

§6 Another Look

We now return to Example 1 and consider the case $\beta = \omega_2$ (hence we assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$). Our forcing then adds a cofinal ω -sequence in ω_2 without adding new reals.

This is, of course, what Namba forcing was designed to accomplish. When we first developed the forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of Example 1 we were enormously proud of having accomplished the same task in a "completely different" way. Indeed the motivation of the construction and the combinatorics of the proof are different, but the forcing turns out to be the same, for we have: $BA(\mathbb{P}) \cong BA(\mathbb{N})$, where \mathbb{N} is the set of Namba conditions. We shall now prove this. Until further notice we assume: $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. (We also let β denote ω_2 .) We first define the set \mathbb{N} of Namba conditions:

Def $\mathbb{N} =$ the set of subtrees $T \neq \emptyset$ of $\omega_2 < \omega$ s.t. $\overline{\{t \mid r \sqsubseteq_T t\}} = \omega_2$ for all $r \in T$.

\mathbb{N} is partially ordered by: $T \leq T' \iff T \subset T'$

Note Call T a strict Namba tree iff

- $\forall r \in T \forall t (r \sqsubseteq_T t \wedge r \neq t \implies t$ is a split pt. of T)
- At r is a split point of T , then r has ω_2 many immediate successors in T .

\mathbb{N} is sometimes defined as the set of strict Namba trees. This is equivalent, however, since the strict Namba trees are dense in the Namba trees.

Call $f: \omega \rightarrow \omega_2$ a branch in T iff $f \upharpoonright m \in T$ for $m < \omega$. If H is \mathbb{N} -generic over V , then $h = \bigcup H$ is a cofinal map of ω to ω_2 . Moreover, $H = \{T \in \mathbb{N} \mid h \text{ is a branch in } T\}$.

We then say that h is a Namba-generic sequence.

Let $\mathbb{P} = \mathbb{P}$ be as in Example 1 with $\beta = \omega_2$. (Thus we assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$.)

Lemma 1 Let G be \mathbb{P} -generic. Let $h \in V[G]$ s.t. $h: \omega \rightarrow \omega_2^V$ cofinally. Then h is Namba-generic.

proof of Lemma 1

Let $h^G = h$. Assume w.l.o.g. that $\Vdash h : \check{\omega} \rightarrow \check{\omega}_2$ cofinally.

Claim 1 $\Delta_h^{\circ} = \{ p \mid |p| = \omega_1 \overset{M^P}{|p|} \wedge \bigwedge \check{h} \in M^P_{|p|} \ p \Vdash h^{\circ} = \pi_{|p|, \omega_1}^{\circ} \check{h} \}$

is dense in \mathbb{P} ,

proof.

Let $r \in \mathbb{P}$. We seek $p \leq r$ s.t. $p \in \Delta_h^{\circ}$.

Let $G \ni r$ be \mathbb{P} -generic, $h = h^G$. Then

there must be $d < \omega_1$ s.t.

$\text{rng}(h) \subset \text{rng}(\pi_{d, \omega_1}^{\circ})$. Hence $h = \pi_{d, \omega_1}^{\circ} \check{h}$

for an $\check{h} \in M_d^G$. But then there must

be a $p \in G$ s.t. $|p| \geq d$, $|p| = \omega_1 \overset{M^P}{|p|}$,

and $p \Vdash h^{\circ} = \pi_{d, \omega_1}^{\circ} \check{h}$. We may

then assume $|p| = d$ (otherwise replace

\check{h} by $\pi_{d, |p|}^P \circ \check{h}$), QED (Claim 1)

Def Let $p \in \Delta_h^{\circ}$. Set

$T^p =$ the set of $\alpha \in \omega_2 < \omega$ s.t.

$\llbracket \check{\varphi}_\alpha \rrbracket \neq \emptyset$ in $BA(\mathbb{P})$, where $\check{\varphi}_\alpha =$

$(\check{p} \in \check{G} \wedge \bigwedge_{i < |\alpha|} h^{\circ}(i^{\check{v}}) = \check{\alpha}_i)$. ($|\alpha| = \text{length}(\alpha)$)

Claim 2 $T^p \in \mathbb{N}$ for $p \in \Delta_h^0$.

pf. Suppose not, let $T = T^p$,

Then there is $\alpha \in T$ s.t. $\overline{T_{(\alpha)}} < \omega_2$,

where $T_{(\alpha)} = \{t \in T \mid \alpha \leq_T t \vee t \leq_T \alpha\}$.

Hence $\{t(i) \mid t \in T_{(\alpha)} \wedge i < |t|\} \subset \delta < \omega_2$

for some δ . Let G be IP-
-generic s.t. $G \cap \mathbb{Q}_\alpha \neq \emptyset$. Then

$p \in G$ and $h^G(i) = \alpha_i$ for $i < |\alpha|$.

But $\sup h^G \omega_1 = \omega_2^V$, hence there is
j s.t. $h^G(j) \geq \delta$. But then

$V[G] \models \varphi_t$, where $t = h^G(j+1)$,

Hence $t \in T_{(\alpha)}$ and $t(j) \geq \delta$,

Contr! QED (Claim 2)

In the following let $p \in \Delta_h^0$ and

let $\bar{M} = M_d^p$, $d = |p|$. Let

$\forall \bar{h} \in \bar{M} \quad \pi_d^{\bar{h}} \circ h^{\bar{h}} = h^0$,

Def For $r \in M$ set $M^{(r)} = L_r^A$, where

$M = L_\beta^A$. Similarly, for $r \in \bar{M} = M_d^p =$

$= L_{\beta_d}^{A_d}$ we set $\bar{M}^{(r)} = L_r^{A_d}$.

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Claim 3 Let $\bar{v} = \bar{h}(i)$ and let $v = \pi(v)$
for an $\pi \in T = T^P$. Set:

$f =$ the M -locus $f: \omega_1 \xrightarrow{\text{onto}} M^{(v)}$

$\bar{f} =$ " \bar{M} - " $\bar{f}: \omega_1 \xrightarrow{\text{onto}} \bar{M}^{(\bar{v})}$.

Set $\pi = \pi(\bar{v}, v) = \{ \langle \bar{f}(\bar{z}), f(z) \rangle \mid \bar{z} < \alpha \}$,

Let $\langle a, \bar{a} \rangle \in FP$. Then

$$\pi: \langle \bar{M}^{(\bar{v})}, \bar{a} \cap \bar{M}^{(\bar{v})} \rangle \prec \langle M^{(v)}, a \cap M^{(v)} \rangle.$$

proof.

Let G be P -generic with $\llbracket \Phi_1 \rrbracket \cap G \neq \emptyset$.

