§4 Subproper Forcing

We recall the definition:

<u>Def</u> Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is <u>subproper</u> iff for sufficiently large cardinals θ the following holds: Let $\mathbb{B} \in H_\theta$. Let $\tau > \theta$ be regular s.t. $H_\theta \subset W = L^A_\tau$. Let $\sigma : \bar{W} \prec W$ where $\bar{W}$ is countable, transitive, and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{a}, \bar{s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_n) = \theta, \mathbb{B}, a, s, \lambda_1, \ldots, \lambda_n$ where $a \in \mathbb{B} \setminus \{0\}$ and λ_i is regular s.t. $\bar{\bar{\mathbb{B}}} < \lambda_i$ for $i = 1, \ldots, n$. Then there is $c \in \mathbb{B} \setminus \{0\}$ s.t. $c \subset a$ and whenever $G \ni c$ is $\mathbb{B}$-generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \bar{W} \prec W$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{a}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, a, s, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma_0{}'' \bar{\lambda}_i = \sup \sigma'' \lambda_i$ $(i = 0, \ldots, n)$

where $\bar{\lambda}_0 = On \cap \bar{W}$

(d) $\bar{G} = \sigma_0^{-1}{}'' G$ is $\bar{\mathbb{B}}$-generic over $\bar{W}$.

As before, we say that $\mathbb{B}$ is <u>weakly subproper</u> if there is a parameter p s.t. for sufficiently large Θ, the above holds whenever $p \in \operatorname{rng}(\sigma)$. As before, weak subproperness implies subproperness.

The <u>two step iteration theorem</u> for subproper forcing says that if $\mathbb{A} \subseteq \mathbb{B}$, $\mathbb{A}$ is subproper, and

$$\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\dot{G} \text{ is subproper},$$

then $\mathbb{B}$ is subproper ($\dot{G}$ being, again, the canonical generic name).

In this section we prove:

<u>Thm 5</u> Thm 1 holds with "subcomplete" replaced by "subproper".

proof

We are given an RCS-iteration $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ satisfying (a) – (c) of Thm 1 (with "subproper" instead of "subcomplete"). By induction on i we prove:

<u>Claim</u> Let $h \leq i$. Let G be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i / G$ is subproper in $V[G]$.

The cases $h = i$, $i = 0$, and $i = j+1$ are again straightforward.

Let $i = \lambda$ where λ is a limit ordinal. We consider the same two cases:

<u>Case 1</u> $cf(\lambda) \leq \overline{\overline{\mathbb{B}}}_i$ for an $i < \lambda$.

Exactly as before it suffices to prove:

<u>Claim</u> Assume $cf(\lambda) \leq \omega_1$ in V. Then $\mathbb{B}_\lambda$ is subproper.

We again fix $f: \omega_1 \to \lambda$ s.t. $\sup f''\omega_1 = \lambda$. Let $\theta > \lambda$ be a cardinal s.t. $\overline{\overline{\mathbb{B}}} < \theta$ and H_θ witnesses the subproperness of $\mathbb{B}_{i'}/\check{G}_i$ for $i \leq i' < \lambda$. Let $W = L^A_\tau$ where $\tau > \theta$ is regular, and $H_\theta \subset W$. Let $\sigma: \overline{W} \prec W$ s.t. $\overline{W}$ is countable, transitive, and full. Suppose moreover that:

$\sigma(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \lambda, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, n)$

(We shall suppose w.l.o.g. that $\bar{s}$ codes $\bar{a}$, so that $\sigma'(\bar{a}) = a$ whenever $\sigma': \overline{W} \prec W$ s.t. $\sigma'(\bar{s}) = s$. This simplifies the notation.) $\lambda_1, \ldots, \lambda_n$ is again a sequence s.t. λ_i is regular and $\overline{\overline{\mathbb{B}}}_\lambda < \lambda_i$. It suffices to show:

<u>Claim</u> There is $c \in \mathbb{B}_\lambda \setminus \{0\}$ s.t. $c \subset a$ and whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic, then there is $\sigma' \in V[G]$ s.t.

(a) $\sigma': \overline{W} \prec W$

(b) $\sigma'(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \lambda, s, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$ where $\bar{\lambda}_0 = On \cap \overline{W}$; $\tilde{\lambda}_i = \sup \sigma''\lambda_i$

(d) $\overline{G} = \sigma'^{-1}{}''G$ is $\overline{\mathbb{B}}_{\bar{\lambda}}$-generic over $\overline{W}$.

As before we fix $\langle \nu_i \mid i<\omega\rangle$ s.t. $\nu_i < \omega_1^{\bar{W}}$ and $\langle \bar{\xi}_i \mid i<\omega\rangle$ is monotone and cofinal in $\bar{\lambda}$, where $\bar{\xi}_i = \bar{f}(\nu_i)$. If $cf(\lambda)=\omega$, we will have $cf(\bar{\lambda})=\omega$ in $\bar{W}$ and we can choose $\langle \nu_i \mid i<\omega\rangle \in \bar{W}$; hence:

(1) $\langle \bar{\xi}_i \mid i<\omega\rangle \in \bar{W}$ if $cf(\lambda)=\omega$.

