§2 The Forcing Construction

In the following let $\omega < \kappa < \beta$, where κ is regular, β is a cardinal, $2^{\underset{\smile}{\kappa}} = \kappa$ and $2^{\underset{\smile}{\beta}} = \beta$. Set $M = L^A_\beta = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set $N = \langle H_{\beta^+}, M, <, \dots \rangle$, where $<$ is a well ordering of N. (Then N is a ZFC$^-$ model.) Let $\mathcal{L}$ be a language on N satisfying the above conditions $(*)$ of §1. Suppose moreover that it has two constants $\dot{M}, \dot{\pi}$ and contains the additional axioms:

$(**)$ • $\underline{\kappa} = \omega_1$

• $\dot{M} = \langle \dot{M}_i \mid i \leq \underline{\kappa} \rangle$ where $\dot{M}_i = L^{\dot{A}_i}_{\beta_i}$ is countable for $i < \underline{\kappa}$ and $\dot{M}_{\underline{\kappa}} = \underline{M}$

• $\dot{\pi} = \langle \dot{\pi}_{ij} \mid i \leq j \leq \underline{\kappa} \rangle$ is a commutative, continuous system of elementary embeddings $\dot{\pi}_{ij} : \dot{M}_i \prec \dot{M}_j$ ("continuous" means that $\dot{M}_\lambda = \bigcup_{i<\lambda} \operatorname{rng}(\dot{\pi}_{i\lambda})$ for limit λ.)

• $\beta_i < \dot{\kappa}_{i+1}$ for $i < \underline{\kappa}$ (where $\dot{\kappa}_i = \operatorname{crit}(\dot{\pi}_{i,i+1})$)

We shall show that if $\mathcal{L}$ satisfies two further conditions, we can form a set generic extension of the universe containing $\langle M_i \mid i \leq \kappa \rangle$, $\langle \pi_{ij} \mid i \leq j \leq \kappa \rangle$ satisfying $(**)$ (and also some other statements which are provable in $\mathcal{L}$). The first of these conditions is consistency. The second is reresectionability:

Def Let $\mathcal{L}$ be a consistent language on N satisfying (*), (**). $\mathcal{L}$ is reresectionable iff the following holds:

In the generic collapse $V[G]$ of β^+ to ω we have:

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $\bar{N}, \bar{\mathcal{L}}, \sigma$ be s.t. $\bar{N}, \bar{\mathcal{L}} \in H_\kappa^{\mathfrak{A}}$, $\sigma \in wfc(\mathfrak{A})$ and $\sigma : \langle \bar{N}, \bar{\mathcal{L}} \rangle \prec \langle N, \mathcal{L} \rangle$ s.t., letting $\alpha = \sigma^{-1}(\kappa)$ we have: $\bar{M} = \dot{M}_\alpha^{\mathfrak{A}}$ and $\sigma \upharpoonright \bar{M} = \dot{\pi}_{\alpha\kappa}^{\mathfrak{A}}$, where $\sigma(\bar{M}) = M$. Let $\bar{\mathfrak{A}} \in H_\kappa^{\mathfrak{A}}$ be a solid model of $\bar{\mathcal{L}}$. Define:

$\tilde{M} = \langle \tilde{M}_i \mid i \leq \kappa \rangle$, $\tilde{\pi} = \langle \tilde{\pi}_{ij} \mid i \leq j \leq \kappa \rangle$ by:

$$\tilde{M} = \begin{cases} \dot{M}_i^{\bar{\mathfrak{A}}} & \text{if } i \leq \alpha \\ \dot{M}_i^{\mathfrak{A}} & \text{if } i \geq \alpha \end{cases} ; \quad \tilde{\pi}_{ij} = \begin{cases} \dot{\pi}_{ij}^{\bar{\mathfrak{A}}} & \text{if } i \leq j \leq \alpha \\ \dot{\pi}_{\alpha j}^{\mathfrak{A}} \dot{\pi}_{i\alpha}^{\bar{\mathfrak{A}}} & \text{if } i \leq \alpha \leq j \\ \dot{\pi}_{ij}^{\mathfrak{A}} & \text{if } \alpha \leq i \leq j \end{cases}$$

Form $\tilde{\mathfrak{A}}$ by reinterpreting $\dot{M}, \dot{\pi}$ as $\tilde{M}, \tilde{\pi}$. Then $\tilde{\mathfrak{A}}$ is a solid model of $\mathcal{L}$.

(Note Taken literally this requires that $\mathcal{L}$ have no non logical predicates, constants, or function symbols other than $\dot{\in}$, $\underline{x}$ $(x \in N)$ and $\dot{M}, \dot{\pi}$. If, in fact, $\mathcal{L}$ has other non logical symbols, we form the reduced language $\mathcal{L}'$ with just these symbols, taking as its axioms the statements of the reduced language which are provable in $\mathcal{L}$.)

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In the following let $\mathcal{L}$ be consistent and resectionable.

We define forcing conditions $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as follows:

Def $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_{\mathcal{L}}$ is the set of $\langle p_0, p_1, p_2 \rangle$ s.t.

(a) p_0 is a finite partial map from n to κ

(b) $\mathrm{dom}(p_1) = \mathrm{dom}(p_0)$ and each $p_1(i)$ is a finite partial map from $p_0(i)$ to β for $i \in \mathrm{dom}(p_1)$

(c) $\mathrm{dom}(p_2) = \mathrm{dom}(p_0)$ and $p_2(i) \subset M$.

If $p \in \tilde{\mathbb{P}}$ we set: $D(p) = \mathrm{dom}(p_0)$, $\beta_i^p = p_0(i)$, $\pi_i^p = p_1(i)$, $a_i^p = p_2(i)$ for $i \in D(p)$. We also set: $D'(p) = \{i \in D(p) \mid a_i^p \neq \emptyset\}$.

