Appendix to §1.2 : $\Sigma_1^{(n)}$- Skolem Functions

Def Let M be acceptable, $n \geq 0$.

$X \prec_{\Sigma_1^{(n)}} M$ iff for all $i \leq n$:

Let $x_1, \ldots, x_m \in X$; let $R(y^i, \vec{v})$ be a $\Sigma_1^{(i)}(M)$ relation. Then

$$\bigvee y\, R(y^i, \vec{x}) \to \bigvee y^i \in X\, R(y^i, \vec{x}).$$

Since each $\Sigma_1^{(i)}(M)$ relation is uniformizable by a $\Sigma_1^{(i)}(M)$ partial map to H_M^i, it follows that $X \prec_{\Sigma_1^{(n)}} M$ iff X is closed under good $\Sigma_1^{(n)}$ fcns.

Def Let $X \subset M$

$h^n(X)$ = the closure of X under good $\Sigma_1^{(n)}(M)$ fcns.
= the smallest $Y \prec_{\Sigma_1^{(n)}} M$ s.t. $X \subset Y$.

We know: If $Y = h^n(X)$ and $\sigma : N \stackrel{\sim}{\longleftrightarrow} M|Y$, N trans., then $\sigma : N \to_{\Sigma_1^{(n)}} M$ iff $\rho_N^i = \sigma^{-1}{}''\rho_M^i$ for $i \leq n$

σ will be $\Sigma_1^{(n)}$ - preserving if $X = Z \cup \{p\}$, $Z \subset J_{\rho^n}^{EM}$.

<u>Def</u> h is a $\Sigma_1^{(n)}$ - <u>Skolem fcn</u> for M iff h is a universal good $\Sigma_1^{(n)}(M)$ fcn:

(a) h is a good $\Sigma_1^{(n)}(M)$ fcn.

(b) $\mathrm{dom}(h) \subset \omega \times M$

(c) Let f be a good $\Sigma_1^{(n)}(M)$ fcn of one argument. Then $f = h_i$ for some i (where $h_i(x) \simeq h(i, x)$).

<u>Properties of $\Sigma_1^{(n)}$ - Skolem fcns:</u>

Let h be a $\Sigma_1^{(n)}$ - Sk. fcn.

(1) $\tilde{h}(\{p\}) = h''(\omega \times \{p\})$ for $p \in M$

(2) Let $\emptyset \neq X \subset M$ be closed under ordered pairs. Then $\tilde{h}(X) = h''(\omega \times X)$.

prf.
$$h''(\omega \times X) = \bigcup_{x \in X} \tilde{h}(\{x\}).$$ But
$$x, y \in X \rightarrow \tilde{h}(x), \tilde{h}(y) \subset \tilde{h}(z),$$
where $z = \langle x, y \rangle \in X$. QED

Similarly:

(3) Let X be as above, $p \in M$. Then
$$\tilde{h}(X \cup \{p\}) = h''(\omega \times (X \times \{p\}))$$

(4) Let $X \subset On_M$ be closed under Gödel pair. Then:

(a) $h^n(X) = h''(\omega \times X)$

(b) $h^n(X \cup \{p\}) = h''(\omega \times (X \cup \{p\}))$

(5) $h^n(X) = \bigcup \{ h''(\omega \times \{a\}) \mid a \subset X, a \text{ finite} \}$

. . . . .

Canonical Skolem fcns:

Def We define a sequence of good $\Sigma_1^{(n)}$ fcns h_M^n as follows

$h^0 = h =$ the canonical Σ_1 Skolem fcn.

$h^{n+1}(i, x) \simeq h^n((i)_0, \langle \hat{h}^n((i)_1, x), x \rangle)$

where $\hat{h}^n$ is the $\Sigma_1^{(n+1)}$ map to H^{n+1} defined by:

$$\hat{h}^n(i, x) \simeq h_{M^{n+1}, p(x)}(i, 0),$$

and $p(x) = \langle x, 0, \text{—}, 0 \rangle$.

Thm h^n is a universal $\Sigma_1^{(n)}$ - Skolem fcn.

prf. Ind. on n.

For $n=0$ it is given.

Assume it for n. We prove it for $n+1$. We first show:

Lemma Every good $\Sigma_1^{(n+1)}$ fcn f has the form: $f(\vec{x}) \simeq f_0(f_1(\vec{x}), \vec{x})$, where f_0 is a good $\Sigma_1^{(n)}$ map and f_1 is a $\Sigma_1^{(n+1)}$ map to H^{n+1}.

prf.

Let $\mathbb{F}$ be the set of such f. Clearly every $\Sigma_1^{(i)}$ map to H_i is in $\mathbb{F}$ for $i \leq n+1$. Hence it suffices to show:

Claim Let $f(\vec{v}), g_i(\vec{x})$ be in $\mathbb{F}$ for $i = 1, \ldots, m$. Then $\tilde{f}(\vec{x}) \simeq f(\vec{g}(\vec{x}$

is in $\mathbb{F}$.

prf. of Claim.

Let $f(\vec{v}) \simeq f_0(f_1(\vec{v}), \vec{v})$

$g_i(\vec{x}) \simeq g_i^0(g_i^1(\vec{x}), \vec{x})$.

Set: $\tilde{f}_0(u, \vec{x}) \simeq f_0((u)_0, g_1^0((u)_1, \vec{x}), \dots, g_m^0((u)_m, \vec{x}$

Then $\tilde{f}_0$ is a good $\Sigma_1^{(n)}$ fcn.

Set: $\tilde{f}_1(\vec{x}) \simeq \langle f^1(\vec{x}), g_1^1(\vec{x}), \dots, g_m^1(\vec{x}) \rangle$.

$\tilde{f}_1$ is a $\Sigma_1^{(n+1)}$ map to H^{n+1}. But

$\tilde{f}(\vec{x}) \simeq \tilde{f}_0(\tilde{f}_1(\vec{x}), \vec{x})$.

QED (Lemma)

Now let f be a good $\Sigma_1^{(n+1)}$ fcn of one argument. By the lemma:

$f(x) \simeq f_0(f_1(x), x)$.

Set: $f'(u) \simeq f_0((u)_0, (u)_1)$. By the ind. hyp $f' = h_i^n$ for some i.

Clearly $f_1 = \hat{h}_j^n$ for some j.

Hence $f(x) \simeq h^n(i, \langle \hat{h}^n(j, x), x \rangle)$

$\simeq h^{n+1}(\langle i, j \rangle, x)$

QED