We again set $\xi_i = \sigma(\bar{\xi}_i) = f(\nu_i)$. We again have:

(2) $\sigma'(\bar{\xi}_i)=\xi_i$ whenever $\sigma' : \bar{W} \prec W$ and $\sigma'(\bar{f})=f$.

As before, our strategy is to define a sequence $c_i \in \mathbb{B}_{\xi_i}$, $\dot{\sigma}_i \in V^{\mathbb{B}_{\xi_i}}$ s.t. $\langle c_i \mid i<\omega\rangle$ is a thread in $\langle \mathbb{B}_{\xi_i} \mid i<\omega\rangle$ and c_i forces that $\sigma_i : \bar{W} \prec W$ and $\bar{G}_i = \sigma_i^{-1}{}''G_i$ is $\mathbb{B}_{\bar{\xi}_i}$ - generic over $\bar{W}$, where $G_i \ni c_i$ is $\mathbb{B}_{\xi_i}$ - generic and $\sigma_i = \dot{\sigma}_i^{G_i}$. We then set $c = \bigcap_i c_i$. We require enough pointwise coherence between the σ_i that we can again define $\sigma' : \bar{W} \prec W$ by:

$\sigma'(x) =_{\mathrm{Df}} \sigma_i(x)$ for sufficiently large i,

where $\sigma_i = \dot{\sigma}_i^{G}$ and $G \ni c$ is $\mathbb{B}_\lambda$ - generic. We also build enough pointwise coherence into the construction to get (b), (c) of the Claim and:

(d') $\bar{G}_i = \sigma'^{-1}{}''G_i$ is $\bar{\mathbb{B}}_{\bar{\xi}_i}$ - generic over $\bar{W}$ $(i<\omega)$.

There are, however, two new problems which we must address:

Problem 1 We must ensure $c \subset a = \sigma(\bar{a})$.

Problem 2 We must ensure that (d') will imply: (d) $\bar{G} = \sigma'^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\bar{W}$.

Problem 1 is the easier one. We fix an $\bar{a}' \subset \bar{a}$ s.t. $\bar{a}' \in X$, where X is a dense set in $\bar{\mathbb{B}}_{\bar{\lambda}}$ defined as follows:

Case A $cf(\lambda) = \omega$.

Then $cf(\bar{\lambda}) = \omega$ in $\bar{W}$ and, in fact, $\langle \bar{\xi}_i \mid i<\omega \rangle \in \bar{W}$, Hence the set X of $b = \bigcap_{i<\omega} b_i$ s.t. $\langle b_i \mid i<\omega \rangle \in \bar{W}$ is a thread in $\langle \bar{\mathbb{B}}_{\bar{\xi}_i} \mid i<\omega \rangle$ is dense in $\bar{\mathbb{B}}_{\bar{\lambda}}$.

Thus $b = \bigcap_{i<\omega} h_{\bar{\xi}_i}(b)$ for $b \in X$.

Case B $cf(\lambda) = \omega_1$

Then $cf(\bar{\lambda}) = \omega_1$ in $\bar{W}$. Hence $X = \bigcup_{i<\bar{\lambda}} \bar{\mathbb{B}}_i \setminus \{0\}$ is dense in $\bar{\mathbb{B}}_{\bar{\lambda}}$. But $b = h_{\bar{\xi}_i}(b)$ for a sufficiently large i for $b \in X$.

Thus, in either case, we need only to fix an $\bar{a}' \subset \bar{a}$ s.t. $\bar{a}' \in X$ and ensure that $c_i \subset h_{\bar{\xi}_i}(a')$ where $a' = \sigma(\bar{a}')$.

We now consider Problem 2. Let X be the dense set in $\overline{\mathbb{B}}_{\bar\lambda}$ define in Case A or B. We define:

<u>Def</u> By a <u>master sequence</u> for $\overline{W}$ we mean a sequence $\langle b_i \mid i<\omega\rangle$ s.t.

(a) $b_i \in X$, $b_i \subset b_h$ and $h_{\bar\xi_h}(b_i) = h_{\bar\xi_h}(b_h)$ for $h \le i$

(b) Whenever $G \subset \overline{\mathbb{B}}_{\bar\lambda}$ is an ultrafilter s.t. $G \cap \overline{\mathbb{B}}_{\bar\xi_i}$ is $\overline{\mathbb{B}}_{\bar\xi_i}$-generic over $\overline{W}$ for $i<\omega$ and $\{b_i \mid i<\omega\} \subset G$, then G is $\overline{\mathbb{B}}_{\bar\lambda}$-generic over $\overline{W}$.

There are many master sequences (though we cannot, of course, expect to find one which is an element of $\overline{W}$). We prove:

(3) There is a master sequence $\langle b_i \mid i<\omega\rangle$ s.t. $b_0 \subset \bar a$.

