§0 Forcing Axioms Compatible with CH

The recent decades have seen great interest in forcing axioms like MM and PFA, which imply the negation of CH. In these notes we survey four axioms which are compatible with CH and have a "natural" motivation similar to that of MM and PFA. These axioms seem quite strong. They imply some of the most striking consequences of MM. Their least known upper consistency bound is a supercompact cardinal.

These notes form a "survey" in the truest sense: We say nothing new. All results are either previously known or follow easily from previously known ones. Two of the axioms we discuss - the complete forcing axiom (CFA) and the Dee-proper forcing axiom (DPFA) -

have in one form or other been known and studied for years. The other two - the subcomplete forcing axiom (SCFA) and the Dee-subproper forcing axiom (DSbPFA) - are based on our recent extensions of Shelah's iteration theories.

In §2 we discuss the complete forcing axiom, which says that Martin's Axiom holds for all forcings which are complete in Shelah's sense. It turns out that these forcings have an equivalent characterization which is more familiar: A complete Boolean algebra $\mathbb{B}$ is a complete forcing iff it is isomorphic to the canonical Boolean algebra BA($\mathbb{P}$) over a set of conditions $\mathbb{P}$ which is ω-closed. To show that CFA is consistent, one follows a standard construction: One iterates "all possible" complete forcings up to a supercompact cardinal κ, using a Laver function to select the components of the iteration. We describe this in some detail, since it forms a template for the later constructions. We call the resulting structure the "natural model". It satisfies CPA + CH. If we made GCH true in the ground model by an application of Silver forcing, then it satisfies GCH. Complete forcings are $\diamond$-preserving - i.e. if $\diamond$ holds in the ground model, it will hold in the extension. Since there are complete forcings which make $\diamond$ true, we conclude that $\diamond$ holds in the natural model. The natural model also makes the slightly stronger form CFA^+ true. In [FMS] it was shown that CFA^+ implies a strong form of Chang's conjecture.

Complete forcings not only add no reals; they add no countable sets of ordinals. In [SPSC] we defined the class of sub-complete (sc) forcings. These add no reals, but among them are forcings, like Namba and Prikry forcing, which add new countable sets of ordinals and thereby change cofinalities. §3 deals with the subcomplete forcing axiom (SCFA) which says that Martin's axiom holds for all SC forcings. The SC forcings are closed under revised countable support iteration, subject to certain standard restraints. Using this, we again force up to a supercompact cardinal to obtain the "natural model". (The definition of 'sc' is modified slightly from earlier notes so as to facilitate this construction.) The natural model again satisfies $SCFA^+ + CH$, and can be consistently supposed to satisfy GCH. Since SC forcing preserves $\diamondsuit$, we again conclude that the natural model satisfies $\diamondsuit$.

SCFA has two of the more impressive consequences of MM:

- $\square$ fails everywhere (and Friedman's principle holds).
- The singular cardinals hypothesis holds at strong limit cardinals.

The proof is quite straightforward: We simply observe that the forcings to which MM was applied to get these results are subcomplete. Finally, we consider the application of SCFA to the following general question:

Let $X \prec H_\tau$ s.t. $\omega_1 \subset X$ and $\overline{\overline{X}} = \omega_1$. For regular $\tau \in X$ set: $cf_X(\tau) = cf(\tau \cap X)$. What forms can the function $cf_X(\tau)$ take?

If we assume $V = L$, the possibilities are very limited (e.g. $cf_X(\tau)$ can change only finitely often as τ ranges over the regular elements of X). One would expect the use of large cardinals to yeild more flexible solutions.

That is, indeed, the case. Foreman and Magidor, by forcing over a model containing a set of measurables, obtain a model in which the measurables remain regular and in which the function $cf_X(\tau)$ can be anything we want as τ ranges over the formerly measurable cardinals. Here we show that SCFA + GCH implies an optimally flexible solution. Even SCFA + CH yields many interesting results in this direction. It implies fr. ins. that, if τ is a strong limit cardinal which is regular or has cofinality ω_1, then for any $A \subset H_{\tau^+}$ there is $X \prec \langle H_{\tau^+}, A\rangle$ s.t. $\omega_1 \subset X$, $\overline{\overline{X}} = \omega_1$, $cf_X(\tau) = \omega_1$, and $cf_X(\kappa) = \omega$ for all regular $\kappa \in (\omega_1, \tau)$. We don't know whether this follows from (or is even consistent with) MM.

In §4 we briefly consider the axioms DPFA and DSbPFA. The treatment is sketchier and we refer the reader to other sources for some of the basic definitions. DPFA says that Martin's axiom holds for all forcings which are both ω_1-proper and (simply) Dee-proper. Since every complete forcing has these properties, DPFA is a strengthening of CFA. The natural model satisfies DPFA$^+$ + CH and can consistently be supposed to satisfy GCH. By a theorem of Shelah, however, DPFA implies that every Aronszajn tree is special. Hence it is incompatible with $\diamondsuit$. In [DSP] we generalized ω_1-proper forcing and (simply) Dee-proper forcing to ω_1-subproper forcing and Dee-subproper forcing. It again turns out that subcomplete forcings have these two properties, so DSbPFA strengthens all three of the previous axioms. The natural model again satisfies $DSbPFA^+ + CH$ and can consistently satisfy GCH. It again implies that all Aronszajn trees are special.

Bibliography

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[S] Shelah *Proper and Improper Forcing* Springer Perspectives in Math. Logic (1991)

[LF] Jensen *$\mathcal{L}$-Forcing* *

[SPSC] Jensen *Subproper and Subcomplete Forcing* *

[EN] Jensen *The Extended Namba Problem* *

[IT] Jensen *Iteration Theorems for Subcomplete and Related Forcings* *

[DSP] Jensen *Dee-Subproper Forcing* *

* These handwritten notes are available on my website. (To find it, enter "Ronald B. Jensen" in Google.)