Def Let $p \in \tilde{\mathbb{P}}$. $\mathcal{L}(p)$ is $\mathcal{L}$ augmented by the further axioms:

(a) $\underline{\beta_i^p} = On \cap \dot{M}_i$; $\underline{\pi_i^p} \subset \dot{\pi}_{\underline{i}\,\underline{n}}$ for $i \in D(p)$

(b) $\bigvee \bar{a}\ \dot{\pi}_{\underline{i}\,\underline{n}} : \langle \dot{M}_i, \bar{a} \rangle \prec \langle \underline{M}, \underline{a_i^p} \rangle$ für $i \in D(p)$

In the following we write: $\dot{\pi}_i^p$ for $\dot{\pi}_{i\underline{n}}^p$.

We also set: $a_i^p(n) = \{x \mid \langle x, n \rangle \in a_i^p\}$ for $n < \omega$, As mentioned above we also write:

$$\beta_i = On \cap M_i \quad , \quad \kappa_i = \pi_i^{-1}(\underline{\kappa}).$$

We are now ready to define $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as a subset of $\tilde{\mathbb{P}}$. We first define:

<u>Def</u> Let $p \in \tilde{\mathbb{P}}$, p is <u>good</u> iff $\mathcal{L}(p)$ is consistent.

<u>Def</u> Let $p \in \tilde{\mathbb{P}}$, $i, j \in D(p)$, $i < j$. i is <u>neat in</u> j iff a_i^p is $\langle M, a_j^p(n)\rangle$ definable in parameters from $\operatorname{rng}(\pi_j^p)$.

p is <u>neat</u> iff i is neat in j whenever $i, j \in D(p)$, $i < j$, and $R_j^p \neq \emptyset$.

<u>Def</u> Let $\mathcal{L}$ be consistent and resectionable.

$\mathbb{P} = \mathbb{P}_{\mathcal{L}} =_{\text{Df}}$ the set of $p \in \tilde{\mathbb{P}}$ which are good and neat.

$p \leq q$ in $\mathbb{P}$ iff $p, q \in \mathbb{P}$, $p_0 \supset q_0$, and $\pi_i^p \supset \pi_i^q$, $a_i^p = a_i^q$ for $i \in D(q)$.

. . . . .

Before proceeding further we prove a number of lemmas on the extendability of conditions in $\mathbb{P}$.

<u>Lemma 1.1</u> Let $p \in \mathbb{P}$. Let u be finite s.t. $D(p) \subset u \subset \kappa$. There is $p' \leq p$ s.t. $D(p') \supset u$.

proof. of Lemma 1.1

Work in a generic collapse $V[G]$ of β^+ to ω. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Set:

$p_0' = \langle \beta_i \mid i \in u \rangle$ where $\beta_i = On \cap M_i^{\mathfrak{A}}$. We then set: $p_h'(i) = \begin{cases} p_h(i) & \text{if } i \in D(p) \\ \emptyset & \text{if } i \in u \setminus D(p) \end{cases}$

for $h = 1, 2$. Then $p' \leq p$ has the desired property. QED (Lemma 1.1)

In similar fashion:

<u>Lemma 1.2</u> Let $p \in \mathbb{P}$, $i \in D(p)$, and let $u \subset \beta_i^p$ be finite. There is $p' \leq p$ s.t. $u \subset dom(\pi_i^{p'})$

prf. (Assume w.l.o.g. $dom(\pi_i^p) \subset u$)

We again let $\mathfrak{A} \in V[G]$ be a solid model of $\mathcal{L}(p)$. Set $p_0' = p_0$. p_1' is defined by: $p_1'(j) = \pi_j^{p'} = \pi_j^p$ if $j \in D(p)$, $j \neq i$,

$p_1'(i) = \pi_i^{p'} = \pi_{ik}^{\mathfrak{A}} \upharpoonright u$; $p_2' = p_2$

Then $p' \leq p$ has the desired property

QED (1.2)

<u>Lemma 1.3</u> Let $p \in \mathbb{P}$, $u \subset \beta$ where u is finite. There is $p' \leq p$ s.t. $u \subset rng(\pi_j^{p'})$ for some j.

prf. of Lemma 1.3.
Let $\mathfrak{A} \in V[G]$ be a solid model of $\mathcal{L}(p)$.
Pick $j < \kappa$ s.t. $D(p) \subset j$ and $u \subset \operatorname{rng}(\pi_{j\kappa}^{\mathfrak{A}})$.
Set: $p_0' = p_0 \cup \{\langle \beta_j^{\mathfrak{A}}, j \rangle\}$ $\quad (\beta_j = On \cap M_j)$

$$p_1'(h) = \begin{cases} \pi_h^p & \text{for } h \in D(p) \\ \pi_{j\kappa}^{\mathfrak{A}} \upharpoonright (\pi_{j\kappa}^{\mathfrak{A}\,-1}{}''u) & \text{for } h = j \end{cases}$$

$p_2' = p_2 \cup \{\langle \emptyset, j \rangle\}$. Then $p' \leq p$ has the desired property. QED(1.3)

Lemma 1.4 Let $p \in \mathbb{P}$, $u \subset \operatorname{rng}(\pi_\lambda^p)$, where λ is a limit ordinal. There is $p' \leq p$ s.t. $u \subset \operatorname{rng}(\pi_j^{p'})$ for a $j < \lambda$.