As a preliminary we show:

(4) Let $b \in X$. Let $\Delta \in \overline{W}$ be strongly dense in $\overline{\mathbb{B}}_{\bar\lambda}$. There is $b' \in X$ s.t. $b' \subset b$, $h_{\bar\xi_i}(b') = h_{\bar\xi_i}(b)$ and the set $\Delta^i = \{a \in \overline{\mathbb{B}}_{\bar\xi_i} \mid a \cap b' \in \Delta\}$ is dense below $h_{\bar\xi_i}(b)$ in $\overline{\mathbb{B}}_{\bar\xi_i}$.

proof.

The set $\{h_{\bar\xi_i}(b') \mid b' \subset b \wedge b' \in \Delta\}$ is certainly dense below $h_{\bar\xi_i}(b)$ in $\overline{\mathbb{B}}_{\bar\xi_i}$. Let A be a maximal antichain in this set. Then A is predense below $h_{\bar\xi_i}(b)$. For $a \in A$ choose $b_a \subset b$ s.t. $b_a \in \Delta$ and $h_{\bar\xi_i}(b_a) = a$. Set: $b' = \bigcup_{a\in A} b_a$. Then b' has the desired property. QED (4)

(3) then follows easily: Let $\langle \Delta_i \mid i<\omega\rangle$ enumerate the $\Delta \in \bar{W}$ which are strongly dense in $\bar{\mathbb{B}}_{\bar{\lambda}}$. Successively pick b_i s.t. $b_0 = \bar{a}'$, $b_{i+1} \subset b_i$, $h_{\bar{\xi}_i}(b_{i+1}) = h_{\bar{\xi}_i}(b_i)$ and Δ_i^i is dense below $h_{\bar{\xi}_i}(b)$ in $\mathbb{B}_{\bar{\xi}_i}$. Let $G \subset \bar{\mathbb{B}}_{\bar{\lambda}}$ be as above. Let $\Delta = \Delta_i$ be strongly dense in $\mathbb{B}_{\bar{\lambda}}$. Then $\Delta^i \cap G_i \neq \emptyset$ where $G_i = G \cap \mathbb{B}_{\bar{\xi}_i}$. Let $a \in \Delta^i \cap G_i$. Then $a \cap b_i \in G \cap \Delta$. QED(3)

We now fix a master sequence $\langle \bar{b}_i \mid i<\omega\rangle$ s.t. $\bar{b}_0 \subset \bar{a}$. Set $\bar{b}_i^d = h_{\bar{\xi}_i}(\bar{b}_i)$. In our construction we shall enforce that if $G_i \ni c_i$ is $\mathbb{B}_{\bar{\xi}_i}$-generic and $\sigma_h = \dot{\sigma}_h^{G_i}$ for $h \le i$, then $\sigma_h(\bar{b}_h) = \sigma_i(\bar{b}_h)$ for $h \le i$ and $\sigma_i(\bar{b}_i^i) \in G_i$. Thus at the end we shall have $\sigma_i(\bar{b}_i^i) \in G$ for $i<\omega$. But $\bar{b}_j^d \subset \bar{b}_i^d$ for $i \le j$. Hence: $\sigma_j(\bar{b}_i^d) \in G$ for $j \ge i$. If Case A holds, we then have $\sigma'(\hat{b}_i) = \sigma'(\bigcap_{j\ge i} \bar{b}_i^d) =$
$\bigcap_{j \ge i} \sigma'(\bar{b}_i^j) \in G$ by genericity. If Case B holds, there is j s.t. $\bar{b}_i = \bar{b}_j^i$ & hence $\sigma'(\bar{b}_i) \in G$. Hence $\{\bar{b}_i \mid i<\omega\} \subset \bar{G} = \sigma'^{-1}{}''G$, which will guarantee the $\mathbb{B}_{\bar{\lambda}}$-genericity of $\bar{G}$ over $\bar{W}$.

From now on let $\langle \bar{b}_i \mid i<\omega\rangle$ be a fixed master sequence s.t. $\bar{b}_0 \subset \bar{a}$ and $\langle x_i \mid i<\omega\rangle$ a fixed enumeration of $\bar{W}$ with infinite repetitions.

We construct $c_k \in \mathbb{B}_{\xi_k}$, $\dot{\sigma}_k \in V^{\mathbb{B}_{\xi_k}}$ s.t. $\langle c_i \mid i<\omega\rangle$ is a thread in $\langle \mathbb{B}_{\xi_i} \mid i<\omega\rangle$ and:

(*) Let $G_k \ni c_k$ be $\mathbb{B}_{\xi_k}$ - generic. Set:

$\sigma_j = \dot{\sigma}_j^{G_k} = \dot{\sigma}_j^{G_j}$ for $j \leq k$ where $G_j = G_k \cap \mathbb{B}_{\xi_j}$.

