§1 Preliminaries

We shall generally use the Boolean algebra version of forcing, employing the notation and conventions of [SPSC] and [IT]. (In particular, we say that X is *dense* (*predense*) in $\mathbb{B}$ iff it is a dense (predense) subset of $\mathbb{B} \setminus \{0\}$.)

Def $\mathbb{B}$ *satisfies Martin's axiom* (in symbols: $MA(\mathbb{B})$) iff whenever $\langle \Delta_i \mid i < \omega_1 \rangle$ is a sequence of dense sets in $\mathbb{B}$, there is a filter G on $\mathbb{B}$ s.t. $G \cap \Delta_i \neq \emptyset$ for $i < \omega_1$.

$\mathbb{B}$ *satisfies the* (+) *form of Martin's axiom* (in symbols: $MA^+(\mathbb{B})$) iff in addition, if $\dot{a} \in V^{\mathbb{B}}$ s.t. $\Vdash (\dot{a} \subset \check{\omega}_1$ is stationary in $\omega_1)$, then G can be so chosen that $\dot{a}^G = \{\xi \mid G \Vdash \check{\xi} \in \dot{a}\}$ is stationary in ω_1, (where $G \Vdash \varphi \underset{Df}{\longleftrightarrow} \bigvee b \in G \; b \Vdash \varphi$),

<u>Lemma 1</u> $MA(\mathbb{B})$ is equivalent to:
Let $\mathbb{B} \in H_\theta \subset N$, where N is a transitive ZFC^- model. Let $\mathbb{B} \in X \prec N$, where $\mathrm{card}(X) \leq \omega_1$. There exist $\pi : \bar{N} \prec N$ and $\bar{G}$ s.t.

- $\bar{N}$ is transitive, $\mathrm{card}(\bar{N}) = \omega_1 \subset \mathrm{rng}(\pi)$, $X \subset \mathrm{rng}(\pi)$
- $\bar{G}$ is $\bar{\mathbb{B}} = \pi^{-1}(\mathbb{B})$ - generic over $\bar{N}$.

proof

($\leftarrow$) Let $\Delta_i \in X$ for $i < \omega_1$. Let G be the filter on $\mathbb{B}$ generated by $\pi''\bar{G}$. Then $G \cap \Delta_i \neq \emptyset$ for $i < \omega_1$

($\rightarrow$) Assume w.l.o.g. that $X \prec N$ and $\omega_1 \subset X$. Let G be a filter on $\mathbb{B}$ s.t. $G \cap \Delta \neq \emptyset$ whenever $\Delta \in X$ is dense in $\mathbb{B}$. Set:

$$\Gamma = \{ t \in X \cap N^{\mathbb{B}} \mid \Vdash t \in \check{N} \},$$

where $[\![t \in \check{N}]\!] = \bigcup_{x \in N} [\![t = \check{x}]\!]$. Then

$$\Delta_t = \{ b \mid \bigvee x \in N \; b \Vdash t = \check{x} \}$$

is a dense element of X. Hence we may define:

Def $t^G =$ that x s.t. $G \Vdash t = \check{x}$ for $t \in \Gamma$. (Hence $\check{x}^G = x$ for $x \in X$.)

Set $Y = \{t^G \mid t \in \Gamma\}$. Then $X \subset Y$, since $\check{x}^G = x$ for $x \in X$. Moreover, $Y \prec N$. (To see this, let:

$$N \models \bigvee v\, \psi(v, t_1^G, \ldots, t_n^G).$$

Then $G \Vdash \bigvee v \in \check{N}\ \psi_{\check{N}}(v, t_1, \ldots, t_n)$.
But there is $t_0 \in \Gamma$ s.t.
$\Vdash (\bigvee v \in \check{N}\ \psi_{\check{N}}(v, t_1, \ldots, t_n) \to \psi_{\check{N}}(t_0, \ldots, t_n))$.
Hence $N \models \psi(t_0^G, \ldots, t_n^G)$.)

We now show:

<u>Claim</u> Let $\Delta \in Y$ be dense in $\mathbb{B}$. Then $G \cap \Delta \cap Y \neq \emptyset$.

proof.
Let $\Delta = \dot{\Delta}^G$, where $\dot{\Delta} \in \Gamma$ and (w.l.o.g.) $\Vdash \dot{\Delta}$ is dense in $\check{\mathbb{B}}$. But then there is $b' \in \Gamma$ s.t. $\Vdash b' \in \dot{\Delta} \cap \dot{G}$, where $\dot{G}$ is the canonical $\mathbb{B}$-generic name. Then $b \in \Delta$, where $b = b'^G \in Y$.

Moreover, $G \Vdash \check{b} \in \dot{G}$. Let $b' \in G$ s.t. $b' \Vdash \check{b} \in \dot{G}$. Then $b' \subset [\![\check{b} \in \dot{G}]\!] = b$. Hence $b \in G$. QED (Claim).

Letting $\pi : \bar{N} \overset{\sim}{\longleftrightarrow} Y$ be the transitivization of Y and $\bar{G} = \pi^{-1}{}''G$, the conclusion is immediate.

QED (Lemma 1)

By an entirely similar proof:

<u>Lemma 2</u> $MA^{+}(\mathbb{B})$ is equivalent to:

Let N be as above. Let $a \in N^{\mathbb{B}}$ s.t. $\Vdash a \subset \check{\omega}_1$ is stationary in $\check{\omega}_1$.

Let $a, \mathbb{B} \in X \subset N$, $\mathrm{card}(X) \le \omega_1$. There exist $\pi : \bar{N} \prec N$, $\bar{G}$ s.t.

- $\bar{N}$ is transitive, $\mathrm{card}(\bar{N}) = \omega_1 \subset \mathrm{rng}(\pi)$, $X \subset \mathrm{rng}(\pi)$
- $\bar{G}$ is $\bar{\mathbb{B}} = \pi^{-1}(\mathbb{B})$-generic over $\bar{N}$
- $\bar{a}^{\bar{G}}$ is stationary in ω_1, where $\bar{a} = \pi^{-1}(a)$.