prf.
Let $\mathfrak{A} \in V[G]$ be a solid model of $\mathcal{L}(p)$.
Pick $j < \lambda$ s.t. $u \subset \operatorname{rng}(\pi_{j\lambda}^{\mathfrak{A}})$ and $j \notin D(p)$. Set:
$p_0' = p_0 \cup \{\langle \beta_j^{\mathfrak{A}}, j \rangle\}$ and

$$p_1'(i) = \begin{cases} \pi_i^p & \text{for } i \in D(p) \\ \pi_{i\lambda}^{\mathfrak{A}} \upharpoonright (\pi_{i\lambda}^{\mathfrak{A}\,-1}{}''u) & \text{for } i = j \end{cases}$$

$p_2' = p_2 \cup \{\langle \emptyset, j \rangle\}$. Then $p' \leq p$ has the desired property. QED(1.4)

Lemma 1.5 Let $p \in \mathbb{P}$. There is $p' \leq p$ s.t. $\operatorname{rng}(\pi_i^{p'}) \subset \operatorname{rng}(\pi_j^{p'})$ whenever $i, j \in D(p')$, $i < j$.

proof.
Set $p_0' = p_0$, $p_2' = p_2$. We define p_1' as follows: Set $u_i = \bigcup_{h \in D(p) \cap i} \operatorname{rng}(\pi_h)$ for $i \in D(p)$.

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Set: $\pi_i^{p'} = \dot{\pi}_{iu}^{\mathfrak{A}} \upharpoonright u_i$ for $i \in D(p)$. Then p' has the desired property. QED (1.5)

<u>Lemma 1.6</u> Let $p \in \mathbb{P}$, $\xi_1, \ldots, \xi_m \in \mathrm{rng}(\pi_i^p)$ s.t. $M \models \bigvee_\eta \varphi(\eta, \vec{\xi})$. There is $p' \leq p$ s.t. $\bigvee \eta \in \mathrm{rng}(\pi_i^{p'})$ $M \models \varphi(\eta, \vec{\xi})$.

prf.
Let $\mathfrak{A} \in V[G]$ be a solid model of $\mathcal{L}(p)$. Let $\dot{\pi}_{iu}^{\mathfrak{A}} : M_i^{\mathfrak{A}} \prec M$. Since $M \models \bigvee_\eta \varphi(\eta, \vec{\xi})$ and there is $\eta \in \mathrm{rng}(\dot{\pi}_{iu}^{\mathfrak{A}})$ s.t. $M \models \varphi(\eta, \vec{\xi})$. Let $\dot{\pi}_{iu}^{\mathfrak{A}}(\bar{\eta}) = \eta$. Set $p' = \langle p_0, p_1', p_2 \rangle$ where
$$\pi_h^{p'} = \begin{cases} \pi_h^p & \text{for } h \neq i \\ \pi_h^p \cup \{\langle \eta, \bar{\eta} \rangle\} & \text{for } h = i \end{cases}$$
Then $p' \leq p$ has the desired property.

<u>Note</u> The last proof also gives: If $\langle M, a_i^p \rangle \models \bigvee_\eta \varphi(\eta, \vec{\xi})$, where $\xi_1, \ldots, \xi_m \in \mathrm{rng}(\pi_i^p)$, then there is $p' \leq p$ s.t. $\bigvee \eta \in \mathrm{rng}(\pi_i^{p'})$ $\langle M, a_i^p \rangle \models \varphi(\eta, \vec{\xi})$.

Lemma 2 Let G be $\mathbb{P}$-generic.

(A) $\bigcup\{p_0 \mid p \in G\} = \langle \beta_i \mid i < \kappa \rangle$, where $\beta_i < \kappa$ for $i < \kappa$.
(set: $\beta_\kappa = \beta$)

(B) Let $i < \kappa$. Set $\bar{\pi}_i = \bigcup\{\pi_i^p \mid p \in G \wedge i \in D(p)\}$
Then $\bar{\pi}_i : \beta_i \to \beta$ is monotone with
$\kappa_i = \mathrm{crit}(\bar{\pi}_i)$ where $\kappa_i = \bar{\pi}_i^{-1}(\kappa)$

(C) $i \leq j \leq \kappa \to \mathrm{rng}(\bar{\pi}_i) \subset \mathrm{rng}(\bar{\pi}_j)$
(letting $\bar{\pi}_\kappa = \mathrm{id} \restriction \beta$)

(D) If $\lambda \leq \kappa$ is a limit ordinal, then
$\mathrm{rng}(\bar{\pi}_\lambda) = \bigcup_{i<\lambda} \mathrm{rng}(\bar{\pi}_i)$

(E) Let $i \leq \kappa$. Set X_i = the smallest $X \prec M$
s.t. $\mathrm{rng}(\bar{\pi}_i) \subset X$. Then $X \cap \beta = \mathrm{rng}(\bar{\pi}_i)$

(F) Set $\pi_i : M_i \xleftrightarrow{\sim} M|X_i$ $(i \leq \kappa)$.
Then $\pi_i : M_i \prec M$, $\pi_i \restriction \beta_i = \bar{\pi}_i$, $\beta_i = \mathrm{On} \cap M_i$,
$\pi_i \restriction \kappa_i = \mathrm{id}$, where $\kappa_i = \pi_i^{-1}(\kappa)$.

(G) Set $\pi_{ij} = \pi_j^{-1}\pi_i$ $(i \leq j \leq \kappa)$. Then
$\pi_{ij} : M_i \prec M_j$ and $\langle \pi_{ij} \mid i \leq j \leq \kappa \rangle$ is a
commutative continuous system of
embeddings.

prf.
We prove (A)–(E), since (F), (G) then
follow trivially.

