§ 1 Preliminaries

We use the notation and terminology of [SPSC] §1. In particular, we write $\mathbb{A} \subseteq \mathbb{B}$ to mean that $\mathbb{A}$ is a complete Boolean algebra completely contained in the Boolean algebra $\mathbb{B}$. If $\mathbb{A} \subseteq \mathbb{B}$ we write: $h_{\mathbb{A}}(b) = \bigcap \{a \in \mathbb{A} \mid b \subset a\}$ for $b \in \mathbb{B}$.
If we are dealing with a sequence $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ s.t. $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$, we often write $h_i(b)$ for $h_{\mathbb{B}_i}(b)$. If $\alpha = \omega$ we call $\langle b_i \mid i < \omega \rangle$ a <u>thread through</u> $\mathbb{B}$ iff $b_0 \neq 0$ and $h_i(b_j) = b_i$ for $i \leq j < \omega$.
We say that a set X is <u>dense</u> in a BA $\mathbb{B}$ iff $X \setminus \{0\}$ is dense in $\mathbb{B} \setminus \{0\}$.

By an <u>iteration</u> we mean a sequence $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ s.t.

- $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$
- $\mathbb{B}_0 = 2$
- If $\lambda < \alpha$ is a limit ordinal, then $\bigcup_{i<\lambda} \mathbb{B}_i$ generates $\mathbb{B}_\lambda$.

This paper will deal mainly with the properties of certain revised countable support (RSC) iterations. The only fact one needs to know for the purposes of this paper is:

Fact Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RSC iteration. Then:

(a) If $\lambda < \alpha$ and $\langle \xi_i \mid i < \omega \rangle$ is monotone and cofinal in λ, then

(i) If $\langle b_i \mid i < \omega \rangle$ is a thread through $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$, then $\bigcap_{i<\omega} b_i \neq 0$ in $\mathbb{B}_\lambda$.

(ii) The set of all such $\bigcap_{i<\omega} b_i$ is dense in $\mathbb{B}_\lambda$.

(b) If $\lambda < \alpha$ and $\Vdash_i \mathrm{cf}(\check{\lambda}) > \omega$ for all $i < \lambda$, then $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$.

(c) If $i < \lambda$ and G is $\mathbb{B}_i$-generic, then the iteration $\langle \mathbb{B}_{i+j}/G \mid j < \alpha - i \rangle$ satisfies (a), (b) in $V[G]$.

Note By (a) we have: If $\langle b_i \mid i < \omega \rangle$ is a thread through $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$ and $b = \bigcap_{i<\lambda} b_i$, then $h_i(b) = b_i$ for $i < \omega$.

There seems to be some interest in considering the class of iterations which we get by omitting (a)(ii) from the above list of properties. In our paper [EN] we used such iterations – for which (a)(ii) demonstrably failed – and found it necessary to prove a special iteration theorem for them. A slight generalisation of that theorem is proven here in § 3 Thm 4.