Then: $\sigma_0 = \sigma$ and

(a) $\sigma_k : \bar{W} \prec W$

(b) $\sigma_k(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \lambda, \mathbb{B}, s, \lambda_i$ $(i=1,\dots,n)$

(c) $\sup \sigma_k{}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i=0,\dots,n)$

where $\bar{\lambda}_0 = On \cap \bar{W}$, $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$

(d) $\bar{G}_k = \sigma_k^{-1}{}'' G_k$ is $\bar{\mathbb{B}}_{\bar{\xi}_k}$ - generic over $\bar{W}$

(f) $\sigma_k(x_l, \bar{b}_l, d_l) = \sigma_l(x_l, \bar{b}_l, d_l)$ for $l \leq k$,

where $d_l = \begin{cases} \text{The } \bar{W}\text{-least } d \in \bar{G}_l \cap x_l \text{ if} \\ \bar{G}_l \cap x_l \neq \emptyset; \\ 0 \text{ if not} \end{cases}$

(g) Let $i = 0,\dots,n$, $k = j+1$ s.t.

$\sigma_j(\bar{\xi}^i_m) \leq \xi^i_h < \sigma_j(\bar{\xi}^i_{m+1})$

Then $\sigma_h(\bar{\xi}^i_l) = \sigma_j(\bar{\xi}^i_l)$ for $l \leq m+1$

(h) $\sigma_k(h_{\bar{\xi}_k}(\bar{b}_k)) \in G_k$

We first show that (*) implies the claim. Let $c = \bigcap_i c_i$. Then $c \in \mathbb{B}_{\tilde{\lambda}} \subset \mathbb{B}_\lambda$. Let $G \ni c$ be $\mathbb{B}_\lambda$ - generic. Set $\sigma_i = \dot{\sigma}_i^G = \dot{\sigma}_i^{G_i}$, where $G_i = G \cap \mathbb{B}_{\xi_i}$. By (f) we can define $\sigma' : \overline{W} \prec W$ by:

$\sigma'(x) = \sigma_l(x)$ for sufficiently large l.

(a), (b) of the claim are immediate. (c) follows exactly as in the proof of Thm 1. It remains to show:

(d) $\bar{G} = \sigma'^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$ - generic over $\overline{W}$.

We first show:

(5) $\bar{G}_i = \sigma'^{-1}{}''G$ is $\mathbb{B}_{\bar{\xi}_i}$ - generic over $\overline{W}$.

$\bar{G}_i$ is obviously an ultrafilter on $\bar{\mathbb{B}}_{\bar{\xi}_i}$. Let $\Delta \in \overline{W}$ be strongly dense in $\bar{\mathbb{B}}_{\bar{\xi}_i}$.

<u>Claim</u> $\Delta \cap \bar{G}_i = \emptyset$

Let $j \geq i$ s.t. $\Delta = x_j$. Then $\Delta \cap \sigma_j^{-1}{}''G_i \neq \emptyset$, $(\sigma_j^{-1}{}''G_j) \cap \bar{\mathbb{B}}_{\bar{\xi}_i}$ is $\bar{\mathbb{B}}_{\bar{\xi}_i}$ - generic. Hence $\sigma_j(d_j) \in G_i$ and $d_j \in \Delta = x_j$. But $\sigma_l(d_j) = \sigma_j(d_j)$ for $j \leq l$. Hence $\sigma'(d_j) = \sigma_j(d_j) \in G_i$. Hence $d_j \in \bar{G}_i \cap \Delta$. QED(5)

But $\sigma_j(h_{\bar{\xi}_j}(\bar{b}_j)) \in G_j$ for all j. By the argument sketched above we then have: $\sigma_j(\bar{b}_j) \in G$ and hence

$\bar{b}_j \in \bar{G}$ for $j < \omega$. $\bar{G}$ is obviously an ultrafilter. Hence $\bar{G}$ is $\mathbb{B}_{\bar{\lambda}}$ - generic over $\bar{W}$, since $\langle \bar{b}_j \mid j < \omega \rangle$ is a master sequence. Note that $\bar{a} \in \bar{G}$ since $\bar{b}_0 \subset \bar{a}$. Hence $a \in G$. Since this holds for every $\mathbb{B}_\lambda$ - generic $G \ni c$, we conclude that $c \subset a$. QED (Claim)

It remains only to construct $\dot{\sigma}_k$, c_k and to verify (*). The construction is virtually the same as in Thm 1. We again proceed by induction on k. For $k = 0$ set: $c_0 = 1$, $\dot{\sigma}_0 = \hat{\sigma}$. Now let $k = j+1$. The construction is essentially a careful repeat of the proof of the two step iteration theorem:

Let $G_j \ni c_j$ be $\mathbb{B}_{\xi_j}$ - generic. Then $\sigma_j = \dot{\sigma}_j^G$ extends uniquely to a $\sigma_j^* : \bar{W}[\bar{G}_j] \prec W[G_j]$ s.t. $\sigma_j^*(\bar{G}_j) = G_j$, where $\bar{G}_j = \sigma_j^{-1} {}'' G_j$.

