $\text{IB}_{j^*}$ -generic  $G' \ni b'$ . The conclusion is immediate. QED (Lemma 1.2)

All of these definitions and results depend on an arbitrarily chosen  $\beta \geq d^\circ$ . Since we have been dealing with a single  $i < d^\circ$ , we could replace  $\beta$  by any  $\nu \leq \beta$  s.t.  $\nu \geq i$ , thus obtaining the notion of " $\nu$ -superbness". It will turn out that  $\nu$ -superbness is equivalent to  $\beta$ -superbness whenever  $\nu \geq i$ . Since we are including the case  $\nu = i$ , however, we must slightly revise our definitions.

Recall that  $\Gamma_i^\nu = \Gamma_i \cap (i+1)$ .

The definitions of " $\Gamma_i^\nu$ -tower" and " $\Gamma_i^\nu$ -pretower" are obvious. In particular,  $\pi \upharpoonright \Gamma_i^\nu$  is a tower.

The definition of "reformation" must be modified as follows:

Def Let  $i < d^0$ ,  $i \leq \nu \leq \beta$ . By a  $\Gamma_i^\nu$ -reformation of  $\pi \upharpoonright (\nu+1)$  we mean a  $\Gamma_i^\nu$ -pretower  $\sigma$  s.t.

(a)  $\sigma \upharpoonright [i+1, \nu]$  is a revision of  $\pi \upharpoonright [i+1, \nu]$  if  $\nu > i$ .

(b)  $\text{rng } \sigma \circ \pi^{i, i+1} \subset \bigcup \text{rng } (\pi^{i, i+1})$  if  $\nu > i$

(c)  $\text{rng } \sigma^0 \subset \bigcup \text{rng } \pi^i$  if  $\nu = i$ .

The definition of " $\sigma'$  respects  $\sigma$  at  $x$ " is virtually unchanged. "Coincidence" must be redefined as follows:

Def Let  $\sigma, \sigma'$  be reformations of  $\pi \upharpoonright (\nu+1)$ , where  $\sigma$  is a  $\Gamma_i^\nu$ -pretower,  $\sigma'$  is a  $\Gamma_j^\nu$ -pretower, and  $i \leq j$ .

Let  $u \in N^{\hat{\nu}}$ , where  $\hat{\nu} = \max \Gamma_j^\nu$ .

$\sigma'$  coincides with  $\sigma$  on  $u$

iff  $\sigma'$  coincides with  $\sigma \upharpoonright \Gamma_j^\nu$  on  $u$ .

(Note  $\hat{\nu} = \nu$  if  $\nu > i$ ;  $\hat{\nu} = 0$  if  $\nu = i$ .)

The definition of "strong" must be altered

The definition of "strong over" is as before with  $\Gamma_i^v, \Gamma_j^v$  in place of  $\Gamma_i, \Gamma_j$  (except that (b) is understood as holding vacuously if  $v=j$ ).

The definition of "witness" is also virtually unchanged, as is the def. of " $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$ ".

The definition of "superb" requires a small subtle change:

Def Let  $i < \alpha$ ,  $i \leq v \leq \beta$ .

$\bar{G} \in IB_i^\alpha$  is  $v$ -superb for  $i$  iff

if  $\bar{G}$  is good and whenever  $l \leq i \leq v$ ,  $\langle \sigma, G \rangle$  witnesses  $\bar{G}_l$  and  $u \in \mathbb{N}^{\hat{v}}$  is finite,

then there is  $\langle \sigma', G' \rangle$  witnessing  $\bar{G}_l$  s.t.  $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$  and

$\sigma'$  coincides with  $\sigma$  on  $u$ , where  $\hat{v} = \max_j \Gamma_j^v$ .

(Hence  $\hat{v} = v$  unless  $v = i = i$  and  $\hat{v} = 0$ .)

In (b), (c) of Lemma 1.2 we must similarly replace " $u \in \mathbb{N}^\beta$ " by " $u \in \mathbb{N}^{\hat{v}}$ ".

The proof of Lemma 1.2 then goes through as before.

Lemma 1.3 Let  $\nu \leq \beta$ . The set:

$$\{ \langle \bar{G}, i \rangle \mid i \leq \nu, i < \alpha^0 \text{ and } \bar{G} \text{ is } \nu\text{-super for } i \}$$

is uniformly  $N$ -definable in the parameters  $\theta, IB, \pi \upharpoonright (\nu+1)$ .

Proof (sketch).

We first note that, since  $IB \in H_\theta \subset N$ , the various terms  $\sigma, \dot{\nu}$  etc. involved in the definition can be taken as lying in  $N$ . However, the definition will involve statements of the form:

$$a \Vdash_{IB} \sigma^k : \check{N}^k < \check{N},$$

which are not expressible in  $N$ . We note that in all such cases it will also be forced that  $a \Vdash \text{rng } \sigma^k \subset \check{N}^*$ ,

where  $N^* = \cup \pi^\nu$ . But  $N^* \in N$  and  $N^* < N$ , so we can replace  $\check{N}$  by  $\check{N}^*$  in the above statement.

QED (Lemma 1.3)

But  $\pi^{\nu+1} : N^{\nu+1} < N$  and  $\sigma^{\nu+1}(\tilde{\pi}) = \pi \upharpoonright (\nu+1)$ , where

$$\tilde{\pi} = \langle \pi^{h, \nu+1} \mid h \leq \nu \rangle,$$

Hence:



Corollary 1.4 Let  $\nu < \beta$ . The set

$\{ \langle \bar{G}, i \rangle \mid i \leq \nu, i < d^0 \text{ and } \bar{G} \text{ is } \nu\text{-superb for } i \}$   
 is uniformly  $N^{\nu+1}$ -definable in the  
 parameters  $\theta^{\nu+1}, B^{\nu+1}, \hat{\pi}$ , where  
 $\hat{\pi} = \langle \pi^h, \nu+1 \mid h \leq \nu \rangle$ .

Using this we prove:

Lemma 1.5 Let  $\nu < \beta$ . Let  $i < d^0$  s.t.  $i \leq \nu$ .  
 Let  $\bar{G} \in B_i^0$ . Then  $\bar{G}$  is superb for  $i$  iff  
 iff  $\bar{G}$  is  $\nu$ -superb for  $i$ .

prf.