Then $\pi_{d\omega_1}^G(\bar{f}) = f$. Hence

$$\pi_{d\omega_1}^G(\bar{f}(\bar{z})) = f(z) \text{ for } \bar{z} < \alpha \text{ + hence}$$

$$\pi = \pi_{d\omega_1}^G \upharpoonright \bar{M}^{(\bar{v})}, \text{ where}$$

$$\pi_{d\omega_1}^G(\langle \bar{M}^{(\bar{v})}, \bar{a} \cap \bar{M}^{(\bar{v})} \rangle) = \langle M^{(v)}, a \cap M^{(v)} \rangle,$$

since $\pi_{d\omega_1}^G(\bar{v}) = v$ and

$$\pi_{d\omega_1}^G: \langle \bar{M}, \bar{a} \rangle \prec \langle M, a \rangle. \quad \square \text{ED (Claim 3)}$$

Claim 4 Let $T \leq T^P$ in \mathbb{N} . There is

$g \leq p$ s.t. $g \upharpoonright h$ is a branch in \bar{T} .

proof of Claim 4,

Let $N^* = \langle H_\delta, M, <, p, T, \bar{P}, \bar{N}, \dots \rangle$ where $\delta > \beta^+$. Let $p' \leq p$ conform to N^* .

Let $\bar{N}^* = \bar{N}^*(p', N^*) = \langle \bar{N}, \bar{M}, <, \bar{p}, \bar{T}, \bar{P}, \bar{N}, \dots \rangle$.

Let \bar{H} be \bar{N} -generic over \bar{N}^* s.t.

$\bar{T} \in \bar{H}$. Set $\bar{h} = U \cap \bar{H}$. Then

$\bar{h} : \omega \rightarrow \bar{\beta}$ cofinally, where $\bar{\beta} = \text{On} \cap \bar{M}$.

(Note that $\bar{M} = M_{(p)}$.) Obviously Claim 3

holds relativized to \bar{N}^* . Thus, for $\bar{v} = \bar{h}(i)$, $\tilde{v} = \bar{h}(i)$ we can define $\pi(\bar{v}, \tilde{v})$ as in Claim 3. It is easily seen that

if $\bar{h}(i) < \bar{h}(j)$, then

$$\pi(\bar{h}(i), \bar{h}(i)) \subset \pi(\bar{h}(j), \bar{h}(j)).$$

Thus, setting $\pi = \bigcup_{i < \omega} \pi(\bar{h}(i), \bar{h}(i))$,

we have:

$\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \bar{M}, \tilde{a} \rangle$ cofinally for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\bar{p}}$. Since $\langle \bar{M}, \tilde{a} \rangle$ is a ZFC⁻ model, we conclude:

(1) $\pi : \langle \bar{M}, \bar{a} \rangle \prec \langle \bar{M}, \tilde{a} \rangle$ for all $\langle \tilde{a}, \bar{a} \rangle \in F^{\bar{p}}$.

Moreover:

(2) $\text{rng}(\pi) = \text{the smallest } X \prec \bar{M}$
s.t. $\text{rng}(\bar{h}) \cup d \subset X$ ($d =_{\bar{p}}(p)$)

proof of (2)

(\supset) is trivial. But if $\text{rng}(\tilde{h}) \subset X < \tilde{M}$, then $f_{\tilde{h}(i)} \in X$ for all $i < \omega$. Hence

$$\text{rng}(\pi(\tilde{h}(i), \tilde{h}(i))) = f_{\tilde{h}(i)} \in X.$$

QED(2)

Now let $\tilde{\alpha} = |P'|$ (hence $\tilde{\alpha} = \omega_1 \bar{N}^* + 1$).

Since \bar{H} is \bar{N}^* -generic over \bar{N}^* , $\bar{N}^*[\tilde{h}]$ is a ZFC-model. An $\bar{N}^*[\tilde{h}]$ we define $\langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle, \langle \tilde{\pi}_i \mid i \leq i \leq \tilde{\alpha} \rangle$ as follows:

For $\beta < \tilde{\alpha}$ let $X_\beta =$ the smallest $X < \tilde{M}$ s.t. $\beta \text{rng}(\tilde{h}) \subset X$. Set:

$C = \{ \beta \geq \alpha \mid \beta = (X_\beta \cap \tilde{\alpha}) \}$. Then $\tilde{\alpha} \in C$ and $C \cap \tilde{\alpha}$ is club in $\tilde{\alpha}$. Set:

$$\tilde{C} = C \cup \{ \omega_1^{M_i^P} \mid i \leq \alpha \} \quad (\alpha = |P|).$$

For $\alpha \leq i \leq \tilde{\alpha}$ set: $\tilde{\pi}_i: \tilde{M}_i \leftrightarrow X_{\tilde{\alpha}_i}$,

where \tilde{M}_i is transitive. For $i \leq \alpha$ set:

$$\tilde{M}_i = M_i^P, \quad \tilde{\pi}_i = \pi_\alpha^P \circ \pi_{i,\alpha}^P.$$

$$\tilde{\pi}_{i,j} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i \quad \text{for } i \leq j \leq \tilde{\alpha}.$$

Define \mathcal{G} by: $M^{\mathcal{G}} = \langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle,$

$$\pi^{\mathcal{G}} = \langle \tilde{\pi}_{i,j} \mid i \leq j \leq \tilde{\alpha} \rangle, \quad F^{\mathcal{G}} = F^{P'}.$$

But then:

(3) $q \in \mathbb{P}$

proof.

Let \mathcal{M} model $\mathcal{L}(p')$. Change \mathcal{M} to $\tilde{\mathcal{M}}$ by replacing $\dot{m}_i^{\mathcal{M}}, \dot{\pi}_{i'}^{\mathcal{M}}$ by $\tilde{m}_i, \tilde{\pi}_{i'}$ for $i \leq \tilde{\alpha} - i.e$

$$\dot{m}_i^{\tilde{\mathcal{M}}} = \begin{cases} \dot{m}_i^{\mathcal{M}} & \text{for } i \geq \tilde{\alpha} \\ \tilde{m}_i & \text{for } i \leq \tilde{\alpha} \end{cases}$$

$$\dot{\pi}_{i'}^{\tilde{\mathcal{M}}} = \begin{cases} \dot{\pi}_{i'}^{\mathcal{M}} & \text{for } \tilde{\alpha} \leq i \leq j \\ \dot{\pi}_{\tilde{\alpha}'}^{\mathcal{M}} = \tilde{\pi}_{i'} & \text{for } i \leq \tilde{\alpha} \leq 1 \\ \tilde{\pi}_{i'} & \text{for } i \leq i' \leq \tilde{\alpha} \end{cases}$$

Then $\tilde{\mathcal{M}}$ models $\mathcal{L}(q)$. QED (3)

But then:

(4) \leq

proof.