But $\mathbb{B}' = \mathbb{B}_{\xi_k} / G_j$ is subproper in $V[G_j]$. Hence there is $c' \in \mathbb{B}' \setminus \{0\}$ s.t. whenever $G' \ni c'$ is $\mathbb{B}'$ - generic, then there is a $\sigma' \in V[G_j][G']$ s.t.

(a) $\sigma' : \bar{W}[\bar{G}_j] \prec W[G_j]$, $\sigma'(\bar{G}_j) = G_j$

(b) $\sigma'(\bar{f}, \bar{\theta}, \bar{\lambda}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = f, \theta, \lambda, \mathbb{B}, s, \lambda_i$
   $(i = 1, \ldots, n)$

(c) $\sup \sigma' {}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$

(d) $\bar{G}' = \sigma'^{-1} {}'' G'$ is $\bar{\mathbb{B}}_{\bar{\xi}_k}$ - generic over $\bar{W}[\bar{G}_j]$

Moreover, since any finite pointwise coherence of the embeddings σ_j^*, σ' can be enforced, we may require:

(e) $\sigma'(x_\ell, \bar{b}_\ell, d_\ell) = \sigma_i(x_\ell, \bar{b}_\ell, d_\ell)$ for $\ell < k$

where $d_\ell = \begin{cases} \text{the } \bar{W}\text{-least } d \in x_\ell \text{ s.t. } \sigma_\ell(d) \in G_\ell \\ \text{if this exists;} \\ 0 \text{ if not.} \end{cases}$

(f) Let $i = 0, m$ s.t. $\sigma_i(\bar{\xi}_m^i) \leq \xi_k^i < \sigma_i(\bar{\xi}_{m+1}^i)$.

Then $\sigma'(\bar{\xi}_\ell^i) = \sigma_i(\bar{\xi}_\ell^i)$ for $\ell \leq m+1$.

(g) $\sigma'(h_{\bar{\xi}_k}(\bar{b}_k)) = \sigma_i(h_{\bar{\xi}_k}(\bar{b}_k))$

(h) $\sigma_i(h_{\bar{\xi}_k}(\bar{b}_k))/G_j \in G'$

(To enforce (h) we pick $c' \subset b' = \sigma_i(h_{\bar{\xi}_k}(\bar{b}_k))/G_i =$
$= \sigma_i^*(h_{\bar{\xi}_k}(\bar{b}_k)/\bar{G}_i)$. We know that $b' \neq 0$, since $h_{\bar{\xi}_i} h_{\bar{\xi}_k}(\bar{b}_k) = h_{\bar{\xi}_i}(\bar{b}_i) \in \bar{G}_j$.)

Set $G = G_i * G' = \{b \in \mathbb{B}_{\xi_k} \mid b/G_i \in G'\}$. Then G is $\mathbb{B}_{\xi_k}$-generic. Similarly $\bar{G} = \bar{G}_j * \bar{G}$ is $\bar{\mathbb{B}}_{\bar{\xi}_k}$-generic over $\bar{W}$. Setting $\sigma_k = \sigma' \upharpoonright \bar{W}$, we have: $\bar{G} = \sigma_k^{-1}{}''G$. It follows easily from (a)–(g) that σ_k satisfies (*)(a)–(h) with $G = G_i * G'$. But the fact that there is a $c' \in \mathbb{B}'$ forcing the existence of such a σ_k is forced by c_j.

Thus we may w.l.o.g. take

$c' = \dot{c}^{G_i}$, where $c' = \dot{c}^{G_i}$ forces this conclusion over $V[G_i]$ whenever $G_i \ni c_i$ is $\mathbb{B}_{\xi_i}$ - generic. We may also assume w.l.o.g. that $[\![\dot{c} \neq 0]\!] = c_i$. Thus $\Vdash \dot{c} \in \check{\mathbb{B}}_{\xi_k}$. Hence there is a unique $c_k \in \mathbb{B}_{\xi_k}$ s.t. $\Vdash_{\xi_i} \check{c}_k/\dot{G}_i = \dot{c}$. But then $h_{\xi_i}(c_k) = [\![\check{c}_k / \dot{G}_i \neq 0]\!] = c_i$. If $G \ni c_i$ is $\mathbb{B}_{\xi_k}$ - generic, then $c_i \in G_k$ and $c_k/G_i = \dot{c}^{G_i} \in \mathbb{B}_{\xi_k}/G_i$ where $G_i = G \cap \mathbb{B}_i$. Set $G' = G/G_i =_{df} \{ b/G_i \mid b \in G \}$. Then G' is $\mathbb{B}' = \mathbb{B}_{\xi_k}/G_i$ - generic over $V[G_i]$ and $V[G_i][G'] = V[G]$. Hence there is $\sigma_k \in V[G]$ satisfying (*)(a)-(h). Since this is forced by c_k, there is a $\dot{\sigma}_k \in V^{\mathbb{B}_{\xi_k}}$ s.t. $\sigma_k = \dot{\sigma}_k^G$ satisfies (*)(a)-(h) whenever $G \ni c_k$ is $\mathbb{B}_{\xi_k}$ - generic. QED

This completes Case 1.