( $\rightarrow$ ) By Cor 1.4 it suffices to show that  
 $\bar{G}$  is  $\nu$ -superb in  $N^{\nu+1}$ . We use criterion  
 (b) in Lemma 1.2. Let  $l \leq j \leq i, l^* \leq j^*$

Let  $a \in B_{j^*}^{\nu+1}, \sigma, u \in (N^{\nu+1}) \upharpoonright B_{l^*}^{\nu+1}$  s.t.

$G \ni a$  is  $B_{l^*}^{\nu+1}$ -generic over  $N^{\nu+1}$ , then,

letting  $\sigma = \sigma \upharpoonright G, u = u \upharpoonright G, u \in N^{\hat{\nu}}$  is finite and  
 $\langle \sigma, G \rangle$  witnesses  $\bar{G}_l$ , and  $\sigma^o(l) = l^*, \sigma^o(j) = j^*$ .

Claim There is  $G' \in B_{j^*}^{\nu+1}$  s.t.  $G'$   
 $B_{j^*}^{\nu+1}$ -generic over  $N^{\nu+1}$  and there is  $\sigma' \in N^{\nu+1}[G']$

s.t.  $\langle \sigma', G' \rangle$  witnesses  $\bar{G}_j$  and,

letting  $G = G' \upharpoonright B_{i^*}^{\nu+1}, \sigma = \sigma \upharpoonright G, u = u \upharpoonright G$ ,

we have:  $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$

and  $\sigma'$  coincides with  $\sigma$  on  $u$ .

Define  $\tilde{l} < \tilde{j} < 2$  by:  $\tilde{l} = \pi^{\nu+1}(l^*)$ ,  $\tilde{j} = \pi^{\nu+1}(j^*)$ .

By  $\beta$ -subproperness there is  $\tilde{a} \in \mathbb{B}_{\tilde{l}}$  and  $\tilde{\sigma} \in N^{\mathbb{B}_{\tilde{l}}}$  s.t. whenever  $\tilde{G} \ni \tilde{a}$  is  $\mathbb{B}_{\tilde{l}}$ -generic and  $\tilde{\sigma} = \tilde{\sigma}'' \tilde{G}$ , then:

- $\tilde{\sigma}$  is a  $\langle \theta, \mathbb{B}, \tilde{l}, \tilde{j} \rangle, \mathbb{B}_{\tilde{l}}, \tilde{G}$ -revision of  $\pi \upharpoonright [\nu+1, \beta]$
- $\tilde{a} \in \tilde{G}^{\nu+1}$
- $\tilde{\sigma}$  coincides with  $\pi \upharpoonright [\nu+1, \beta]$  on  $\tilde{u}$ , where  $\tilde{u} = \pi^{\nu+1, \beta} \circ \tilde{\sigma}^{\wedge} \circ u$ .

Extend the domain of  $\tilde{\sigma}$  to  $\Gamma_{\tilde{l}}$  by setting:  $\tilde{\sigma}^h = \tilde{\sigma}^{\nu+1} \circ \sigma^h$  for  $h \in \Gamma_{\tilde{l}}^{\nu+1}$ . Then  $\langle \tilde{\sigma}, \tilde{G} \rangle$  witnesses  $\bar{G}_{\tilde{l}}$ .

By superbness there is  $\langle \tilde{\sigma}', \tilde{G}' \rangle$  s.t.  $\langle \tilde{\sigma}', \tilde{G}' \rangle$  witnesses  $\bar{G}_{\tilde{l}}$  and is strong over  $\langle \tilde{\sigma}, \tilde{G} \rangle$  and  $\tilde{\sigma}'$  coincides with  $\tilde{\sigma}$  on  $\tilde{u}$ . But then  $\langle \tilde{\sigma}', \tilde{G}' \rangle$  has the desired properties, where  $\tilde{\sigma}' = (\tilde{\sigma}')^{\nu+1}$ ,  $\tilde{G}' = (\tilde{G}')^{\nu+1}$ .

QED ( $\rightarrow$ )

( $\leftarrow$ ) Let  $\langle G, \sigma \rangle$  witness  $\bar{G}_\ell$ , where  $\sigma^o(l) = l^*$ ,  $\sigma^o(j) = j^*$   
 Let  $u \subset N^\beta$  be finite.

Claim There are  $G', \sigma'$  s.t.  $G'$  is  $B_{j^*}^{\nu}$ -generic,  
 $\sigma' \in N[G']$ ,  $\langle \sigma', G' \rangle$  witness  $\bar{G}_j$  and is  
 strong over  $\langle \sigma, G \rangle$ . Moreover  $\sigma'$  coincides with  
 $\sigma$  on  $u$ .

Prf.

Set  $\tilde{G} = G^{\nu+1}$ ,  $\tilde{\sigma} = \langle \sigma^{h, \nu+1} \mid h \in \Gamma_\ell^{\nu+1} \rangle$

Then  $\langle \tilde{G}, \tilde{\sigma} \rangle$  witness  $\bar{G}_\ell$  in the sense of  
 $N^{\nu+1}$ . Set  $\tilde{u} = (\sigma^{\hat{\nu}})^{-1} \upharpoonright u$ , where

$\hat{\nu} = \max \Gamma_j^\nu$ . We apply Lemma 1.2 (d) to

$\langle \tilde{G}, \tilde{\sigma} \rangle$ . Let  $\tilde{\sigma}^o(l) = \tilde{l}$ ,  $\tilde{\sigma}^o(j) = \tilde{j}$ . There

is  $\tilde{b} \in B_{\tilde{j}}^{\nu+1}$  s.t. whenever  $\tilde{G}' \ni \tilde{b}'$  is  $B_{\tilde{j}}^{\nu+1}$ -

generic over  $N^{\nu+1}$  and  $\tilde{G}' \supset \tilde{G}$ , there is

$\tilde{\sigma}' \in N^{\nu+1}[\tilde{G}']$  s.t.  $\langle \tilde{\sigma}', \tilde{G}' \rangle$  witness  $\bar{G}_j$ , is strong

over  $\langle \tilde{\sigma}, \tilde{G} \rangle$  and  $\tilde{\sigma}'$  coincides with  $\tilde{\sigma}$  on  $\tilde{u}$ .

Moreover  $h_{\tilde{l}}^{\nu+1}(\tilde{b}') = 1$ .

Subclaim There is  $\langle \sigma', G' \rangle$  s.t.