(A) follows by 1.1. By 1.2 it follows that $\overline{\pi}_i : \beta_i \to \beta$. $\overline{\pi}_i$ is monotone however, since each π_i^p is monotone for $p \in G$, since $\mathcal{L}(p)$ is consistent. This gives (B). (C) follows by Lemma 1.5. (D) follows by Lemma 1.3 for $\lambda = \kappa$ and Lemma 1.4 for $\lambda < \kappa$. We prove (E), using Lemma 1.6. Let $f : \beta \xrightarrow{\text{onto}} M$ be M-definable. Set $X = f''\mathrm{rng}(\overline{\pi}_i)$. Then

<u>Claim 1</u> $X \prec M$

prf.
Let $x_1, \ldots, x_n \in X$, $M \models \bigvee y\, \varphi(y, \vec{x})$.
It suffices to show:
<u>Claim</u> $\bigvee y \in X \; M \models \varphi(y, \vec{x})$
Let $x_i = f(\xi_i)$, $\xi_i \in \mathrm{rng}(\overline{\pi}_i)$ $(i = 1, \ldots, n)$.
Then $M \models \bigvee \eta\, \varphi(f(\eta), f(\xi_1), \ldots, f(\xi_n))$.
Hence there is such an $\eta \in \mathrm{rng}(\overline{\pi}_i)$, and we take: $y = f(\eta)$. QED (Claim 1)

<u>Claim 2</u> $\mathrm{rng}(\overline{\pi}_i) \subset X$.

prf.
Let $\xi \in \mathrm{rng}(\overline{\pi}_i)$. Then $M \models \bigvee \eta\; \xi = f(\eta)$.
Hence there is such an $\eta \in \mathrm{rng}(\overline{\pi}_i)$.
Hence $\xi = f(\eta) \in X$. QED (Claim 2)

X is then the smallest $X \prec M$ s.t. $\mathrm{rng}(\bar{\pi}_i) \subset X$, since any such X is closed under f. It remains only to note:

<u>Claim 3</u> $X \cap \beta \subset \mathrm{rng}(\bar{\pi}_i)$.

prf.
Let $\eta \in X$, $\eta = f(\xi)$, $\xi \in \mathrm{rng}(\bar{\pi}_i)$.
Then $M \models \bigvee \eta\ \eta = f(\xi)$. Hence $\eta \in \mathrm{rng}(\bar{\pi}_i)$ by Lemma 1.6. QED (Lemma 2)

<u>Lemma 3</u> Let G be $\mathbb{P}$-generic. Then κ is regular in $V[G]$.

prf.
Let $p \Vdash \mathring{f} : \check{\delta} \to \check{\kappa}$, where $\delta < \kappa$.

<u>Claim</u> There is $p' \leq p$, $\alpha < \kappa$ s.t.
$p' \Vdash \mathrm{rng}(\mathring{f}) \subset \check{\alpha}$.

Set $N^+ = \langle H_{\beta^{++}}, N, <, \mathcal{L} \rangle$, where $<$ is a wellordering of $H_{\beta^{++}}$.
We can assume w.l.o.g. that $\mathring{f} \subset N$
(e.g. we can take
$\mathring{f} = \{ \langle \langle \check{\xi}, \check{\zeta} \rangle, r \rangle \mid p \geq r \Vdash \mathring{f}(\check{\xi}) = \check{\zeta} \}$).
Then $\mathring{f} \in N^+$. Let $\langle a(n) \mid n < \omega \rangle$ enumerate the $a \subset M$ which are N^+-definable from the parameters $\mathring{f}, p, \delta, \kappa, M$.

Set $a = \{\langle x, n\rangle \mid n<\omega \wedge x \in a(n)\}$.
If $\mathfrak{A}$ is a model and X a set, we write $X \prec \mathfrak{A}$ to mean: $X \subset \mathfrak{A}$ and $\mathfrak{A}|X \prec \mathfrak{A}$.

<u>Fact</u> For any $X \subset M$ the following are equivalent:
(a) $X \prec \langle M, a(n)\rangle$ for $n<\omega$
(b) Let $Y =$ the smallest $Y \prec N^+$ s.t. $X \cup \{\mathring{f}, p, \delta, \kappa\} \subset Y$. Then $Y \cap M = X$.
((b) $\to$ (a) is trivial. (a) $\to$ (b) follows from the fact that each $z \in Y$ is N^+-definable in parameters from $X \cup \{\mathring{f}, p, \delta, \kappa\}$.)

Now collapse β^{++} generically to ω. Work in the resulting model $V[G]$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\langle M, a\rangle \subset N \subset \mathfrak{A}$. Since κ is regular in $\mathfrak{A}$, there must be an $\alpha<\kappa$ s.t. $\alpha = \operatorname{crit}(\dot{\pi}^{\mathfrak{A}}_{\alpha\kappa}) = \dot{\kappa}^{\mathfrak{A}}_{\alpha} \supset D(p)$ and there is $\bar{a} \subset \dot{M}^{\mathfrak{A}}_{\alpha}$ s.t. $\dot{\pi}^{\mathfrak{A}}_{\alpha\kappa} : \langle \dot{M}^{\mathfrak{A}}_{\alpha}, \bar{a}\rangle \prec \langle M, a\rangle$. We define $p' \leq p$ as follows: $p' = \langle p'_0, p'_1, p'_2\rangle$, where: $D(p') = D(p) \cup \{\alpha\}$, $p'_h \upharpoonright \alpha = p_h$ $(h=0,1,2)$, $p'_0(\alpha) = \dot{\beta}^{\mathfrak{A}}_{\alpha}$, $\pi^{p'}_{\alpha} = \{\langle \kappa, \alpha\rangle\}$, $a^{p'}_{\alpha} = \bar{a}$

Then p' is good, since $\mathfrak{A}$ models $\mathcal{L}(p')$.

p' is neat, since each a_i^p $(i \in D(p))$ is $\langle M, a(n)\rangle$-definable in no parameters for some n (in fact $a_i^p = a(n)$). Thus $p' \leq p$ and it suffices to prove:

<u>Claim</u> $p' \Vdash \mathrm{rng}(\mathring{f}) \subset \check{\alpha}$.