$M^P = M^Q \upharpoonright (|P|+1)$, $\pi^P = \pi^Q \upharpoonright (|P|+1)^2$ by the construction of q . But if

$\langle a, \bar{a} \rangle \in F^P$, then, since $P \leq P'$,

there is a' s.t. $\langle a, a' \rangle \in F^{P'}$ and

$$\pi_{|P|, |P'|}^{P'} : \langle M_{|P|}^P, \bar{a} \rangle \prec \langle M_{|P'|}^{P'}, a' \rangle,$$

But then, whenever $G \ni P'$ is $|P|$ -generic, we have:

$$\pi_{|P'|, |P|}^G : \langle M_{|P'|}^{P'}, a' \rangle \prec \langle M, a \rangle,$$

Since $\pi_{|P'|, |P|}^a$ extends uniquely

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to $\sigma: \bar{N}^* \hookrightarrow N^*$ with $\pi_{|P', |P|}^c \cup F^{P'} \subset \sigma$,

it follows that $F^{P'}(a') = \sigma(a') = a$,

hence $a' = \tilde{a} = \sigma^{-1}(a)$. Then we

have $\pi_{|P', |Q|}^g: \langle M_{|P|}^P, \tilde{a} \rangle \hookrightarrow \langle M_{|P'|}^{P'}, \tilde{a} \rangle$

by (1), where $\langle a, \tilde{a} \rangle \in F^g$.

□ E D (4)

It remains only to show:

(5) $g \mid h$ is a branch in T' .

Let $G \ni g$ be P -generic. Let

$h = h^G$. Then $\pi_{\tilde{a}, \omega_1}^G$ extends to a

unique $\sigma: \bar{N}^* \hookrightarrow N^*$ with $F^g \subset \sigma$.

Since $p \in G$, we know that

$h = \pi_{\tilde{a}, \omega_1}^c \circ \bar{h}$, but $\tilde{h} = \pi_{\tilde{a}, \tilde{a}}^c \circ \bar{h}$.

Hence $h = \pi_{\tilde{a}, \omega_1}^G \circ \tilde{h}$. Since \tilde{h} is

a branch in \tilde{T}' , h must be a

branch in $T' = \sigma(\tilde{T}')$.

□ E D (Claim 4)

We now prove Lemma 1.

Let G be P -generic, $h = h^G$. We

must show that

$$H = \{T \mid h \text{ is a branch in } T\}$$

is N -generic.

Let Δ be dense in \mathbb{N} . It suffices to show that Δ^* is dense in \mathbb{P} , where:

$$\Delta^* = \{p \mid \exists T \in \Delta \text{ } p \Vdash \dot{h} \text{ is a branch in } \check{T}\}$$

is dense in \mathbb{P} . Let $r \in \mathbb{P}$. By Claim 1

there is $p \leq r$ s.t. $p \in \Delta_h^0$. Let $T \leq T_p$

s.t. $T \in \Delta$. By Claim 4 there is

$p' \leq p$ s.t. $p' \Vdash \dot{h}$ is a branch in \check{T} .

□ E D (Lemma 1)

Let $A = BA(\mathbb{N})$, $B = BA(\mathbb{P})$. We

wish to show that $A \cong B$. For

$a \in A$ let $A|a$ be the restriction

of A to $\{a' \mid a' \leq a\}$. Then $A|a$ is a

complete BA and $\|_{A|a} = a$. We also

have: $A|a = BA(\mathbb{P}|a)$. Similarly

for B . As a corollary of Claim 4 in

the foregoing proof we have:

Lemma 2.1 Let $p \in \Delta_h^0$. Then

$$B|_{[p]_{\mathbb{P}}} = A|_{[T_p]_{\mathbb{N}}}.$$

(Here $[p]_{\mathbb{P}}$ is the smallest $b \in \mathbb{P}$ s.t. $p \in b$.

Similarly for $[T]_{\mathbb{N}}$.)

prf. of Lemma 2.1

Let $G \ni p$ be IP-generic. Set:

$H_G = \{T \in \mathbb{N} \mid h^G \text{ is a branch in } T\}$. Then

H_G is \mathbb{N} -generic by Claim 4. But

$h^G = \bigcup \bigwedge_{H_G \in V[H_G]} \text{ and } \langle M_i^G \mid i \leq \omega_1 \rangle,$

$\langle \pi_{i_i}^G \mid i \leq i \leq \omega_1 \rangle$ are uniformly $V[H_G]$ -

definable in M, p, h^G . But

then G is uniformly $V[H_G]$ -definable

in M, p, H_G , since

$$q \in G \iff (M^q = M^G \cap (|q|+1) \wedge \pi^q = \pi^G \upharpoonright (|q|+1)^{2-1}$$

$$\wedge \bigwedge \langle a, \bar{a} \rangle \in F^q \left(\frac{\pi^G}{|q|, \omega_1} : \langle M_{|q|}^G, \bar{a} \rangle \prec \langle M, a \rangle \right).$$

Hence there is a canonical $\check{G} \in V^{\mathbb{N}}$

s.t. $\check{G} \upharpoonright H_G = G$, whenever $G \ni p$ is IP-

-generic. For $a \in \mathbb{B}[\mathbb{P}]_{\mathbb{P}}$ set

$$\sigma(a) = \sigma(\llbracket \check{a} \cap \check{G} \neq \emptyset \rrbracket_{\mathbb{P}}) = \llbracket \check{a} \cap \check{G} \neq \emptyset \rrbracket_{\mathbb{N}}$$

σ is easily seen to be a homomor-

phism of $\mathbb{B}[\mathbb{P}]$ into $\mathbb{A}[\mathbb{T}_p]$. But

σ is injective, since if $\sigma(a) = \emptyset$,

then $a \cap G = \emptyset$ for all IP-generic

$G \ni p$. Hence $a = \emptyset$. It remains only

to show that σ is onto. It is enough

to show that $[T] \in \text{rng}(\sigma)$ for

each $T \leq T_p$ in \mathbb{N} . Let $a = \llbracket T \in H_G \rrbracket_{\mathbb{P}}$.

We claim: $\sigma(a) = [T]$, or in other words?

$\sigma(a) \cap H \neq \emptyset \iff T \in H$ for \mathbb{N} -generic T with $T_p \in H$
 If not there is $T' \leq T_p$ which forces
 the negation of this equivalence,

Let $G \ni p$ be \mathbb{P} -generic w.t. $T' \in H_G$.

Then $G = \check{G}^H$, where $H = H_G$. Hence

$$\sigma(a) \cap H \neq \emptyset \iff a \cap \check{G}^H = a \cap G \neq \emptyset \iff T \in H,$$

Contrs! QED (Lemma 2.1)

Using this we prove:

Lemma 2.2 $A \simeq B$ ($A = BA(\mathbb{N}), B = BA(\mathbb{P})$)

prf.