(a)  $\sigma'$  is a  $\langle \theta, B, i^*, j^* \rangle, B_{j^*}^{\nu}, G$ -revision  
 of  $\sigma \upharpoonright [j+1, \beta]$

(b)  $\sigma'$  coheres with  $\sigma \upharpoonright [\nu+1, \beta]$  with  $B_{j^*}^{\nu}$

(c)  $\sigma'$  coincides with  $\sigma \upharpoonright [\nu+1, \beta]$  on  $u$ .

Moreover:

$$(d) C_{\delta}^N(\text{rng}(\sigma')^{\nu+1}, \bar{\sigma}^{\nu}) = C_{\delta}^N(\text{rng} \sigma^{\nu})$$

where  $\delta = \delta(B_{j^*})$

$$(e) C_{\delta_h}^{N^h}(\text{rng}(\sigma')^{\nu+1, h}, \bar{\sigma}^{\nu, h}) = C_{\delta_h}^{N^h}(\text{rng} \sigma^{\nu, h})$$

for  $\nu < h \leq \beta$ , where  $\delta_h = \delta(\sigma^{\circ h}(B_{j^*}))$ .

proof

We first note that for any  $t \in N^{\beta}$ , there is  $\langle \sigma', G' \rangle$  satisfying (a) - (c) with the additional property that  $\sigma'$  coincides with  $\sigma \upharpoonright [\nu+1, \beta]$  at  $t$ . This follows by §2 Lemma 2. (Let  $a \in G, \bar{\sigma} \in N^{B_{j^*}}$  s.t.

a forces  $\sigma = \bar{\sigma} \upharpoonright G$  to be s.t.,  $\langle G, \sigma \rangle$  is a witness for  $\bar{\sigma}$  and  $\sigma^{\circ}(l) = l^*, \sigma^{\circ}(j) = j^*$ .)

There is  $a' \in B_{j^*}$  s.t.  $h_{j^*}(a') = a$  and  $a'$  forces that if  $G' \ni a$  is  $B_{j^*}$ -generic, there is  $\sigma' \in N^{B_{j^*}}$  satisfying (a) - (c) and coinciding with  $(\sigma \upharpoonright [\nu+1, \beta])$  at  $t$ . Set  $b = a' \cup (1 \upharpoonright a)$ . Let  $G' \supset G$  be  $B_{j^*}$ -generic s.t.  $b \in G'$ . Then  $a' \in G'$  and the conclusion follows.)

Note that  $\bar{c} = C_{\delta_{\nu+1}}^{N^{\nu+1}}(\text{rng} \sigma^{\circ, \nu+1}) \in$

$N^{\nu+1}[G^{\nu+1}]$  and there is

$\bar{t} \in N^{\nu+1} B_{j^*}^{\nu+1}$  s.t.  $\bar{t} \upharpoonright G^{\nu+1} = \bar{c}$ . Let

$\sigma'$  coincide with  $\sigma \upharpoonright [\nu+1, \beta]$  on  $\bar{t} = \sigma^{\circ, \nu+1}(\bar{t})$ .

then, letting  $C = C_{\sigma}^N(\text{rang } \sigma^0)$ , we have:

$$(1) C = \sigma^{*0}(\bar{C}) = \sigma^{*0}(t^{-G^{v+1}}) = t^G$$

where  $\sigma^*$  is the canonical completion of  $\sigma$ . However:

$$(2) \bar{C} = C_{\sigma_{v+1}}^{N^{v+1}}(\text{rang } \tilde{\sigma}'_0)$$

Letting  $\sigma'^*$  be the canonical completion of  $\sigma'$  we have:

$$(3) (\sigma'^*)^{v+1}(\bar{C}) = C_{\sigma}^N(\text{rang}((\sigma'^*)^{v+1}(\tilde{\sigma}'_0))) = C_{\sigma}^N(\text{rang}(\sigma'^{v+1} \circ \tilde{\sigma}'_0))$$

But:

$$(4) (\sigma'^*)^{v+1}(\bar{C}) = (\sigma'^*)^{v+1}(t^{-G^{v+1}} \cap B_{\ell}^{v+1}) = \sigma'^{v+1}(t^{-G} \cap B_{\ell}^v) = t^G = C.$$

This proves (d). The proof of (e) is entirely similar. QED (Subclaim)

Extend the domain of  $\sigma'$  to  $\Gamma_j$  by setting  $\sigma'^h = \sigma'^{v+1} \tilde{\sigma}'^h$  if  $h \in \Gamma_j^v$ .

$\langle \sigma', G' \rangle$  is easily seen to have the desired properties. QED (Lemma 1.5)

Combining Lemmas 1.4 and 1.5 we easily get:

Lemma 1.6

(a)  $\{ \langle \bar{G}, i \rangle \mid i < \alpha, \bar{G} \text{ is superb for } i \}$  is uniformly  $H_{\omega_1}$  definable in the parameter

$$\langle \pi^{h\beta} \mid h < \beta \rangle,$$

(b)  $\{ \langle \bar{G}, i \rangle \mid i < \nu, \bar{G} \text{ is superb for } i \}$

is uniformly  $H_{\omega_1}^{N^{\nu+1}}$  - definable in the parameter

$$\langle \pi^{hr} \mid h \leq \nu \rangle \text{ for } \nu \leq \beta$$

We are now ready to prove the Main Claim.  
 We proceed by induction on  $i$ . The case  $i=0$  is trivial. There remain two cases.

Case 1  $i = \gamma + 1$

By the induction hypothesis it suffices to prove the claim for  $l = \gamma$ . Let  $\bar{G}$  be superb for  $\gamma$ . We know:

$H_\theta \models \exists \bar{z}, \bar{D} \langle \bar{z}, \bar{D} \rangle$  witnesses the Dec -  
 - subproperness of  $\mathbb{B}_{\pi^0(i)} / \bar{G}$

Hence the same holds for  $H_{\theta_0}^{N^0}$ , and there are  $\bar{z}, \bar{D} \in H_\theta^{N^0[\bar{G}]}$  s.t.

$H_\theta^{N^0[\bar{G}]} \models \langle \bar{z}, \bar{D} \rangle$  witnesses the subproperness of  $\mathbb{B}_i / \bar{G}$ .