Suppose not.

Then there are $q \leq p'$, $\xi < \delta$ s.t.

(1) $q \Vdash \mathring{f}(\check{\xi}) \geq \check{\alpha}$.

Working in $V[G]$, where G collapses β^+, pick a solid $\mathfrak{A}$ modelling $\mathcal{L}(q)$. Then

(2) $\dot{\pi}^{\mathfrak{A}}_{\alpha\kappa} : \langle M^{\mathfrak{A}}_\alpha, \bar{a}\rangle \prec \langle M, a\rangle$ for

for $\bar{a} = \dot{\pi}^{\mathfrak{A}\,-1}_{\alpha\kappa}{}'' a$ and $a = a^q_\alpha$.

Let $X = \mathrm{rng}(\dot{\pi}^{\mathfrak{A}}_{\alpha\kappa})$ and $Y =$ the smallest $Y \prec N^+$ s.t. $X \cup \{\mathring{f}, p, \delta, \kappa\} \subset Y$. Then $Y \cap M = X$ by the above Fact and (2).

Set $\pi : \bar{N}^+ \overset{\sim}{\longleftrightarrow} N^+|Y$, where $\bar{N}^+$ is transitive. Then $\pi \supset \dot{\pi}^{\mathfrak{A}}_{\alpha\kappa}$ and $\pi(\dot{M}^{\mathfrak{A}}_\alpha) = M$.

(3) $\bar{N}^+ \in \mathfrak{A}$

proof. Let $\tilde{Y} =$ the smallest $Y \prec N^+$ s.t. $M \cup \{\mathring{f}, p, \delta, \kappa, \beta\} \subset Y$. Let $\tilde{\pi} : \tilde{N}^+ \overset{\sim}{\longleftrightarrow} N|\tilde{Y}$. Then $\tilde{N}^+ \in N \subset \mathfrak{A}$, since $\tilde{N}^+$ is transitive and of cardinality $\leq \beta$. Let

Note that $\tilde{\pi} \upharpoonright M = \mathrm{id}$. Let

$$\tilde{\pi}(\tilde{f}, \tilde{p}, \tilde{\delta}, \tilde{n}, \tilde{\beta}) = f, p, \delta, n, \beta .$$

Let $Y^* =$ the smallest $Y^* \prec \tilde{N}^+$ s.t.

$$X \cup \{\tilde{f}, \tilde{p}, \ldots, \tilde{\beta}\} \subset Y^*.$$

Since $X \in \mathfrak{A}$, it follows that $Y^* \in \mathfrak{A}$.
But $\bar{N}^+$ is the transitivization of $\tilde{N}^+ | Y^*$;
Hence $\bar{N}^+ \in \mathfrak{A}$ by Mostowski's isomorphism
Theorem in $\mathfrak{A}$. QED(3)

Now let $\pi(\bar{N}, \bar{\mathcal{L}}, \bar{\mathbb{P}}, \bar{p}, \bar{f}) = N, \mathcal{L}, \mathbb{P}, p, f$.
Then $\bar{\mathcal{L}}$ is a consistent infinitary language on $\bar{N}$ and $\bar{\mathbb{P}} = \mathbb{P}_{\bar{\mathcal{L}}}$ in $\bar{N}^+$.
Note that $\pi(\alpha) = n$, $\pi \upharpoonright \alpha = \mathrm{id}$. Set:

$$q' = \langle q_0 \upharpoonright \alpha, q_1 \upharpoonright \alpha, q_2 \upharpoonright \alpha \rangle.$$

Then $q' \in \mathbb{P}$ and $q \leq q'$.

(4) $q' \in Y$.

proof.

- $i \in D(q') \rightarrow \beta_i^{q'} = \beta_i^{\mathfrak{A}} < \mu_\alpha^{\mathfrak{A}} = \alpha \subset Y$

 Hence $q_0' \subset \alpha^2 \subset Y$ is finite. Hence $q \in Y$

- $i \in D(q') \rightarrow \mathrm{rng}(\pi_i^{q'}) \subset \mathrm{rng}(\pi_i^{\mathfrak{A}}) \subset$
 $\subset \mathrm{rng}(\pi_{\alpha k}^{\mathfrak{A}}) \subset X$. Hence $q' \subset \alpha \times Y \subset Y$
 is finite.

- $i \in D(q') \to a_i^{q'}$ is $\langle M, a(m)\rangle$ - definable in parameters from $\mathrm{rng}(\pi_\alpha^q) \subset \mathrm{rng}(\pi_{\alpha n}^{\circ\mathfrak{A}}) = X$. Hence $a_i^{q'} \in Y$, since $a(m) \in Y$. Hence $q_2' \subset Y \times \alpha \subset Y$ is finite. QED(4)

Set: $\bar{q}' = \pi^{-1}(q')$. Then $\bar{q}' \leq \bar{p}$ in $\bar{\mathbb{P}}$, since $q' \leq p$ in $\mathbb{P}$. Since $\Vdash_{\bar{\mathbb{P}}}^{\bar{N}^+} \bar{f} : \check{\alpha} \to \check{\alpha}$, there is $\bar{r} \leq \bar{q}'$ in $\bar{\mathbb{P}}$ s.t.