We first note some facts about \mathbb{N} . We recall that the strict Mamba trees are dense in \mathbb{N} .

(1) Let $T \in \mathbb{N}$ be strict. Then

$$A|[T] \simeq A$$

proof.

$A|[T] \simeq BA(\{T' \mid T' \leq T\})$. But forcing with subtrees of T is the same as forcing with subtrees of the set S of split points in T .

$$\text{But } S \simeq 2^{<\omega_1}. \quad \text{QED(1)}$$

(2) Let $a \in A \setminus \{\emptyset\}$. Then $A|a \simeq A$

prf. of (2)

$|A|a = BA(\{\tau \mid \tau \in a\})$. But $\{\tau \mid \tau \in a\}$ then collapses 2^{ω_2} , hence cannot satisfy the 2^{ω_2} -chain condition. Thus there is an max antichain $\langle T_r \mid r < 2^{\omega_2} \rangle$ in $\{\tau \mid \tau \in a\}$.

We may w.l.o.g. assume that each T_r is strict. Similarly there is such a ^{maximal} antichain $\langle T'_r \mid r < 2^{\omega_2} \rangle$ in \mathbb{N} . Let $\sigma_r : |A|[T_r] \xrightarrow{\sim} |A|[T'_r]$.

Then $a = \bigcup_r [T_r]$, $\mathbb{1} = \bigcup_r [T'_r]$ and we can define $\sigma : |A|a \xrightarrow{\sim} |A|$ by $\sigma(b) = \bigcup_{r < 2^{\omega_2}} \sigma_r(b \cap [T_r])$. QED(2)

Since Δ_h^o is dense in \mathbb{P} , there is a max. antichain $\langle P_r \mid r < 2^{\omega_2} \rangle$ in \mathbb{P} s.t. $P_r \in \Delta_h^o$ for all r . Hence we may pick $\sigma_r : |B|[P_r] \xrightarrow{\sim} |A|[T'_{P_r}]$

by Lemma 2.1. We then define $\sigma : |B| \xrightarrow{\sim} |A|$ by $\sigma(b) = \bigcup_r \sigma_r(b \cap [P_r])$.

QED (Lemma 2.2)

We now show that, even if $\beta > \omega_2$, the forcing $\mathbb{P} = \mathbb{P}_\beta$ of Example 1 is equivalent to a variant of Namba forcing. We define:

Def Let $\beta > \omega_1$ be regular. By a Namba amoeba on β we mean a subtree T

of $\mathcal{P}_{\omega_2}(\beta) < \omega$ s.t. if $s \in T$, then

(a) $s(i) \subset s(j)$ for $i \leq j < |s|$

(b) If $u \in \mathcal{P}_{\omega_2}(\beta)$, then

$$\{t \mid s \leq_T t \wedge \forall i \ u \cap t(i) \neq \emptyset\} \neq \emptyset.$$

From now on we let \mathbb{N} be the set of Namba amoebas ordered by:

$$T \leq T' \iff T \subset T' \text{ for } T, T' \in \mathbb{N}.$$

We develop the main properties of Namba amoebas with a view to proving Lemma 3 below.

Def In any forcing extension of V we call h a meat sequence iff

$h: \omega \rightarrow \mathcal{P}_{\omega_2}(\beta)^V$, $h(i) \subset h(j)$ for $i \leq j < \omega$,
and for every $u \in \mathcal{P}_{\omega_2}(\beta)^V$ there is i s.t. $u \subset h(i)$.

It is easily seen that if G is \mathbb{N} -generic and $h = \bigcup G$, then h is a meat sequence. We shall also show that \mathbb{N} does not add new reals.

We shall prove:

Lemma 3 Let G be \mathbb{P} -generic over V .

- (a) $V[G] = V[h]$, where h is a neat sequence.
- (b) If $h \in V[G]$ is a neat sequence, then h is \mathbb{N} -generic over V and $V[G] = V[h]$.

Note We do not know whether

$$BA(\mathbb{P}) \cong BA(\mathbb{N}).$$

Def Let $T \in \mathbb{N}$, $r \in T$ is a big split point in T iff $\text{card}\{\{u : r \hat{<} u\} \in T\} \geq \beta$.

(1) Let $r \in T$. Then there is a big split point $t \geq r$ in T .

prf. Suppose not.

By ind. on n there are fewer than β $t \in T$ s.t. $r \leq_T t$ and $|t| \leq n$. Hence $T_{(r)} < \beta$, which contradicts (b)

(Here $T_{(r)} = T \setminus \{t \mid r \leq_T t \vee t \leq_T r\}$)

As in the case of ordinary Namba amoeba trees we get an amalgamation lemma for Namba amoeba:

Def By an amalgamation sequence we mean a sequence $\langle \langle T_u, \tau_u \rangle \mid u \in \beta^{<\omega} \rangle$ s.t.

(a) $T_u \in \mathbb{N}$ and $\tau_u \in T_u$ is a big split pt. in T_u s.t. $T_{u \langle i \rangle} \subset T_u(\tau_u)$

(b) $\tau_u \subseteq \tau_v$ if $u \subseteq v$

(c) There is a 1-1 enumeration $\langle \tau_u^i \mid i < \beta \rangle$ of the immediate successors of τ_u s.t. $\tau_u^i \leq_{T_u} \tau_{u \langle i \rangle}$ for $i < \beta$.

(d) If $v \in \mathbb{N}_{\omega_2}^{(m)}$, then $\forall i \forall m \ v \subset \tau_{u \langle i \rangle}^{(m)}$.

(2) Let $\langle \langle T_u, \tau_u \rangle \mid u \in \beta^{<\omega} \rangle$ be an amalgamation sequence. Then

$$\bigcap_{m < \omega} \bigcup_{|u|=m} T_u = \bigcup_{h: \omega \rightarrow \beta} \bigcap_{m < \omega} T_{hm}$$

is a Namba amoeba.

Note If such a sequence is defined for $|u| < m$, it can be extended to $|u| \leq m$.

Note At $T^* = \bigcap_{m < \omega} \bigcup_{|u|=m} T_u$, then

the τ_u ($u \in \beta^{<\omega}$) are exactly the split points of T^* . Hence every split pt. of T^* is a big split pt.

Using this we get:

(3) Let G be \aleph -generic over V , Then $\#(\omega)$ is absolute in $V[G]$.

prf.

Let $\Vdash f: \check{\omega} \rightarrow \check{2}$. It suffices to show:

Claim $\Delta = \{T \mid \forall f \ T \Vdash \check{f} = \check{f}^\vee\}$ is dense in \aleph

Let $T \in \aleph$. We first construct an amalgamation sequence $\langle T_u, \check{\alpha}_u \rangle (u \in \beta^{<\omega})$

s.t. $T_u \in T$, $T_u \Vdash \check{f}(\check{\alpha}_u) = \check{n}$ for some n .