$\bar{G}$  adds no new reals to  $N^0$ , since  $\gamma$  is superb. Thus  $H_{\omega_1}^{N^0} = H_{\omega_1}^{N^0[\bar{G}]}$  and

$\bar{D} \in N^0$  is defined by:

$$\bar{D}(M, A) = \{ A_u \mid u \in H_{\omega_1}^{N^0} \} \text{ where}$$

$$A_u = \{ G \mid H_{\omega_1}^{N^0} \models \varphi[M, A, G, u, a] \}$$

where  $a \in H_{\omega_1}^{N^0}$ .

But then  $\bar{D} = \pi^0(\bar{D})$  has the same definition from  $\varphi, a$ , with  $H_{\omega_1}^{N^0}$  in place of  $H_{\omega_1}^{N^0}$ .

It is clear that, in fact,  $\sigma(\mathbb{D}) = \mathbb{D}$   
for any  $\sigma: N^0 \prec N$ .

Set:  $\tilde{N}^0 = L_{\tilde{\sigma}^0}^{A^0} / \bar{G}$ , where  $N^0 = L_{\sigma^0}^{A^0}$ .

Set:  $\tilde{B}^0 = B_i^0 / \bar{G}$ . Then  $\tilde{N}^0, \tilde{B}^0 \in N^i$ ,  
since  $\bar{G} \in N^i$ . We know:

$$\mathbb{D}(\tilde{N}^0, \tilde{B}^0) = \{A_u \mid u \in H_{\omega_1}\} \text{ where}$$

$$A_u = \{G \mid H_{\omega_1} \models \varphi[\tilde{N}^0, \tilde{B}^0, \bar{G}, u, a]\}.$$

Set  $X = \bigcap \{A_u \mid u \in N^i\}$ . Since  $N^i$   
is countable we know that  $X \neq \emptyset$   
and, in fact, for every  $a \in \tilde{B}^0 \setminus \{0\}$

there is  $G \in X$  with  $a \in G$ . Fix  
 $a \in \tilde{B}^0 \setminus \{0\}$  with  $h_{\eta}(a) \in \bar{G}$  and set

$\tilde{G}^0 =$  the  $N$ -least  $G \in X$  s.t.  $a/\bar{G} \in G$ .

Since  $\pi^i: N^{i+1} \prec N$ , it follows that  
 $\tilde{G}^0 \in N^{i+1}$ . Set:

$$\bar{G}' = \bar{G} * \tilde{G}^0 = \bigcap \{b \in B_i^0 \mid b/\bar{G} \in \tilde{G}^0\}.$$

Then  $\bar{G}'$  is  $B_i^0$ -generic over  $N^0$ .  
Moreover,  $a \in \bar{G}'$ .

Claim  $\bar{G}'$  is superb for  $i$ .

By Lemma 1.5 it suffices to show  
that  $\bar{G}'$  is  $i$ -superb for  $i$ .



We must show that if  $l \leq i' \leq c$ ,  $\langle \sigma, G \rangle$  is an  $i'$ -witness for  $\bar{G}_j'$  and  $u \in N^{\hat{j}}$  is finite, then there is an  $i'$ -witness  $\langle \sigma', G' \rangle$  for  $\bar{G}_j'$  which is strong over  $\langle \sigma, G \rangle$  and s.t.  $\sigma'$  coincides with  $\sigma$  on  $u$ . By the induction hypothesis, however, we need only prove this for  $l = \gamma, i' = i$ . (Hence  $\hat{j} = \max \Gamma_{i'}^{i'} = 0$ .)

Let  $\langle \sigma, G \rangle$  be a  $\Gamma_{\gamma}^{i'}$ -witness for  $\bar{G}$ . (Hence  $\sigma = \{ \langle \sigma^0, 0 \rangle, \langle \sigma^i, i \rangle \}$ .) Let  $u \subset N^0$  be finite. Let  $\tilde{\sigma}$  = the canonical completion of  $\sigma$ . Then:

$$\tilde{\sigma}^0: N^0[\bar{G}] \prec N[G], \quad \tilde{\sigma}^0(\bar{G}) = G.$$

Set:  $\tilde{N} = L_{\tau}^{A/G}$ . Hence  $\tilde{\sigma}^0: \tilde{N}^0 \prec \tilde{N}$ .

We know that  $\sigma^0(\mathbb{D}) = \mathbb{D}$  and that there is  $\zeta \in \theta^0$  s.t.

$$H_{\theta^0}^{\tilde{N}^0} = \langle \zeta, \mathbb{D} \rangle \text{ witnesses the Dec-} \\ \text{-subproperness of } \tilde{B}^0$$

$$\text{Hence in } H_{\theta}^{\tilde{N}} = H_{\theta}^V[G] = H_{\theta}[G]$$

we have:

$$\langle \sigma^0(\zeta), \mathbb{D} \rangle \text{ witnesses the Dec-} \\ \text{-subproperness of } \tilde{B} = \tilde{B}_{\sigma^0(i)} / G.$$

But then this statement holds in  $V[G]$ , since  $H_{\theta}^{\tilde{N}} \leq \sum_n H_{\theta}^{V[G]}$  for

a large  $n$ . Since  $\theta > (2^{\sigma^0(3)})^+$ ,

we conclude that  $\langle \theta, \mathbb{D} \rangle$  witnesses the  $\mathbb{D}$ -subproperness of  $\tilde{B}$

in  $V[G]$ . But then there is  $v \in H_{\omega_1}$

s.t. for all  $g \in A_v$  and all finite

$u' \in \tilde{N}^0$  there is  $b \in \tilde{B} = \tilde{B}_{\sigma^0(1)} / G$  s.t.

whenever  $\tilde{G} \ni b$  is  $\tilde{B}$ -generic, then

there is  $\tilde{\sigma}' \in V[G][\tilde{G}]$  with:

- $\tilde{\sigma}' : \tilde{N}^0 \rightarrow \tilde{N}$

- $\tilde{\sigma}' \upharpoonright u' = \tilde{\sigma}^0 \upharpoonright u'$

- $C_{\tilde{\sigma}'}^{\tilde{N}}(\text{rng } \tilde{\sigma}') = C_{\tilde{\sigma}^0}^{\tilde{N}}(\text{rng } \tilde{\sigma}^0)$

where  $\tilde{\sigma} = \delta(\tilde{B})$  in  $V[G]$

- $\tilde{\sigma}' \upharpoonright g \in \tilde{G}$ .