<u>Case 2</u> Case 1 fails.

Then λ is regular and $\overline{\overline{\mathbb{B}}}_i < \lambda$ for $i<\lambda$. We closely follow the proof of Thm 1. Let $\bar{W}, W, \theta, \sigma$ be as before with; $\sigma(\bar\theta, \bar{\mathbb{B}}, \bar{s}, \bar{a}, \bar\lambda, \bar\lambda_i) = \theta, \mathbb{B}, s, a, \lambda, \lambda_i$ $(i=1,\dots,n)$, where $\lambda_1,\dots,\lambda_n$ are as before and $a\in\mathbb{B}_\lambda$. We set: $\lambda_{n+1}=\lambda$, $\bar\lambda_{n+1}=\bar\lambda$, $\bar\lambda_0 = On\cap\bar{W}$ and $\tilde\lambda_i = \sup\sigma''\bar\lambda_i$ $(i=0,\dots,n+1)$. (We also write $\tilde\lambda = \tilde\lambda_{n+1}$.) We again fix an enumeration $\langle x_i \mid i<\omega\rangle$ of $\bar{W}$ with infinite repetition and a master sequence $\langle \bar{b}_i \mid i<\omega\rangle$ with $\bar{b}_0 \subset \bar{a}$. (Note that this time $\bigcup_{i<\lambda}\mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$, so we can take $b_i \in \bigcup_{i<\bar\lambda}\bar{\mathbb{B}}_i$.) We claim:

<u>Claim</u> There is $c\in\mathbb{B}_\lambda$ s.t. $c\subset a$ and whenever $G\ni c$ is $\mathbb{B}_\lambda$-generic, there is $\sigma'\in V[G]$ s.t.

(a) $\sigma': \bar{W} \prec W$

(b) $\sigma'(\bar\theta,\bar{\mathbb{B}},\bar{s},\bar\lambda_i) = \theta,\mathbb{B},s,\lambda_i$ $(i=1,\dots,n+1)$

(c) $\sup\sigma'{}''\bar\lambda_i = \tilde\lambda_i$ $(i=0,\dots,n+1)$

(d) $\bar{G} = \sigma'^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar\lambda}$-generic over $\bar{W}$.

We choose $\langle \bar\xi_i^j \mid i<\omega\rangle$ monotone and cofinal in $\bar\lambda_j$ for $j=0,\dots,n+1$ and set $\xi_i^j = \sigma(\bar\xi_i^j)$. We also set: $\xi_i = \xi_i^{n+1}$, $\bar\xi_i = \bar\xi_i^{n+1}$.

We inductively construct $c_k \in \mathbb{B}_{\xi_k}$, $\dot\sigma_k \in V^{\mathbb{B}_{\xi_k}}$ s.t. I of Case 1 in the proof of Thm 1 holds and:

$\underline{\text{II}}$ Let $G \ni c_k$ be $\mathbb{B}_{\xi_k}$-generic. Set:

$G_\eta = G \cap \mathbb{B}_\eta$ $(\eta \leq \xi_k)$, $\sigma_j = \dot\sigma_j^G = \dot\sigma_j^{G_{\xi_j}}$ $(j \leq k)$.

Then:

(a) $\sigma_k : \bar{W} \prec W$

(b) $\sigma_k(\bar\theta, \bar{\mathbb{B}}, \bar{s}, \bar\lambda_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, m, m+1)$

(c) $\sup \sigma_k{}'' \bar\lambda_i = \tilde\lambda_i$ $(i = 0, m, m+1)$

(d) Let $\sigma_k(\bar\xi_m) \leq \xi_k < \sigma_k(\bar\xi_{m+1})$. Then

$\bar{G} = \sigma_k^{-1}{}'' G_{\sigma_k(\bar\xi_m)}$ is $\bar{\mathbb{B}}_{\bar\xi_m}$-generic over $\bar{W}$

(e) Let $k = j+1$. Then

$\sigma_k(x_i, \bar{b}_i, d_i) = \sigma_j(x_i, \bar{b}_i, d_i)$ for $i \leq j$

where d_i is defined by: Let $\sigma_i(\bar\xi_m) \leq \xi_i < \sigma_i(\bar\xi_{m+1})$

$$d_i = \begin{cases} \text{the } \bar{W}\text{-least } d \in x_i \text{ s.t. } \sigma_i(d) \in G_{\sigma_i(\bar\xi_m)} \\ \text{if such a } d \text{ exists,} \\ 0 \text{ if not} \end{cases}$$

(f) Let $k = j+1$, $i = 0, m, m+1$. If

$\sigma_j(\bar\xi_m^i) \leq \xi_k^i < \sigma_j(\bar\xi_{m+1}^i)$, then

$\sigma_k(\bar\xi_l^i) = \sigma_j(\bar\xi_l^i)$ for $l \leq m+1$.

(g) Let $\sigma_k(\bar\xi_m) \leq \xi_k < \sigma_k(\bar\xi_{m+1})$.