(We construct $\langle T_u, \check{\alpha}_u \rangle (u \in \beta^n)$ by induction on n .) Let $T^* = \bigcap_{n \in \mathbb{N}} \bigcup_{|u|=n} T_u$.

Then $T^* \Vdash \check{f}(\check{\alpha}_u) = \check{n}$ for some $n < \omega$

for all $u \in \beta^{<\omega}$. For each $f: \omega \rightarrow 2$ define a game G_f by:

I chooses $\check{\sigma}_i \in \mathcal{P}(\beta)_{\omega_2}$ in the i -th step
 s.t. $\check{\sigma}_i \supset \check{\sigma}_h$ for all $h < i$.

II then chooses $\check{\zeta}_i < \beta$ s.t. $\check{\sigma}_i \subset \check{\alpha}_{\check{\zeta}_0 \dots \check{\zeta}_i}^{(h)}$ for an $h < |\check{\alpha}_{\check{\zeta}_0 \dots \check{\zeta}_i}|$.

II wins iff $T^* \Vdash \check{f}(\check{\alpha}_{\check{\zeta}_0 \dots \check{\zeta}_{i-1}}) = \check{f}(i)$

for all $i < \omega$.

Clearly, I can only win at a finite stage. Hence one player has a winning strategy.

Subclaim II has a winning strategy.

Suppose not. For each $f: \omega \rightarrow 2$ let S_f be I's winning strategy. Set:

$$S(\bar{\alpha}_0, \dots, \bar{\alpha}_{i-1}) = \bigcup_{f: \omega \rightarrow 2} S_f(\bar{\alpha}_0, \dots, \bar{\alpha}_{i-1}).$$

Then

$S(u) \in \mathcal{P}_{\omega_2}(\beta)$ for $u \in \beta^{<\omega}$, since $2^\omega = \omega_1$.

But then S wins every G_f . Now let

$\bar{\alpha}_i =$ the least $\bar{\alpha}$ s.t. $S(\bar{\alpha}_0, \dots, \bar{\alpha}_{i-1}) \subset \mathcal{P}_{\bar{\alpha}_0, \dots, \bar{\alpha}_{i-1}} \bar{\alpha}$.

Let $f(m) =$ that m s.t. $T^* \upharpoonright_{(\mathcal{P}_{\bar{\alpha}_0, \dots, \bar{\alpha}_{m-1}})}$ If $f^{\check{}}(m) = m^{\check{}}$,

Then II wins G_f against S with the play $\langle \bar{\alpha}_i \mid i < \omega \rangle$. Contr! QED (Subclaim)

Now let S be II's winning strategy for G_f . Let \tilde{T} be the maximal Namba amoeba

(i.e. the set of all $\sigma: m \rightarrow \mathcal{P}_{\omega_2}(\beta)$ s.t. $m < \omega$ and $\sigma_i \subset \sigma_j$ for $i \leq j < m$). Then \tilde{T} is the tree

of all possible plays by I. For $\sigma = \langle \sigma_0, \dots, \sigma_{m-1} \rangle \in \tilde{T}$ s.t. $S(\sigma) = \langle S(\sigma_0, \dots, \sigma_{j-1}) \mid j < m \rangle$.

Then $\check{A}_{S(\sigma)}$ is a point of length $|\sigma|$ in T^* .

It is easily seen that

$$T' = \left\{ \check{a} \mid \forall \sigma \in \tilde{T} \check{a} \in \frac{\check{a}}{T^*} \check{A}_{S(\sigma)} \right\}$$

is a Namba amoeba. But

$$T' \upharpoonright_{(\mathcal{P}_{S(\sigma)})} \subset T^* \upharpoonright_{(\mathcal{P}_{S(\sigma)})} \quad \text{If } f^{\check{}}(1^{\check{}}) = f(1^{\check{}}).$$

Hence $T' \upharpoonright_{(\mathcal{P}_{S(\sigma)})} = T^* \upharpoonright_{(\mathcal{P}_{S(\sigma)})}$. QED (3)

We are assuming $2^\omega = \omega_1$ and $2^\beta = \beta$ in V .
 If G is \aleph -generic over V , then β has
 cofinality ω in $V[G]$, where $\aleph(\omega)$ is
 absolute in $V[G]$. Hence 2^β is not a
 cardinal in $V[G]$, by §4 Lemma 4.1
 Since $\aleph = 2^\beta$ in V , however, 2^{β^+} remains
 regular. Hence $\text{cf}(2^{\beta^+}) = \omega_1$ in $V[G]$

We note that we could also have used
 the set of strict Namba amoebas in the
 above proof, where we define:

Def T is a strict Namba amoeba iff
 T is a Namba amoeba and whenever ι is
 a split point of T , then for each
 $\sigma \in \aleph_{\omega_2}(\beta)$ we have $\sigma \subset u$ for a u s.t.
 $\iota(u) \in T$.

Open question Does forcing with strict
 Namba amoebas yield the same model?
 We doubt it.

Now let $IP = IP_{\mathcal{L}}$ be as in Example 1.

Recalling the definition of neat sequence we first show:

(4) Let G be IP -generic over V , Then $V[G]$ contains a neat sequence.

pf.

Letting $M^G = \langle M_i \mid i \leq \omega_1 \rangle$, $\pi^G = \langle \pi_i \mid i \leq \omega_1 \rangle$, we know that $\langle M_{\omega_1}, \pi_{\omega_1} \rangle$ is the liftup of $\langle M_0, \pi_{0, \omega_1} \upharpoonright M_{\omega_1}^{M_0} \rangle$. Hence each $x \in M = M_{\omega_1}$ is M -definable in parameters from $\omega_1 \cup \text{rng}(\pi_{0, \omega_1})$.

Let $\langle u_i \mid i < \omega \rangle \in V$ s.t. u_i is finite and

$M_0 = \bigcup_i u_i$, $u_i \subseteq u_j$ for $i \leq j < \omega$. Set:

$v_i =$ the set of $\bar{z} < \beta$ s.t. $\{\bar{z}\}$ is M -definable in parameters from $\omega_1 \cup \pi_{0, \omega_1}^{-1} u_i$.

Then $\langle v_i \mid i < \omega \rangle$ is a neat sequence.

(To see this, note that if $v \in \mathcal{P}_{\omega_2}^V(\beta)$, then $\{v\}$ is M -definable in $\omega_1 \cup u_i$ for some i , parameters from for some i .

Hence $v \subseteq v_i$. QED (4)

We also note:

(5) Let $h = \langle v_i \mid i < \omega \rangle \in V[G]$ be a neat sequence. Then $V[G] = V[h]$

proof.