This means, in particular, that  $\text{rng } \tilde{\sigma}' \subset \bigcup \text{rng } (\tilde{\sigma}^0) = N^*$  where

$N^* \in N$ ,  $N^* \prec N$ . Hence the

condition  $\tilde{\sigma}' : \tilde{N}^0 \rightarrow \tilde{N}$  can be

equivalently formulated as:

$$\tilde{\sigma}' : \tilde{N}^0 \rightarrow N^*$$

Since  $\tilde{\sigma}^i: \tilde{N}^i \rightarrow \tilde{N}$  with  $\tilde{\sigma}^i(\tilde{\sigma}^{0,i}) = \tilde{\sigma}^0$   
 and  $\tilde{\sigma}^i(\cup \text{rng } \tilde{\sigma}^{0,i}) = N^*$ , it follows  
 that there is such a  $v \in H_{w_1}^{N^*}$ . But  
 then  $X \subset A_v$ . Hence, letting  
 $u' = u \cup \{i, \gamma, \theta, \beta^0\}$ , there is  $b \in \tilde{G}$   
 s.t. whenever  $\tilde{G} \ni b$  is  $\tilde{B}$ -generic,  
 then the above holds with  $g = \tilde{G}$ .

Then  $\tilde{\sigma}' \in V[G][\tilde{G}] = V[G']$ ,

where  $G' = G * \tilde{G} = \{b \in \mathcal{B}_{\tilde{\sigma}^{0,i}} \mid b/G \in \tilde{G}\}$ .

Set:  $\sigma'^0 = \tilde{\sigma}' \upharpoonright N^0$ ;  $\sigma' = \{\langle \sigma'^0, 0 \rangle\}$ .

It suffices to show:

Claim  $\langle \sigma' \upharpoonright G' \rangle$  witnesses  $\tilde{G}$ ,  
 is strong over  $\langle \sigma, G \rangle$ , and  
 $\sigma'$  coincides with  $\sigma$  on  $u$ .

The verification is straight-  
 forward. We first note that  
 by the usual proof:

$$C_{\tilde{\sigma}}^N(\text{rng } \sigma'^0) = C_{\tilde{\sigma}}^N(\text{rng } \sigma^0)$$

where  $\tilde{\sigma} = \tilde{\sigma}(\mathcal{B}_{\tilde{\sigma}^{0,i}})$ , (cf. proof of §1 Lemma 1)

Hence  $\text{rng}(\sigma'^0) \subset \cup \text{rng}(\sigma^0) \in$

$$\cup \text{rng}(\sigma^i) = \cup \text{rng}(\pi^i)$$

Thus  $\sigma'$  is an  $i$ -reformulation of  $\tau \circ \sigma$ .  
 $\langle \sigma', G' \rangle$  witnesses  $\bar{G}$ , since  $\bar{G} = G'^{\circ} =$   
 $= (\sigma^{\circ})^{-1} \circ G$ ,  $\sigma'$  is obviously an  
 $\langle \theta^{\circ}, B^{\circ}, \gamma, i \rangle$ -reformulation of  $\sigma$ ,  
since  $\sigma'^{\circ}(\theta^{\circ}, B^{\circ}, \gamma, i) = \theta, B, \sigma^{\circ}(\gamma), \sigma^{\circ}(i)$ ,  
But  $\sigma'$  coheres with  $\sigma$  wrt.  $B_{\sigma^{\circ}(\gamma)}$ ,  
since  $G' \supset G$  and  $\bar{G}' \supset \bar{G}$ . Hence  
 $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$ . But  
 $\sigma'^{\circ} \upharpoonright u = \sigma^{\circ} \upharpoonright u$ . Hence  $\sigma'$  coincides  
with  $\sigma$  on  $u$ . QED (Case 1)

Case 2  $i = \lambda$ ,  $\text{Lim}(\lambda)$ .

Def  $A = A_\lambda =$  the set of  $\bar{G} \in \mathcal{B}_\lambda^0$  s.t.  $\bar{G}$  is  $\mathcal{B}_\lambda^0$ -generic over  $N^0$  and  $\bar{G}_\ell$  is superb for  $\ell$  for all  $\ell < \lambda$ , where  $\bar{G}_\ell = \mathcal{B}_\ell^0 \cap \bar{G}$ .

We prove:

Claim 1 Let  $\ell < \lambda$  and let  $\bar{G}$  be superb for  $\ell$ . Let  $a \in \mathcal{B}_\lambda^0 \setminus \{0\}$  s.t.  $h_\ell(a) \in \bar{G}$ . There is  $\bar{G}' \in A$  s.t.  $a \in \bar{G}'$ .

Claim 2 Every element of  $A$  is superb for  $\lambda$ .

We first prove Claim 1. Let  $\langle \bar{\beta}_i \mid i < \lambda \rangle$  be monotone and cofinal in  $\lambda$  s.t.  $\bar{\beta}_0 = \ell$ . Since we are doing an RCS-iteration, the set:

$$\Delta = \{b \in \mathcal{B}_\lambda^0 \mid b = \bigcap_{r < \lambda} h_r(b)\}$$

is dense in  $\mathcal{B}_\lambda^0$ . Call an ultrafilter  $G$  on  $\mathcal{B}_\lambda^0$  good if  $G^+ \subset G$ , where  $G^+ = \{b \in \Delta \mid \bigwedge r < \lambda h_r(b) \in G\}$ . Exactly as in §1 Thm 3 (Case 1) we choose a "master sequence"  $\langle b_i \mid i < \omega \rangle$  s.t.

(a)  $b_i \in \mathcal{B}_{\bar{\beta}_i}^0 \setminus \{0\}$ ,  $b_i = h_{\bar{\beta}_i}(b_j)$  for  $i \leq j$ ,  $b_i \in h_{\bar{\beta}_i}(a)$  (where w.l.o.g.  $a \in \Delta$ )

(b) If  $G$  is a good ultrafilter and  $b_i \in G$  for  $i < \omega$ , then  $G$  is  $\mathcal{B}_\lambda^0$ -generic.

We then successively pick  $\bar{G}_i \subset \mathbb{B}_{\bar{z}_i}^0$  s.t.

$\bar{G}_0 = \bar{G}$ ,  $\bar{G}_{i+1} \supset \bar{G}_i$  is superb for  $\bar{z}_{i+1}$  s.t.