Then $\sigma_k(h_{\bar\xi_m}(\bar{b}_m)) \in G$.

Thus (a)–(c), (f) are unchanged, (d), (e) are altered to fit the new situation, and (g) is new.

<u>Note</u> By (e), $\sigma_k(x_i, \bar{b}_i, d_i) = \sigma_i(x_i, \bar{b}_i, d_i)$ for $i \leq k$.

<u>Note</u> We shall again arrange that if $\sigma_i(\bar{\xi}_m) \leq \xi_i < \xi_k < \sigma_i(\bar{\xi}_{m+1})$, then $\sigma_k = \sigma_i$.

We show now that I, II imply the <u>Claim</u>. The proof is virtually the same as in Thm 1. We set $c = \bigcap_{i<\omega} c_i$. Then $c \in \mathbb{B}_\lambda^\sim \subset \mathbb{B}_\lambda$. Let $G \ni c$ be $\mathbb{B}_\lambda$ - generic. Set $\sigma_i = \dot{\sigma}_i^G$ for $i < \omega$. We define $\sigma' : \bar{W} \prec W$ by: $\sigma'(x) = \sigma_k(x)$ for sufficiently large k. (a)–(c) follow exactly as in Thm 1. We prove (d). We first show:

(6) Let $\sigma'(\bar{\xi}_n) = \xi$. Then $\bar{G} = \sigma'^{-1}{}''G_\xi$ is $\bar{\mathbb{B}}_{\bar{\xi}_n}$ - generic over $\bar{W}$.

proof.

Let Δ be dense in $\mathbb{B}_{\bar{\xi}_n}$. Let $\Delta = x_j$ for a j chosen large enough that $m \geq n$ where $\sigma_j(\bar{\xi}_m) \leq \xi_j < \sigma_j(\bar{\xi}_{m+1})$. Set $\xi' = \sigma_j(\bar{\xi}_m)$. Then $\bar{G}' = \sigma_j^{-1}{}''G_{\xi'}$ is $\bar{\mathbb{B}}_{\bar{\xi}_m}$ - generic over $\bar{W}$. Hence $\bar{G} = \bar{G}' \cap \mathbb{B}_{\bar{\xi}_n}$ is $\bar{\mathbb{B}}_{\bar{\xi}_n}$ - generic over $\bar{W}$ and

and $\Delta = x_{i'} \subset \bar{\mathbb{B}}_{\bar{\xi}_n}$ is dense in $\bar{\mathbb{B}}_{\bar{\xi}_n}$. Hence $\Delta \cap \bar{\mathbb{B}}_{\bar{\xi}_m} \neq \emptyset$. Hence $d_{i'} \in \Delta \subset \bar{\mathbb{B}}_{\bar{\xi}_n}$ and $\sigma_{i'}(d_{i'}) \in G_{\bar{\xi}'} \cap \mathbb{B}_{\xi} \subset G_{\xi}$. But $\sigma_k(d_{i'}) = \sigma_{i'}(d_{i'})$ for $i' < k$. Hence $\sigma'(d_{i'}) = \sigma_{i'}(d_{i'}) \in G_{\xi}$ where $d_{i'} \in \Delta$. Hence $d_{i'} \in \bar{G} \cap \Delta$, QED (6)

But then we need only show:

(7) $\bar{b}_i \in \bar{G}$ for $i < \omega$.

proof

Pick $k > i$ s.t. $m > i$ where $\sigma_k(\bar{\xi}_m) \leq \xi_k < \sigma_k(\bar{\xi}_{m+1})$ and $h_{\bar{\xi}_m}(\bar{b}_i) = \bar{b}_i$. Then $\sigma_k(h_{\bar{\xi}_m}(\bar{b}_k)) \in G$, where $h_{\bar{\xi}_m}(\bar{b}_k) \subset h_{\bar{\xi}_m}(\bar{b}_i) = \bar{b}_i$. Hence $\sigma_k(\bar{b}_i) \in G$ for sufficiently large k. Hence $\sigma'(\bar{b}_i) \in G$ and $\bar{b}_i \in \bar{G}$, QED (7)

Hence (d) holds, since $\langle \bar{b}_i \mid i < \omega \rangle$ is a master sequence. We must also show that $c \subset a$, but this is trivial since $\bar{b}_0 \subset \bar{a}$ + hence $a = \sigma'(\bar{a}) \in G$ whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic.

This verifies the <u>Claim</u>. It remains only to define c_k, σ_k and verify <u>I</u>, <u>II</u>. Here, too, we closely follow the proof in Thm 1.