(5) $\bar{r} \Vdash_{\bar{\mathbb{P}}}^{\bar{N}^+} \bar{f}(\check{\xi}) = \check{\nu}$ for a $\nu < \alpha$.

Set $r = \pi(\bar{r})$. Then $r \in \mathbb{P}$ and

(6) $r \Vdash_{\mathbb{P}}^{N^+} \mathring{f}(\check{\xi}) = \check{\nu}$. Hence:

(7) r is incompatible with q in $\mathbb{P}$,

since $q \Vdash \mathring{f}(\check{\xi}) \geq \check{\alpha}$.

We derive a contradiction by showing that r is, in fact, compatible with q.

Note that $\bar{\mathcal{L}}(\bar{r})$ is consistent, since $\bar{r} \in \bar{\mathbb{P}}$. Since $\bar{N}$ is countable in $\mathfrak{A}$, there is an $\bar{\mathfrak{A}} \in \mathfrak{A}$ which is a solid model of $\bar{\mathcal{L}}(\bar{r})$. By resectionability we can then define a new model $\tilde{\mathfrak{A}}$, interpreting $\dot{M}, \dot{\pi}$ by $\tilde{M} = \langle \tilde{M}_i \mid i \leq n\rangle$, $\tilde{\pi} = \langle \tilde{\pi}_{ij} \mid i \leq j \leq n\rangle$,

where:

$$\tilde{M}_i = \begin{cases} \dot{M}_\nu^{\bar{\mathfrak{A}}} & \text{for } \nu \leq \alpha \\ \dot{M}_\nu^{\mathfrak{A}} & \text{for } \nu \geq \alpha \end{cases}$$

$$\tilde{\pi}_{\nu\tau} = \begin{cases} \dot{\pi}_{\nu\tau}^{\bar{\mathfrak{A}}} & \text{for } \nu \leq \tau \leq \alpha \\ \dot{\pi}_{\alpha\tau}^{\mathfrak{A}}\,\dot{\pi}_{\nu\alpha}^{\bar{\mathfrak{A}}} & \text{for } \nu \leq \alpha \leq \tau \\ \dot{\pi}_{\nu\tau}^{\mathfrak{A}} & \text{for } \alpha \leq \nu \leq \tau. \end{cases}$$

$\tilde{\mathfrak{A}}$ is then a solid model of $\mathcal{L}$. But $\tilde{\mathfrak{A}}$ also models $\mathcal{L}(r)$, since $\beta_i^r = \beta_i^{\tilde{\mathfrak{A}}}$ for $i \in D(r)$, $\pi_i^r \subset \tilde{\pi}_{i\kappa}$ for $i \in D(r)$ and $\tilde{\pi}_{i\kappa} : \langle \dot{M}_i^{\tilde{\mathfrak{A}}}, \bar{a} \rangle \prec \langle M, a_i^r(n) \rangle$ for $i \in D(r)$, $n < \omega$, where $\pi_{i\alpha}^{\bar{\mathfrak{A}}} : \langle \dot{M}_i^{\bar{\mathfrak{A}}}, \bar{a} \rangle \prec \langle \dot{M}_\alpha^{\mathfrak{A}}, a_i^{\bar{r}}(n) \rangle$. Since $r \leq q'$ in $\mathbb{P}$, it follows that $\tilde{\mathfrak{A}}$ models $\mathcal{L}(q')$. But since in $\tilde{\mathfrak{A}}$ nothing was changed $\geq \alpha$, we have: $\tilde{\mathfrak{A}}$ models $\mathcal{L}(q \upharpoonright (\kappa \setminus \alpha))$. Hence $\tilde{\mathfrak{A}}$ models $\mathcal{L}(q)$, since $q = q' \cup q \upharpoonright (\kappa \setminus \alpha)$. Hence $\tilde{\mathfrak{A}}$ models $\mathcal{L}(r \cup q)$ and $r \cup q$ is good, $r \cup q$ might not be neat, however, so we increase its middle component to obtain an $s \leq r, q$. Note that $r = \pi(\bar{r}) \in Y = \pi$ "$\bar{N}^+$. But then r is N^+-definable in a parameter $w \in X = \operatorname{rng}(\dot{\pi}_{\alpha\kappa}^{\mathfrak{A}}) = \operatorname{rng}(\dot{\pi}_{\alpha\kappa}^{\tilde{\mathfrak{A}}})$. Thus every

a_i^{r} ($i \in D(r)$) is $\langle M, a(m)\rangle$ definable in w.

Let $\pi_{h\kappa}^{\tilde{\mathfrak{U}}}(w_h) = w$ for $h \in D(q)$, $h \geq \alpha$.

Set: $s_0 = r_0 \cup q_0$, $s_2 = r_2 \cup q_2$, and

$$\pi_i^{s} = \begin{cases} \pi_i^{r} & \text{for } i < \quad\quad) \\ \pi_i^{q} \cup \{\langle w, w_i\rangle\} & \text{for} \end{cases}$$

where $i \in D(r) \cup D(q)$.

It follows easily that $\tilde{\mathfrak{U}}$ models $\mathcal{L}(s)$ and s is neat. Hence $s \leq r, q$.

Contr! QED (Lemma 3)

We had promised to define a forcing $\mathbb{P}$ s.t. if G is $\mathbb{P}$-generic, then $V[G]$ contains $\langle M_i \mid i \leq \kappa\rangle$, $\langle \pi_{ij} \mid i \leq j \leq \kappa\rangle$ satisfying (**). The only clause in (**) not yet verified is: $\kappa = \omega_1$. We have shown, however, that κ is regular in $V[G]$, so if necessary we can do an additional forcing which makes all $\alpha < \kappa$ countable. (In many cases this step is unnecessary.)