$b_{i+1} \in \bar{G}$ . Set  $\tilde{G} = \bigcup_i \bar{G}_i$ . Then

$$\tilde{G}^{++} = \{ b \in \mathbb{B}_\lambda^0 \mid \forall b' \in \tilde{G}^+ \quad b' \subset b \} \in A.$$

QED (Claim 1).

We now turn to the proof of Claim 2.

We shall closely imitate the proof in Case 2 of §1 Thm 3. Let  $\bar{G} \in A$  be given.

By Lemma 1.5 it suffices to show that  $\bar{G}$  is  $\lambda$ -superb for  $\lambda$ . We employ criterion (b) of Lemma 1.2. Let  $l < j \leq \lambda$ ,  $l^* \leq j^* < d$ .

Let  $a \in \mathbb{B}_{l^*}$ ,  $\sigma \in V^{\mathbb{B}_{l^*}}$  s.t. whenever  $G \ni a$  is  $\mathbb{B}_{l^*}$ -generic and  $\sigma = \dot{\sigma} \upharpoonright G$ , then

$\langle \sigma, G \rangle$  witnesses  $\bar{G}_l$  and  $\sigma(l) = l^*$ ,  $\sigma(j) = j^*$ .

Let  $u \in V^{\mathbb{B}_{j^*}}$  s.t.  $\text{alt}(u \in \mathbb{N}^{\hat{v}})$  is finite. ( $\hat{v} = \max\{j^*, \lambda\}$ )

Claim There is  $\langle \sigma', G' \rangle$  s.t.

- $G'$  is  $\mathbb{B}_{j^*}$ -generic and  $a \in G'$
- $\langle \sigma', G' \rangle$  witnesses  $\bar{G}_j$
- $\langle \sigma', G' \rangle$  is strong over  $\langle \sigma, G \rangle$ , where  $G = G' \cap \mathbb{B}_{l^*}$ ,  $\sigma = \dot{\sigma} \upharpoonright G$
- $\sigma'$  coincides with  $\sigma$  on  $u = \dot{u} \upharpoonright G$ .

Since  $\bar{G}_j$  is superfluous for  $j$  when  $j < \lambda$ , it suffices to prove the claim for  $j = \lambda$ . <sup>Hence  $\hat{v} = 0$ .</sup> Similarly, we can choose  $l < \lambda$  as large as we like, so we ensure that

(\*)  $\forall \alpha \text{ cf}(\lambda) < \tilde{\delta} = \sup_{\nu < \lambda} \delta(|B_\nu^\circ|)$  in  $N^\circ$ , then

$\text{cf}(\lambda) < \delta(|B_\nu^\circ|)$  for a  $\nu < l$  in  $N^\circ$

We are assuming  $\forall \alpha \text{ } (\sigma^\circ(\check{\alpha}) = \check{\alpha}^* \wedge \sigma^\circ(\lambda^\nu) = \lambda^{\check{\alpha}*})$ .

We may also assume w.l.o.g. that there are  $u, \gamma$  int.  $\lambda \in u$  and:

$\forall \alpha \text{ } (u = \check{u} \wedge \sup \sigma^\circ \text{ " } \lambda^\nu = \gamma^\nu)$ ,

We then get:

(1) Let  $\langle \sigma', G' \rangle$  be as in the above Claim for a  $j' < \lambda$ . Then  $\sup \sigma' \text{ " } \lambda = \gamma$ .

proof:

Case A  $\text{cf}(\lambda) < \delta(|B_\nu^\circ|)$  in  $N^\circ$  for a  $\nu < l$ .

Then  $\text{cf}(\lambda) \leq \omega_1$  in  $N^\circ[\bar{G}_\nu]$ . Let  $\bar{f} =$

$=$  the  $N^\circ[\bar{G}_\nu]$ -least  $\bar{f}: \omega_1 \rightarrow \lambda$  int.

$\lambda = \sup \bar{f} \text{ " } \omega_1$ . Let  $\langle \sigma, G \rangle, \langle \sigma', G' \rangle$  be as in

the above claim and let  $f =$

$=$  the  $N[G]$ -least  $f: \omega_1 \rightarrow \lambda$  int.

$\lambda = \sup f \text{ " } \omega_1$ . Let  $\sigma^*$  be the unique extension of  $\sigma^\circ$  int.  $\sigma^*: N[\bar{G}_\nu] \times N[G]$

and  $\sigma(\bar{G}_\nu) = G$ . Let  $\sigma^{*'} \text{ be the}$

unique extension of  $\sigma'^0$  s.t.  $\sigma^{*'}: N[\bar{G}_1] \rightarrow N[G]$   
 and  $\sigma^{*'}(\bar{G}_2) = G$ . (This exists since  $\sigma'^0 \bar{G}_2 \subset G$ .)

Clearly  $\sigma^*(\bar{f}) = \sigma^{*'}(\bar{f}) = f$ . Hence

$$\sup \sigma^0 \text{''} \lambda = \sup \sigma'^0 \text{''} \lambda = \sup f \text{''} \omega_1^{N^0}.$$

QED (Case A)

Case B Case A fails. Then  $\lambda$  is regular in  $N^0$  and  $\lambda > \delta(B_{j^*}^0)$  for  $v < \lambda$ . It follows easily

$$\text{that } \sup \sigma^0 \text{''} \lambda = \sup \lambda^* \cap C_{\delta(B_{j^*}^0)}^N (\text{rng } \sigma^0) =$$

$$= \sup \lambda^* \cap C_{\delta(B_{j^*}^0)}^N (\text{rng } \sigma'^0) = \sup \sigma'^0 \text{''} \lambda.$$

QED (1)

Fix a sequence  $\langle \gamma_i \mid i < \omega \rangle$  monotone and  
 cofinal in  $\gamma$  s.t.  $\gamma_0 = l^*$ . We also  
 fix a sequence  $\langle \bar{\gamma}_i \mid i < \omega \rangle$  monotone  
 and cofinal in  $l$  s.t.  $\bar{\gamma}_0 = l$ .



We construct  $a_i, \dot{\tau}_i$  ( $i < \omega$ ) s.t.