Pick $\bar{z} < \omega_1$ s.t. $v_i \in \text{rng}(\pi_{\bar{z}, \omega_1}^G)$ for $i < \omega$.

Then $\text{rng}(\pi_{\bar{z}, \omega_1}^G) =$ the smallest $X \prec M$

s.t. $\bar{z} \cup \{v_i \mid i < \omega\} \subset X$.

But $M^G \nmid ((\omega_1 + 1) - 3)$, $\pi^G \nmid ((\omega_1 + 1) - 3)^2$ are definable from M , $\text{rng}(\pi_{\bar{3}, \omega_1}^G)$. Hence $M^G, \pi^G \in V[H]$, where G is definable from M^G, π^G by §4 Cor. 2.8. Hence $V[G] = V[H]$. QED (5)

Def A neat sequence \bar{h} is \mathbb{N} -generic over V iff $G_{\bar{h}} = \{T \in \mathbb{N} \mid \bar{h} \text{ is a branch in } T\}$ is \mathbb{N} -generic over V .

(Note If \bar{h} is \mathbb{N} -generic, then $\bar{h} = \bigcup G_{\bar{h}}$. Hence $V[G_{\bar{h}}] = V[\bar{h}]$. Conversely, if G is \mathbb{N} -generic and $\bar{h} = \bigcup G$, then $G = G_{\bar{h}}$.)

Lemma 3(a) follows by (4), (5), as does the last clause in Lemma 3(b). We must still prove:

Sublemma 3.1 Let G be \mathbb{P} -generic over V and $\bar{h} \in V[G]$ a neat sequence. Then \bar{h} is \mathbb{N} -generic over V .

We shall closely imitate the proof of Lemma 1.

Let $\bar{h} = \bar{h}^G$. We assume wlog that \bar{h} is a neat sequence.

Claim 1 $\Delta_{\bar{h}}^{\circ}$ is dense in \mathbb{P} , where

$\Delta_{\bar{h}}^{\circ} =$ the set of $p \in \mathbb{P}$ s.t. $|p| = \omega_1^{M_{|p|}^{\mathbb{P}}}$ and

$\forall \bar{h} \in M_{|p|}^{\mathbb{P}} \quad p \Vdash \bar{h} = \pi_{|p|, \omega_1}^{\circ} \circ \bar{h}^{\vee}$.

proof of Claim 1: Exactly as before.

Def Let $p \in \Delta_h^0$. Set:

$$T^p = \{t \in \mathcal{P}_{\omega_2}(\beta) < \omega \text{ s.t. } [\varphi_t] \neq \emptyset \text{ in } BA(\mathbb{P}),$$

$$\text{where } \varphi_t = (\check{p} \in G \wedge \bigwedge_{i < |t|} h(i^v) = \check{r}_i)$$

Claim 2 $T^p \in \mathbb{N}$ for $p \in \Delta_h^0$.

prf. Suppose not.

Let $T = T^p$. Then there is $t \in T$ and $v \in \mathcal{P}(\beta)_{\omega_2}$ s.t. for all $t' \geq t$ we have $\bigcap_{i < |t|} v \not\subset t'(i)$.

Let G be \mathbb{P} -generic s.t. $G \cap [\varphi_t] \neq \emptyset$.

Then $p \in G$ and $h(|t|) = t$, where $h = h^{\circ G}$. But h is neat. Hence there is $j' \geq |t|$ s.t. $v \subset h(j')$. Let $t' = h(|j'|+1)$. Then $t' \in T^p$ and $v \subset t'(i)$. Contr!

QED (Claim 2)

In the following let $p \in \Delta_h^0$, $\alpha = |p|$,

$$\text{where } p \Vdash \check{\pi}_{\check{\alpha}} \check{\omega}_1 \circ \check{h} = \check{h}.$$

Def For $u \in M$ s.t. $\bar{u} \leq \omega_1$ set:

$M^u =$ the smallest $X < L_\nu^A$ s.t. $u \subset X$, where $\nu = \sup(u)$ and $M = L_\beta^A$.

Similarly for $u \in \bar{M} = M_a^p$ s.t. $\bar{u} \leq \alpha$ we set $M^u =$ the smallest $X < L_\nu^{\bar{A}}$ s.t. $u \in X$, where $\nu = \sup(u)$, $\bar{M} = L_\beta^{\bar{A}}$.

Claim 3 Let $\bar{u} = \bar{h}(i)$, $u = h(i)$, where $i \in T^P$.

Set: $f =$ the M -least $f: \omega_1 \xrightarrow{\text{onto}} M^u$

$\bar{f} =$ " \bar{M} - " $f: \alpha \xrightarrow{\text{onto}} \bar{M}^{\bar{u}}$

Set $\pi = \pi^{\bar{u}u} = \{ \langle \bar{f}(\bar{z}), f(z) \rangle \mid \bar{z} < \alpha \}$.

Let $\langle a, \bar{a} \rangle \in F^P$. Then

$\pi: \langle \bar{M}^{\bar{u}}, \bar{a} \cap \bar{M}^{\bar{u}} \rangle \prec \langle M^u, a \cap M^u \rangle$.

prf. Exactly as in the proof of Lemma 1

Claim 4 Let $T \subseteq T^P$ in \mathbb{N} . There is $q \leq p$ s.t. $q \Vdash h$ is a branch in \check{T} .

prf.

We imitate the earlier proof. Let

$N^* = \langle H_\delta, M, <, p, T, IP, IN, \dots \rangle$ where

$\delta > \beta^+$. Let $p' \leq p$ conform to N^* . Set:

$\bar{N}^* = \bar{N}^*(p', N^*) = \langle \bar{N}, \bar{M}, <, \bar{p}, \bar{T}, \bar{IP}, \bar{IN}, \dots \rangle$

Let \bar{H} be \bar{N} -generic over \bar{N}^* s.t. $\bar{T} \in \bar{H}$.

Set $\tilde{h} = \cup \bar{H}$. Then \tilde{h} is a meet sequence

in \bar{N}^* . But Claim 3 holds relativized

to \bar{N}^* . Thus if $\bar{u} = \bar{h}(i)$, $\tilde{u} = \tilde{h}(i)$, we

can define $\pi^{\bar{u}u}$ as in Claim 3. It

is easily seen that if $\bar{h}(i) \in \bar{M}^{\bar{h}(i)}$,

then $\pi^{\bar{h}(i), \tilde{h}(i)} \subset \pi^{\bar{h}(i), \tilde{h}(i)}$.

But $\bigcup_i \bar{M}^{\bar{h}(i)} = \bar{M}$ since $\bigcup_i \bar{h}(i) = \bar{\beta}$.