§2.1 Mitchell's Problem

We now prove the theorem announced at the outset by giving a positive answer to the following question (posed by Mitchell):

Assume GCH. Let U be a normal ultrafilter on κ. Let $\beta > \kappa$ be a cardinal. Is there a generic extension $V[G]$ in which $\kappa = \omega_1$ and some countable structure $\langle \bar{H}, \bar{u} \rangle$ iterates up to $\langle H_\beta, U \rangle$?

We let $M = L^A_\beta$, $N = L^A_{\beta^+}$, $N^+ = L^A_{\beta^{++}}$. Our language $\mathcal{L}$ has the axioms (*), (**) augmented by:

(i) $\bigvee \bar{u}\ \dot{\pi}_{i\underline{\kappa}} : \langle \dot{M}_i, \bar{u} \rangle \prec \langle \underline{M}, \underline{U} \rangle$ for all $i < \kappa$

Set $\dot{u}_i =$ that $\bar{u}$ s.t. $\dot{\pi}_{i\underline{\kappa}} : \langle \dot{M}_i, \bar{u} \rangle \prec \langle M, \underline{U} \rangle$

(ii) $\dot{\pi}_{i,i+1} : \dot{M}_i \xrightarrow[\dot{u}_i]{} \dot{M}_{i+1}$ for $i < \kappa$

(i.e. $\dot{M}_{i+1} = \mathrm{Ult}(\dot{M}_i, \dot{u}_i)$ and $\dot{\pi}_{i,i+1}$ is the canonical embedding).

Clearly, $\mathcal{L}$ is rerectionable. Moreover

<u>Lemma 4.1</u> $\mathcal{L}$ is consistent.

proof. Let $\bar{\kappa} > \beta^{++}$ be regular. Iterate $\langle N^+, U \rangle$ to $\langle \bar{N}^+, \bar{U} \rangle$ in $\bar{\kappa}$ many steps. Let $\pi : N^+ \to \bar{N}^+$ be the iteration map. Then $\pi(\kappa) = \bar{\kappa}$, $\pi(U) = \bar{U}$. Let $\bar{\mathcal{L}} = \pi(\mathcal{L})$. By absoluteness it suffices to show:

$N^+ \models \bar{\mathcal{L}}$ is consistent.

Hence it suffices to show:

$\bar{N}^+ \models \bar{\mathcal{L}}$ is consistent.

Again by absoluteness it suffices to show that $\bar{\mathcal{L}}$ is consistent in $V[g]$, where g is $\mathrm{coll}(\omega, <\bar{\kappa})$-generic (i.e. g is generic by the conditions which collapse every $\alpha < \bar{\kappa}$ to ω). Thus $\bar{\kappa} = \omega_1$ in $V[g]$. Let $\langle\langle \bar{N}_i^+, \bar{U}_i\rangle \mid i \leq \bar{\kappa}\rangle$ be the iteration of $\langle N^+, U\rangle$ with iteration maps $\bar{\pi}_{i,j}$. Set $M_i = \bar{\pi}_{0i}(M)$, $\pi_j = \bar{\pi}_{0j} \restriction M$. Then $\langle H_{\bar{\kappa}^+}[g], \langle M_i \mid i \leq \bar{\kappa}\rangle, \langle \pi_i \mid i \leq \bar{\kappa}\rangle\rangle$ is a model of $\bar{\mathcal{L}}$. QED (4.1).

We now prove a few more lemmas on extendability of conditions:

<u>Lemma 4.2</u> Let $p \in \mathbb{P}$, $i \in D(p)$, $X \in M$ s.t. $X \subset \kappa$ and X is M-definable in $\xi_1, \ldots, \xi_m \in \mathrm{rng}(\pi_i^p)$. Let $\pi_i^p(\bar{\kappa}) = \kappa$. Then $\bar{\kappa} \in X \longleftrightarrow X \in U$.

prf.
This holds in $\mathfrak{A}$ whenever $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$. QED (4.2)

<u>Lemma 4.3</u> Let $p \in \mathbb{P}$, $i \in D(p)$. There is $p' \leq p$ s.t. $\kappa \in \mathrm{rng}(\pi_i^{p'})$.

proof of Lemma 4.2

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Let $\dot{\pi}_i^{\mathfrak{A}}(\bar{\kappa}) = \kappa$. Set $p' = \langle p_0, p_1', p_2 \rangle$, where

$$p_1'(h) = \begin{cases} p_1(h) & \text{if } h \neq i \\ p_1(h) \cup \{\langle \kappa, \bar{\kappa} \rangle\} & \text{if } h = i \end{cases}$$

Then $p' \leq p$. QED (4.3)

<u>Lemma 4.4</u> Let $p \in \mathbb{P}$, $i \in D(p)$. Let $\xi = f(\varsigma_1, \ldots, \varsigma_n, \bar{\kappa})$ where $\varsigma_1, \ldots, \varsigma_n \in \mathrm{rng}(\pi_i^p)$ and $\pi_i^p(\bar{\kappa}) = \kappa$. There is $p' \leq p$ s.t. $\xi \in \mathrm{rng}(\pi_{i+1}^{p'})$.

prf. Assume w.l.o.g. that $i+1 \in D(p)$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\xi \in \mathrm{rng}(\dot{\pi}_{i+1}^{\mathfrak{A}})$. Let $\dot{\pi}_{i+1}^{\mathfrak{A}}(\bar{\xi}) = \xi$. Set $p' = \langle p_0, p_1', p_2 \rangle$ where

$$p_1'(h) = \begin{cases} p_1(h) & \text{if } h \neq i+1 \\ p_1(h) \cup \{\langle \xi, \bar{\xi} \rangle\} & \text{if } h = i+1. \end{cases}$$