I (a)  $a_i \in \text{IB}_{\gamma_i}$ ,  $h_{\gamma_i}(a_i) = a_j$  for  $i \leq j$ ,  $a_0 = a$

(b)  $\dot{\tau}_i \in \mathcal{V}^{\text{IB}_{\gamma_i}}$  with all  $\dot{\tau}_i = \dot{\sigma}_i^0$

II Let  $G \ni a_i$  be  $\text{IB}_{\gamma_i}$ -generic. Set:

$G_\nu = G \cap \text{IB}_{\gamma_\nu}$  ( $\nu \leq \gamma_i$ ),  $\bar{\tau}_\ell = \dot{\tau}_\ell^{G_\nu}$  ( $\ell \leq i$ ). Then:

(a)  $\bar{\tau}_i : \mathbb{N}^0 \prec \mathbb{N}$

(b)  $\bar{\tau}_i(\theta^0, \text{IB}_i, \lambda) = \theta, \text{IB}_i, \lambda^*$

(c)  $\sup \bar{\tau}_i \lambda = \gamma$

(d)  $\bar{\tau}_i(\bar{\tau}_\ell) = \bar{\tau}_h(\bar{\tau}_\ell)$  for  $h \leq i, \ell \leq m_h$ , where

$m_h =$  the least  $m$  s.t.  $\bar{\tau}_h(\bar{\tau}_m) \succ \gamma_{h+1}$

Simultaneously we define  $\dot{\tau}_i \in \mathcal{V}^{\text{IB}_{\gamma_i}}$  s.t.

III Let  $G \ni a_i$  be as in II. Set:  $\Gamma_\ell = \dot{\tau}_\ell^{G_\nu}$  ( $\ell \leq i$ ).

Then  $\Gamma_i = \langle \langle \bar{\tau}_j, \dot{\sigma}_j, b_j, \dot{u}_j \rangle \mid j \leq m_i \rangle$  where

(a)  $\Gamma_h \subset \Gamma_i$  for  $h < i$ .

(b)  $\bar{\tau}_j = \bar{\tau}_i(\bar{\tau}_j)$  for  $j < m_i$

(c)  $b_j \in \text{IB}_{\bar{\tau}_j} \setminus \{0\}$ ,  $b_h = h_{\bar{\tau}_h}(b_j)$  for  $h \leq j \leq m_i$

(d)  $\dot{\sigma}_j, \dot{u}_j \in \mathcal{V}^{\text{IB}_{\bar{\tau}_j}}$  for  $j \leq m_i$

Moreover, we have:

IV Let  $G, \Gamma_i$  be as in III. Let  $G' \supset G$  be

$\mathbb{B}_{m_i}$ -generic s.t.  $b_{m_i} \in G'$ . Set:

$$G'_\nu = G' \cap \mathbb{B}_\nu \quad (\nu \leq \sum_{m_i} 1), \quad \sigma_h = \sigma_h^{\dot{G}'_{\sum_{m_i}}}, \quad u_h = \dot{u}_h$$

for  $h \leq m_i$ . Then for all  $j \leq m_i$ :

(a)  $\sigma_j$  is a  $\Gamma_j^\lambda$ -reformation of  $\pi \uparrow (\lambda+1)$ ,

and  $\sigma_0 = \sigma = \dot{\sigma}^{G'_2^*}$

(b)  $\langle \sigma_j, G'_{\sum_{m_i}} \rangle$  witnesses  $\bar{G}_j$ .

(c)  $\langle \sigma_j, G'_{\sum_{m_i}} \rangle$  is strong over  $\langle \sigma_h, G'_h \rangle$  for  $h \leq j$

(d)  $u_j \subset N^0$  is finite

(e)  $\sigma_j$  coincides with  $\sigma_h$  on  $\sigma_h^0 \cap u_h$  for  $h \leq j$

(f)  $u \cup \bigcup_{h < j} u_h \subset u_j$

(g)  $x_h, w_h \in u_j$  for  $h < j$ , where  $\langle x_i, i < \omega \rangle$  is a fixed enumeration of  $N^0$  and

$w_j^0 =$  the  $N^0$ -least  $w$  s.t.  $\bar{w} = \delta = \delta(\mathbb{B}_j^0)$

in  $N^0$  and  $\sigma^0(x_j) \in \sigma_j^0(w_j^0)$ .

(h)  $\bar{z}_0, \dots, \bar{z}_{m_h} \in u_j$  if  $h \leq j, j \leq m_h$

(i) If  $j = m_i$ , then  $\bar{z}_0, \dots, \bar{z}_{m_i^+} \in u_j$ ,

where  $m_i^+ =$  the least  $n$  s.t.

$$\sigma_{i-1}(\bar{z}_n) > \gamma_n \quad \text{where}$$

$$\gamma_{h-1} < \sigma_{i-1}(\bar{z}_{m_i^+}) \leq \gamma_h$$

Finally:

V Let  $G$  be as in II where  $i = k+1$ . Then:

(a)  $h_{\eta_i}^{m_k}(b_{m_k}) \in G$

(b)  $\tau_i = \left( \sigma_{m_k-1}^i G \right)^{\circ} \quad \text{*)}$

Note  $\sigma_{m_k-1}^i \in V^{B_{\eta_i}}$  since  $\xi_{m_k-1} \leq \eta_i < \xi_{m_k}$ .

Moreover  $b_{m_k-1} = h_{\xi_{m_k-1}}^{m_k}(b_{m_k}) \in G$ . Hence

IV (a) - (h) hold at  $i \leq m_k - 1$  with  $h$ :

$G$  in place of  $G'$ , since  $G$  extends to a

$G' \supset G$  as in IV.

\*) Note: We recall once more our convention that, if  $A \subseteq B$ , we arrange that  $V^A \subseteq V^B$  - i.e.,

if  $t \in V^A$ , then  $t \in V^B$  and

$t^B = t^A$  if  $B$  is  $B$ -generic and

$A = B \cap A$ . (If the reader is uncomfortable with this convention, he can, of course,

define:  $\tau_i = \left( \sigma_{m_k-1}^i G \cap B_{\xi_{m_k-1}} \right)^{\circ}$ )

Assuming  $\text{I} - \text{V}$ , we now finish the proof of the Main Claim. Set:  $\tilde{a} = \bigcap_{i < \omega} a_i$ .

Then  $\tilde{a} \in \text{IB}_\gamma \subset \text{IB}_{\lambda^*}$ ,  $h_{\mathcal{L}^*}(\tilde{a}) = a$ . Let  $G \ni \tilde{a}$  be  $\text{IB}_{\lambda^*}$ -generic. Set  $G_\nu = G \cap \text{IB}_\nu$  for  $\nu \leq \gamma$ . Set  $\sigma = \dot{\sigma}^{G_{\mathcal{L}^*}}$ . We know that  $u = \dot{u}^{G_{\mathcal{L}^*}}$ .