Similarly $\bigcup_i \tilde{M}^{\tilde{h}(i)} = \tilde{M}$, Hence we have

$$\pi = \bigcup_i \pi^{\tilde{h}(i), \tilde{h}(i)} : \langle \tilde{M}, \tilde{a} \rangle \leq_{\Sigma_0} \langle \tilde{M}, \tilde{a} \rangle$$

cofinally for all $\langle \tilde{a}, \tilde{a} \rangle \in F\tilde{P}$. Since

$\langle \tilde{M}, \tilde{a} \rangle$ is a ZFC model, we conclude

(1) $\pi \upharpoonright \langle \tilde{M}, \tilde{a} \rangle \leq \langle \tilde{M}, \tilde{a} \rangle$ for all $\langle \tilde{a}, \tilde{a} \rangle \in F\tilde{P}$,

(2) $\text{rng}(\pi) = \text{the smallest } X \subseteq \tilde{M} \text{ s.t.}$
 $\text{rng}(\tilde{h}) \cap X \subseteq X$.

The proof of (2) is as before.

Now let $\tilde{\alpha} = |p'| = \omega_1^{\tilde{N}^*}$. Since \tilde{H} is \tilde{M} -
 -generic over \tilde{N}^* and $\tilde{h} = \bigcup \tilde{H}$,

$\tilde{N}^*[\tilde{h}]$ is a ZFC⁻ model. For $\tilde{N}^*[\tilde{h}]$

we define $\langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle, \langle \tilde{\pi}_i \mid i \leq \tilde{\alpha} \rangle$

exactly as before, noting that

$$\tilde{M}_i = M_i^P, \tilde{\pi}_i = \pi_i^P \text{ for } i \leq \tilde{\alpha}.$$

As before, we define q as follows:

$$M^q = \langle \tilde{M}_i \mid i \leq \tilde{\alpha} \rangle, \pi^q = \langle \tilde{\pi}_i \mid i \leq \tilde{\alpha} \rangle,$$

$F^q = F^P$. Just as before we then get:

(3) $q \in IP$

(4) $q \leq p$

(5) $q \Vdash \tilde{h}$ is a branch in T .

QED (Sublemma 3.1) Lemma 3 then

follows exactly as Lemma 1 did, QED

We now again let $\beta = \omega_1$. There is a variant of Namba forcing which Shelah calls Nm' and we shall consequently call IN' . We shall show that IN' is reshapable and that, in fact, $BA(IN') \simeq BA(IP)$ where IP is a variant of Example 1. We first define IN' and develop its properties.

Def IN' = the set of $T \in IN$ s.t.

for some $s \in T$ we have:

- $T = T_{(s)}$

- If $t \in T$ and $|t| \geq |s|$, then t has ω_2 many immediate successors.

s is then unique and is called the stem of T , denoted by $stm(T)$.

(Thus T consists of a single stem followed by a tree isomorphic to $\omega_2^{<\omega}$.)

AIG is IN' -generic and $b = \bigcup AIG$, then b is a branch in $\omega_2^{<\omega}$ and

$$G = G_b = \{ T \in IN' \mid b \text{ is a branch in } T \}$$

We then say that b is \mathbb{N}' -generic.

It is known that forcing with \mathbb{N}' adds no new reals. Magidor and Shelah have shown, however, that forcing with \mathbb{N}' adds no \mathbb{N} -generic sequence $b: \omega \rightarrow \omega_1$ and conversely. We sketch the argument:

Lemma 4.1 Let b be \mathbb{N}' -generic over V . Let $F \in V$, $F: \omega_2 \rightarrow \omega_2$. Then

$$(*) \quad \forall n \ \exists i \geq n \ \delta_i > \sup_{h < i} F(\delta_h),$$

where $b = \langle \delta_i \mid i < \omega \rangle$.

proof.

We show that the set of conditions which force $(*)$ is dense in \mathbb{N}' . Let $T \in \mathbb{N}'$, $\kappa = \text{stem}(T)$, $n = |\kappa|$. Set

$$T' = \{ \tau \in T \mid \exists i \geq n \ \tau(i) > \sup_{h < i} F(\tau(h)) \}.$$

Then $T' \in \mathbb{N}'$, $T' \leq T$ in \mathbb{N}' and every branch thru \mathbb{N}' satisfies $(*)$.

QED(4.1)

Lemma 4.2 Let $b = \langle \delta_i \mid i < \omega \rangle$ be \mathbb{N} -generic over V . For some $F: \omega_2 \rightarrow \omega_2$ in V there are arbitrarily large $c < \omega_1$ s.t.

$$\delta_c \leq \sup_{h < c} F(\delta_h).$$

prf. of 4.2

Let $T \in \mathbb{N}$ be (w.l.o.g.) a strict Namba tree. We construct $T' \leq T$, all of whose branches must satisfy the conclusion.

As usual, let $N = \langle H_{\omega_3}, M, <, \nu \rangle$, where $M = L_{\omega_2}^A = H_{\omega_2}$ and $<$ well orders N .

For $\bar{3} < \omega_2$ set:

$Y_{\bar{3}} =$ the least $Y < N$ s.t. $\omega_1 \cup Y$

$$F(\bar{3}) = \omega_2 \cap Y_{\bar{3}}$$

Then $\bar{3} < F(\bar{3}) < \omega_2$. Note that if

$\alpha \in F(\bar{3})^{<\omega}$ is a split pt. of T , then

$$(1) \sup \{ \beta < F(\bar{3}) \mid \alpha \langle \beta \rangle \in T \} = F(\bar{3})$$

For $\alpha \in T$ let $\langle l_0^{\alpha}, \dots, l_{m-1}^{\alpha} \rangle$ be the monotone enumeration of the $l < |\alpha|$

s.t. $\alpha \cup l$ is a split point of T .

Set: $T' =$ the set of $\alpha \in T$ s.t.

$$\alpha(l_i^{\alpha}) \leq \sup_{h < l_i^{\alpha}} F(\alpha(h))$$

whenever i is odd.

Claim $T' \in \mathbb{N}$.

prf.

Clearly $\emptyset \in T'$. Now let $\alpha \in T'$.

We show that α can be extended to $t \in T'$ which has ω_2 immediate successors in T' .

Case 1 m_α is even.

Let $t \geq \alpha$ in T be minimal s.t. t is a split point in T . Then all immediate successors of t lie in T' .

Case 2 $m = m_\alpha$ is odd.

Let $t \geq \alpha$ in T be as above. Let $d = \sup_{h < |t|} F(t(h)) = F(\sup_{h < |t|} t(h))$.