Just as in Thm 1 we shall, in an intermediate step, define b_k, $\dot{\sigma}_k$, where $b_k \in \mathbb{B}_{\xi_k}$, and then define $c_k \subset b_k$.
(Note The b_k we shall define now are not to be confused with the elements of the master sequence $\langle \bar{b}_k \mid k < \omega \rangle$. We apologize for having used the same letter.)
We shall inductively verify I – IV, where III, IV are exactly as before.
Suppose first that I – IV hold below k and b_k, $\dot{\sigma}_k$ are given satisfying III (a)–(c) and IV. We define c_k and verify I, II, III (d). We first define:

$$a^{i\nu\mu} \quad (\nu \leq \xi_k < \mu < \tilde{\lambda}, \ \sup_{i<k} \xi_i < \nu)$$

exactly as before and let $A = A_k$ be the set of $a^{i\nu\mu} \neq 0$, as before. IV then gives us $\dot{\sigma}_a$ for $a = a^{i\nu\mu} \in A_k$ s.t.

(8) $\dot{\sigma}_a \in \mathbb{B}_\nu$ and $\dot{\sigma}_a^G = \dot{\sigma}_k^G$ whenever $G \ni a$ is $\mathbb{B}_{\xi_k}$-generic.

Arguing as before – and imitating the construction of $\dot{\sigma}_{i+1}$, c_{i+1} from σ_i, $\dot{\sigma}_i$ in Case 1, we get:

(9) Let $a \in A_k$, $a = a^{i\nu\mu}$. There exist $\tilde{a} \in \mathbb{B}_\mu$, $\dot{\sigma}_a' \in V^{\mathbb{B}_\mu}$ s.t. $h_\nu(\tilde{a}) = a$ and whenever $G \ni \tilde{a}$ is $\mathbb{B}_\mu$-generic, $\sigma_a = \dot{\sigma}_a^G$, $\sigma_a' = \dot{\sigma}_a'^G$, and $\sigma_l = \dot{\sigma}_l^G$ for $l < k$, then:

(a) $\sigma_a' : \bar{W} \prec W$

(b) $\sigma_a'(\bar{\theta}, \bar{\mathbb{B}}, \bar{\tau}, \bar{\lambda}_i) = \theta, \mathbb{B}, \tau, \lambda_i$ $(i = 1, \ldots, n)$

(c) $\sup \sigma_a'{}''\bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, n)$

(d) $\bar{G} = \sigma_a'^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar{\xi}_{i+1}}$-generic over $\bar{W}$

(e) Let r be least s.t. $\mu \leq \xi_r$. Then $\sigma_a'(x_l, \bar{b}_l, d_l) = \sigma_a(x_l, \bar{b}_l, d_l)$ for $l < r$, where d_l is as above for $l < k$ and

$$d_l = \begin{cases} \text{the } \bar{W}\text{-least } d \in x_l \text{ s.t. } \sigma_a(d) \in G_\nu \\ \quad \text{if such exists,} \\ 0 \text{ if not} \end{cases}$$

for $k \leq l < r$.

(f) Let r be as above. Let $j = 0, \ldots, n+1$ and let $\sigma_a(\bar{\xi}^j_m) \leq \xi_r < \sigma_a(\bar{\xi}^j_{m+1})$. Then $\sigma_a'(\bar{\xi}^j_l) = \sigma_a(\bar{\xi}^j_l)$ for $l \leq m+1$. (Note that $m = i$ for $j = n+1$)

(g) $\sigma_a'(h_{\bar{\xi}_{i+1}}(\bar{b}_{i+1})) = \sigma_a(h_{\bar{\xi}_{i+1}}(\bar{b}_{i+1})) \in G$

We fix $\tilde{a}, \dot{\sigma}_a'$ for $a \in A_k$ and again define c_k by:

Def Set $\bar{b} = b_k \setminus \bigcup A_k$. $c_k =_{Df} \bar{b} \cup \bigcup_{a \in A_k} h_{\xi_k}(\tilde{a})$.

The verifications are as before.

Now let I – IV hold below k. We must define $b_k, \dot{\sigma}_k$ and verify III (a)–(c) and IV.

For $k = 0$ again set: $b_k = 1$, $\dot{\sigma}_k = \check{\sigma}$.

Now let $k = j+1$. Since $A_l, \langle \tilde{a} \mid a \in A_l \rangle$ has been defined for $l \leq j$ we again set:

Def $\hat{A}_j =$ the set of $a = a^{i \nu \mu} \in \bigcup_{l \leq j} A_l$ s.t. $\xi_j < \mu$.

Def $b_k = \bigcup \{ h_{\xi_k}(\tilde{a}) \mid a \in \hat{A}_j \}$

Def Set $\tilde{A} =$ the set of $a^{i \nu \mu} \in \hat{A}_j$ s.t. $\mu \leq \xi_k$

$\dot{\sigma}$ is an element of $V^{\mathbb{B}}$ s.t.

$[\![\dot{\sigma}_k = \dot{\sigma}_a']\!]$ if $a \in \tilde{A}$

$[\![\dot{\sigma}_k = \dot{\sigma}_j]\!] \cap b_k = b_k \setminus \bigcup \tilde{A}$.

The verifications are exactly as before.

QED (Thm 5)

It is not hard to reformulate and reprove Thm 2 – Thm 4 for "subproper" instead of "subcomplete".