Claim There is  $\sigma' \in V[G]$  s.t.

- $\langle \sigma', G \rangle$  witnesses  $\bar{G} = \bar{G}_\lambda$
- $\langle \sigma', G \rangle$  is strong over  $\langle \sigma, G_{\mathcal{L}^*} \rangle$
- $\sigma'$  coincides with  $\sigma$  on  $u$ ,

where  $\sigma'$  is a  $\Gamma_\lambda^\lambda$ -reformation of  $\pi(\lambda+1)$ .

(Note that then  $\sigma' = \{ \langle \sigma' \upharpoonright 0, 0 \rangle \}$ .)

Set:  $\tau_i = \dot{\tau}_i^{G_{\gamma_i}}$ ,  $\Gamma_i = \dot{\Gamma}_i^{G_{\gamma_i}}$  for  $i < \omega$ .

Then  $\text{II}$  (a) - (d) hold for all  $i$ , and

do  $\text{III}$  (a) - (d). Set:

$$\Gamma = \bigcup_{i < \omega} \Gamma_i = \langle \langle \dot{\zeta}_j, \dot{\sigma}_j, b_j, \dot{u}_j \rangle \mid i < \omega \rangle,$$

By  $\text{V}$  we have  $b_{m_k-1} = h_{\dot{\zeta}_{m_k-1}}(b_{m_k}) \in G$

for all  $k < \omega$ . Hence  $b_j \in G_{\dot{\zeta}_j}$  for  $j < \omega$  and, letting:

$\sigma_j = \sigma_j^{\circ} \circ \tau_j$ ,  $\text{IV (a)-(c)}$  hold for  $j < \omega$ .

Since  $x_j \in U_j$ , we can define a new map  $\tilde{\sigma} : N^{\circ} \rightarrow N$  by:

$$\tilde{\sigma}(x) = \sigma_j^{\circ}(x) \text{ if } \sigma_j^{\circ}(x) = \sigma_h^{\circ}(x) \text{ for all } h \geq j.$$

Then:

$$(1) \text{rang}(\tilde{\sigma}) \subset C_{\tilde{\delta}}^N(\text{rang} \sigma^{\circ}),$$

$$\text{where } \tilde{\delta} = \sup_{j < \omega} \delta(B_{x_j}) \leq \delta(B_{x^*}),$$

$$\text{since } \tilde{\sigma}(x) = \sigma_j^{\circ}(x) \in C_{\delta(B_{x_j})}^N(\text{rang} \sigma^{\circ}),$$

$$(2) C_{\tilde{\delta}}^N(\text{rang} \tilde{\sigma}) = C_{\tilde{\delta}}^N(\text{rang} \sigma^{\circ})$$

follows by the usual proof, using that  $w_i \in U_i$ .

Set  $\sigma' = \{ \langle \tilde{\sigma}, 0 \rangle \}$ . Since  $\sigma$  is a reformulation of  $\pi \uparrow (\lambda+1)$ , we have:  $\text{rang} \tilde{\sigma} \subset \cup \text{rang}(\sigma^{\circ}) \subset \cup \text{rang} \pi^{\lambda}$  by (1), and hence:

(3)  $\sigma'$  is a reformulation of  $\pi \uparrow (\lambda+1)$ ,

(4)  $\tilde{\sigma}(l) = l^*$ ,  $\tilde{\sigma}(\lambda) = \lambda^*$ ,  $\tilde{\sigma} \upharpoonright u = u$ .

since  $l = \bar{z}_0 \in u_0$  and  $\lambda^* \in u \subset u_0$ ;

(6)  $\tilde{\sigma} \upharpoonright G_{\bar{z}_i} \subset G_{z_i}$  for  $i < \omega$ .

proof.

Let  $b \in G_{\bar{z}_i}$ , let  $\tilde{\sigma}(b) = \sigma_j^0(b)$  for

a  $j \geq i$  s.t.  $b \in u_j$ . Then

$\tilde{\sigma}(b) = \sigma_j^0(b) \in G_{z_i} \cap B_{z_i} = G_{z_i}$ , since

$\tilde{\sigma}(B_{\bar{z}_i}) = B_{z_i}$ . QED (6)

(7)  $\tilde{\sigma}^{-1} \upharpoonright G \subset \bar{G}$

proof.

Let  $\Delta = \{b \in B_{\lambda^*} \mid b = \bigcap_{\nu < \lambda^*} h_\nu(b)\}$ ,  $\Delta$  is

dense in  $G$ ; hence it suffices to show

that  $\tilde{\sigma}^{-1} \upharpoonright \Delta \subset \bar{G}$ . Let  $b \in \Delta$ ,  $b = \tilde{\sigma}(\bar{b})$

Then  $\bar{b} = \bigcap_{\nu < \lambda} h_\nu(\bar{b})$  and  $\tilde{\sigma}(h_{\bar{z}_i}(\bar{b})) =$

$h_{z_i}(b) \in G_{z_i}$ . By genericity it follows,

(6) gives us  $\tilde{\sigma}^{-1} \upharpoonright G_{z_i} \subset \bar{G}_{\bar{z}_i}$ ; hence

$h_{\bar{z}_i}(\bar{b}) \in \bar{G}_{\bar{z}_i}$  for  $i < \omega$ . Hence

$\bar{b} \in \bar{G}$ , QED (7)

Note By genericity we of course then get  $\tilde{\sigma}^{-1} \ulcorner G = \bar{G}$  and hence  $\tilde{\sigma} \ulcorner \bar{G} \subset G$ ,

By (7) we have:

(8)  $\langle \sigma', G \rangle$  witnesses  $\bar{G} = \bar{G}_\lambda$ ,

By (6) we have:

(9)  $\langle \sigma', G \rangle$  coheres with  $\langle \sigma, G_{\beta^*} \rangle$

wrt.  $\mathbb{B}_{\beta^*}$ .

Hence by (4)

(10)  $\langle \sigma', G \rangle$  is strong over  $\langle \sigma, G_{\beta^*} \rangle$ ,  
and  $\sigma'$  coincides with  $\sigma$  on  $u$ .

This completes the proof of the  
Main Claim.

It remains only to define  $a_i, t_i, \Gamma_i$   
and verify I-V. We proceed  
by induction on  $i$ ,