Then $t \in d^{\omega}$ and there is $i < d$ s.t. $t^{(i)} \in T$ by (1). Hence,

letting $t' = t^{(i)}$ we have:
 $l_m^{t'} = |t'|$ and $t^{(i)}(l_m^{t'}) \leq \sup_{h < |t'|} F(t'(h)) = d$. Thus $t' \in T'$ and $m_{t'} = m + 1$

is even. We can then apply Case 1. QED (Claim)

Thus $T' \leq T$ in \mathbb{N} has the desired property. QED (4.2)

We know, however, that if b is \mathbb{N} -generic then every $b' \in V[b]$ which is a cofinal ω -sequence in ω_2^V is \mathbb{N} -generic. Hence:

Lemma 4.3 Let b be \mathbb{N} -generic. Then $V[b]$ contains no \mathbb{N}' -generic sequence.

But then we get the converse:

Lemma 4.4 Let b' be \mathbb{N}' -generic. Then $V[b']$ contains no \mathbb{N} -generic sequence.

proof.

Let $b \in V[b']$ be \mathbb{N} -generic. Let $f \in V[b]$ biject ω_1 onto ω_2^V . Then

$f^{-1} \circ b' \in H_{\omega_1} \subset V$. But then

$b' = f \circ (f^{-1} \circ b) \in V[b]$, contradicting

Lemma 4.3. QED (4.4).

We now develop some other basic properties of \mathbb{N}' . We have a weak amalgamation lemma:

Lemma 4.5 Let $T \in \mathbb{N}'$, $\kappa = \text{stem}(T)$,
 let $\langle T_u \mid u \in \omega_2^{<\omega} \rangle$ be a t.s.,
 $T_u \in \mathbb{N}'$ and, letting $\kappa_u = \text{stem}(T_u)$,
 we have: $T_\emptyset = T$, $|\kappa_u| = |\kappa| + u$,

$T_{u \langle i \rangle} \subset T_u$ for $i < \omega_2$,

~~$\kappa_{u \langle i \rangle} \neq \kappa_{u \langle j \rangle}$ for $i < j < \omega_1$.~~

Then $T' \in \mathbb{N}'$, where

$$T' = \bigcap_{m < \omega} \bigcup_{|u|=m} T_u = \bigcup_{f: \omega \rightarrow \omega_2} \bigcap_{m < \omega} T_{f \upharpoonright m}$$

The proof is left to the reader.

We also have the refinement

lemma:

Lemma 4.6 Let $T \in \mathbb{N}'$. Let $f: T \rightarrow \omega_1$.

There is $T' \leq T$ in \mathbb{N}' s.t.,

$|z| = |z'| \rightarrow f(z) = f(z')$ for all $z, z' \in T'$,

Proof.

For each $g: \omega \rightarrow \omega_1$ we play a game

G_g defined by Shelah: let $\kappa = \text{stem}(T)$,

At the i -th move, player I picks an $\alpha_i < \omega_2$. Player II must then pick a β_i s.t. $\beta_i \geq \alpha_i$ and $\lambda(\beta_{0,i}, \beta_i) \in T$ and $f(\lambda(\beta_{0,i}, \beta_i)) = g(i)$. If at any point II cannot move, then I wins.

Otherwise I wins. Thus one of the players has a winning strategy. Claim There is g for which II has a winning strategy.

pf. Suppose not.

Let S_g be I's winning strategy for the game G_g . Set $S(t) = \bigcup_{g: \omega \rightarrow \omega_q} S_g(t)$.

Then $S(t) < \omega_2$. (We assume of course $2^\omega = \omega_1$.) Then S wins all of the games. Now pick successively

β_i ($i < \omega$) s.t. $\lambda(\beta_{0,i}, \beta_i) \in T$ and $\beta_i > S(\lambda(\beta_{0,i}, \beta_{i-1}))$. Set

$g(i) = f(\beta_i)$. This play wins G_g , defeating S . Contr! QED(Claim)

Let S be a winning strategy for II for $G_{\vec{d}}$.
 Let T' be the tree of all $t \in \omega^{\omega}$ such that $t \upharpoonright n \in S(\vec{d} \upharpoonright n)$
 where \vec{d} is any finite sequence of plays
 by I . Then $T' \leq T$ has the desired
 property. QED (4, 6)

(Note. Using the weak amalgamation
 lemma and refinement lemma in tandem,
 it is not too hard to show that \mathbb{N}'
 adds no new reals.)

The following can be regarded as a
 strengthening of Lemma 4.1:

Lemma 4.6 Let W be a transitive ZF^+
 model s.t. $2^{\omega} = \omega_1$ & $2^{\omega_1} = \omega_2$ in W and
 $d = (2^{\omega_2})^W$ exists and is countable in V .
 Let $\bar{\mathbb{N}} = \mathbb{N}'^W$. Let $F: \omega_2^W \rightarrow \omega_2$. For
 each $T \in \bar{\mathbb{N}}$ there is an \mathbb{N} -generic
 $G \ni T$ s.t. for $b = \langle \delta_i, i < \omega \rangle = \bigcup G$:

$$\forall m \ \exists i \geq m \ \delta_i > \sup_{h < i} F(\delta_h).$$

(Note. We do not require $F \in W$)

(Note. If $F \in W$ it follows from this
 that $\text{If } \bar{\mathbb{N}} \ \forall m \ \exists i \geq m \ \delta_i > \sup_{h < i} F(\delta_h)$.)

Lemma 4.6 follows from:

Lemma 4.7 Let $T \in \mathbb{N}'$, $s = \text{stem}(T)$,
 $n = |s|$. Let Δ be dense in \mathbb{N}' . There
 is $T' \leq T$ in \mathbb{N}' with $s' = \text{stem}(T')$
 s.t. $\forall i \geq n (i < |s'| \rightarrow s'(i) > \sup_{h < i} F(s'(h)))$,
 and $T' \in \Delta$.

proof.

For $t \in T$, $|t| \geq n$ define:

$$f(t) = \begin{cases} 1 & \text{if there is } T' \in \Delta \text{ with } T' \leq T \text{ and} \\ & t = \text{stem}(T') \\ 0 & \text{if not.} \end{cases}$$

By the refinement lemma there is $T' \leq T$
 s.t. $\text{stem}(T') = s$ and $f(t) = g(|t|)$ for
 all $t \in T'$ s.t. $|t| \geq n$, (here $g: \omega \rightarrow \omega_2$).

But then there is $m \geq n$ s.t. $g(m) = 1$,
 since there is certainly a $T'' \leq T'$ s.t.
 $T'' \in \Delta$. We can certainly pick

$t \in T'$ s.t. $|t| = m$ and

$$\forall i \geq n (i < m \rightarrow t(i) > \sup_{h < i} F(t(h))).$$

Since $f(t) = 1$ there is $T' \leq T$ s.t.
 $t = \text{stem}(T')$ and $T' \in \Delta$.

□ E D (4.7)