More on Iterability

In our earlier notes [ANFS] (Addendum to "A New Fine Structure...") we showed that under appropriate assumptions various notions of iterability for 1-small premice coincide. Let Θ be a strongly inaccessible cardinal. We made the further assumptions:

A1 Either no $\tau < \Theta$ is Woodin in an inner model or else V_Θ is closed under $\#$.

A2 Let $M \in V_\Theta$ be a 1-small premouse and $\mathcal{I}$ a normal iteration of M of length Θ. Then $\mathcal{I}$ has a cofinal branch.

A3 Θ is Mahlo.

From these assumptions we showed that normal iterability implies full iterability in V_Θ. Moreover, weak iterability implies weak iterability. (M is said to be <u>weakly</u> (normally) iterable iff whenever $\sigma : \bar{M} \xrightarrow[\Sigma^*]{} M$ and $\bar{M}$ is countable, then $\bar{M}$ is countably (normally) iterable.) We now show that these results follow from A1, A2 alone without A3. Our main lemma does not involve A1 or A2 but needs a certain weakening of A2:

(+) Every countably normally iterable 1-small premouse is ω_1+1 iterable.

(Note (+) will hold if ω_1 is not Woodin in an inner model or if $A^\#$ exists for all $A\subset\omega_1$. Similarly A2 holds if θ is not Woodin in an inner model ~~or~~ $A^\#$ exists for $A\subset\theta$.)

The main lemma states that if θ is inaccessible, (+) holds, and $Q\in V_\theta$ is $\theta+1$-normally iterable, then Q is weakly iterable. (This is proven for 1-small Q. The proof will also work for any premouse whose normal iterations always have unique cofinal branches.) In proving this we follow Steel's basic technique of constructing a "background array" of weakly iterable premice N_i, M_i ($i\leq\xi<\theta$) and showing that Q embeds into some N_i. In this case, however, the N_i are obtained directly by normal iterations of Q. We construct a sequence of normal iterations $\mathcal{I}_i$ of Q s.t.

$lh(\mathcal{I}_i) = i+1$. The $\mathcal{I}_i$ have a tree structure under the relation: $(i<j \wedge \mathcal{I}_i = \mathcal{I}_j | (i+1))$. Letting Q_i be the ultimate model in $\mathcal{I}_i$, we define N_i to be a "segment" of Q_i (i.e. either $N_i = Q_i \| \beta$ or $N_i = \langle J_\beta^{E^{Q_i}}, \emptyset \rangle$ where $E_\beta^{Q_i} \neq \emptyset$). The process terminates only when $N_\xi = Q_\xi$ and Q_ξ is a simple iterate of Q. We first show that the process does terminate below Θ. This gives us an iteration map from Q to N_ξ. We then use Steel's arguments to show that each N_i is weakly iterable. (However, in §1 we do not carry out the full iterability proof but only sketch a special case which illustrates several of the main ideas. The full proof is given in §3, which is based straight-forwardly on Steel's proof in [S] §9.)

In [ANFS] we considered a weaker notion of iterability ("MS-iterability") which seems to be implicit in the work of Mitchell and Steel. The difference is that MS-iterations do not allow unrestricted linear "cascading" of normal iterations. It followed that MS-iterability coincides with full iterability under the assumptions A1–A3 (similarly for "weakly MS-iterable" and "weakly iterable"). We now, of course, have the same result under the weakened assumption. The Mitchell-Steel iterations differ from ours, however, in another essential respect: Because their extenders are indexed differently, the pattern in which they are applied in the course of a normal iteration differs from ours. It is possible, however, using our premice, to define an alternative notion of "normal iteration" in which the extenders are applied in the Mitchell-Steel fashion. In these iterations $\mathcal{I} = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{ij}\rangle, T\rangle$, we define $s_i = s(\nu_i)$ to be the "natural length" of $E^{M_i}_{\nu_i}$ in Steel's sense, and use s_i in place of $\lambda_i = lh(E^{M_i}_{\nu_i})$. In place of ν_i we then have s_i^+ = the successor of s_i in the ultrapower by $E^{M_i}_{\nu_i}$. We call such $\mathcal{I}$ a <u>normal</u> s-<u>iteration</u>. The notion of <u>good</u> s-<u>iteration</u> is then defined as before. It is possible to do s-coiterations. (In fact the whole of [MS] could easily be recast in terms of s-iterations of our premice.) In §2 we develop the basic theory of s-iterations and then show, under the assumptions A1 and A2, that for 1-small premice in V_θ the notions "iterable", "normally s-iterable" and "s-iterable" coincide.

References

[MS] Mitchell, Steel Fine Structure and Iteration Theory (Lecture Notes in Logic)

[S] Steel The Core Model Iterability Problem (Lecture Notes in Logic)

[NFS] Jensen A New Fine Structure Theory for Higher Core Models (Handwritten Notes)

[ANFS] Jensen Addendum to NFS (Handwritten Notes)