Then $p' \leq p$. QED (4.4)

Now let G be $\mathbb{P}$-generic. Define $\langle M_i \mid i \leq n \rangle$, $\langle \pi_{ij} \mid i \leq j \leq n \rangle$ as before. By the above lemmas it follows easily that $\pi_{i,i+1} : \langle M_i, U_i \rangle \longrightarrow_{U_i} \langle M_{i+1}, U_{i+1} \rangle$ where $U_i = \pi_{i,n}^{-1}{}''U$. Moreover $\langle M_n, U_n \rangle = \langle M, U \rangle$. Hence $\langle M_0, U_0 \rangle$ iterates to $\langle M, U \rangle$ in n many steps.

QED

§2.2 Variants

The question stated at the outset can easily be stated for extenders rather than normal measures: Let $M = L^u_\beta = \langle L_\beta[u], u\rangle$ be iterable where u is an extender on κ in M. Let $M_i = L^{u_i}_{\beta_i}$ be the i-th iterate, with u_i an extender on κ_i. Assume V_α u_α is an extender on κ_α in L^{u_α}. What is the least such α? The answer is as before, by exactly the same methods. In particular we get:

Assume GCH. Let U be an extender on κ in V. Let $U \in H_\beta$ where $\beta > \kappa^+$. There is a generic extension $V[G]$ in which $\kappa = \omega_1$ and a countable structure $\langle \bar{H}, \bar{U}\rangle$ iterates up to $\langle H_\beta, U\rangle$.

The proof is virtually unchanged. The details are left to the reader. We can even do better than this: Suppose e.g. that $\pi : V_\beta \prec V_\beta$ with critical point κ, and $\langle V_\beta, \pi\rangle$ is iterable. (This will be the case if π extends to a $\tilde{\pi} : V \prec M$.) Then there is a generic extension $V[G]$

s.t. $\kappa = \omega_1$ in $V[G]$ and there is a countable $\langle \bar{V}, \bar{\pi} \rangle$ in $V[G]$ which iterates up to $\langle V_\beta, \pi \rangle$. (Note that κ^+ acquires cofinality ω_1 in this model.)

. . . . . .

It is also possible to generically collapse ω_2 to ω_1, giving it cofinality ω, even in the absence of an inner model with a measure. Let $\kappa = \omega_1$, $\beta = \omega_2$. Assume GCH. Add to (**) the axiom:

(A) $\bigwedge i < \underline{\kappa}$ rng $(\mathring{\pi}_{i\underline{\kappa}})$ is cofinal in κ.

Let $\mathcal{L}$ be the language with the axioms (*), (**), (A). We show that $\mathcal{L}$ is consistent. Let $N = J^A_{\omega_3}$ be as before. Define $\langle X_i \mid i < \omega \rangle$ by: $X_0 =$ the smallest $X \prec N$ s.t. $\omega_1 \subset X$

$X_{i+1} =$ the smallest $X \prec N$ s.t. $X_i \cup \{\sup(X_i \cap \omega_2)\} \subset X$.

Set $\tilde{X} = \bigcup_{i<\omega} X_i$. Let $\sigma : \bar{N} \xrightarrow{\sim} \tilde{X}$, where $\bar{N}$ is transitive. Then $\sigma : \bar{N} \prec N$ and $\omega_2^{\bar{N}} = \mathrm{crit}(\sigma)$. Let $\sigma(\bar{\mathcal{L}}) = \mathcal{L}$. Clearly it suffices to show that $\bar{\mathcal{L}}$ is consistent, since then:

$\bar{N} \models \bar{\mathcal{L}}$ is consistent

by absoluteness & hence:

$N \models \mathcal{L}$ is consistent.

But $\bar{\mathcal{L}}$ trivially has a model, since ω_1^N is ω-cofinal. Hence $\bar{\mathcal{L}}$ is consistent. QED

$\mathcal{L}$ trivially satisfies the criterion of resectionability.

Let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ be defined as before and let G be $\mathbb{P}$-generic over V. Then ω_1 is absolute and $\omega_2^V = \bigcup_{\xi < \omega_1} \dot{\pi}^G_{in}\, ``\beta_i^G$. Hence ω_2 is collapsed to ω_1. It suffices to show:

<u>Claim</u> $\cancel{\mathrm{rng}}(\dot{\pi}^G_{o\,\omega_1})$ is cofinal in ω_2^V.

To see this we prove another extension lemma:

<u>Subclaim</u> Let $p \in \mathbb{P}$, $\xi < \beta$, $i \in D(p)$. There is $p' \leq p$ s.t. $\forall \eta < \beta_i^p \;\; \pi_i^p(\eta) > \xi$.

prf. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\mathrm{rng}(\dot{\pi}^{\mathfrak{A}}_{i\underline{n}})$ is cofinal in β. Let $\tilde{\eta} > \xi$ s.t. $\tilde{\eta} < \beta$ and $\tilde{\eta} \in \mathrm{rng}(\dot{\pi}^{\mathfrak{A}}_{i\underline{n}})$. Let $\dot{\pi}^{\mathfrak{A}}_{i\underline{n}}(\eta) = \tilde{\eta}$. Define p' by:

$$p_0' = p_0 \;;\; p_1'(j) = \begin{cases} p_1(j) & \text{for } j \neq i \\ p_1(j) \cup \{\langle \tilde{\eta}, \eta \rangle\} & \text{for } j = i \end{cases} \;;\; p_3' = p_3 .$$

QED