§3.3.2 Upward Extensions of Embeddings

As a preliminary to proving the weak covering lemma, we develop some fine structural lemmas.

Suppose we are given acceptable $\bar{M} = \langle J_{\bar{\beta}}^{\bar{A}}, \bar{B} \rangle$ and $\bar{Q} = J_{\bar{\gamma}}^{A}$, where $\bar{\nu} = \omega\bar{\beta}$ or $\bar{\nu} \in \bar{M}$ is regular in $\bar{M}$. Suppose furthermore that $\sigma : \bar{Q} \xrightarrow[\Sigma_0]{} Q$ cofinally. Under what circumstances can we "lift" σ to a "mice" imbedding $\tilde{\sigma} : \bar{M} \longrightarrow M$. In particular, when do we find $\tilde{\sigma} \supset \sigma$ and $M = \langle J_{\beta}^{A}, B \rangle$ s.t. $Q = J_{\gamma}^{A}$ and $\tilde{\sigma} : \bar{M} \xrightarrow[\Sigma_1^{(n)}]{} M$ for all n s.t. $\bar{\nu} \leq \omega\rho_{\bar{M}}^{n}$?

We try to construct such $\tilde{\sigma}$, M by imitating the Σ^*-ultraproduct construction.

Def $\Gamma = \Gamma_{\bar{Q},\bar{M}} = \Gamma_{\bar{\nu},\bar{M}}$ is the set of maps f s.t. dom(f) is bounded in $\bar{Q}$ (i.e. $\forall u \in \bar{Q}$ dom(f) $\subset u$ and either f is $\Sigma_1^{n}(\bar{M})$ for an n s.t. $\bar{\nu} \le \omega\rho_{\bar{M}}^{n+1}$ or $f \in H_{\bar{M}}^{n}$ where $\bar{\nu} \le \omega\rho_{\bar{M}}^{n}$.

(Note If f is $\Sigma_1^{(n)}(\bar{M})$ and dom(f), rng(f) are bounded in $H_{\bar{M}}^{n}$, then $f \in H_{\bar{M}}^{n}$.)

(Note dom(f) $\in \bar{Q}$ for $f \in \Gamma$)

Def $D = \{\langle a, f\rangle \mid f \in \Gamma \wedge a \in \sigma(\mathrm{dom}(f))$
We define a pseudo $\in$-relation e and a pseudo-identity I on D as follows:

<u>Def</u> $\langle a, f\rangle \underset{I}{e} \langle b, g\rangle$ iff

$$\langle a, b\rangle \in \sigma(\{\langle u, v\rangle \mid f(u) \underset{=}{\in} g(v)\})$$

Similarly;

<u>Def</u> $\underset{\mathring{B}}{\mathring{A}}(\langle a, f\rangle)$ iff $a \in \sigma(\{u \mid f(u) \in \underset{B}{A}\})$

(This obviously generalizes to the case of many predicates or predicates with many argument places).

<u>Def</u> $\mathbb{D} = \mathbb{D}_{\bar{Q}, \sigma, \bar{M}} = \mathbb{D}_{\bar{\gamma}, \sigma, \bar{M}} =$

$= \langle D, I, e, \mathring{A}, \mathring{B} \rangle.$

Exactly as in §1.3 Lemma 1.1 we get Łoś Theorem for Σ_0 formulae:

<u>Lemma 1.1</u> Let $\langle a_1, f_1\rangle, \dots, \langle a_n, f_n\rangle \in$
$\in D$ + let φ be a Σ_0 formula.

$\mathbb{D} \models \varphi[\overrightarrow{\langle a, f\rangle}] \longleftrightarrow$

$\longleftrightarrow \vec{a} \in \sigma(\{\vec{u} \mid \bar{M} \models \varphi[\vec{f}(\vec{u})]\})$

In place of §1.3 Lemma 1.2 we use:

<u>Lemma 1.2</u> Let $R(y^m, x^{j_1}, \dots, x^{j_r})$ be $\Sigma_1^{(m)}(\bar{M})$ where $\vec{j} \leq m$. Let $n \geq m$ s.t. $\alpha p_{\bar{M}}^{n+1} \geq \bar{\nu}$ and let $f_1, \dots, f_r \in \Gamma$ be good $\underline{\Sigma}_1^{(n)}(\bar{M})$ maps, where f_i is to H^{j_i}. Then there is a good $\underline{\Sigma}_1^{(n)}(\bar{M})$ map $g \in \Gamma$ s.t. $\mathrm{dom}(g) = \mathrm{dom}(f_1) \times \dots \times \mathrm{dom}(f_r)$ and

$$\bigvee y^m R(y^m, \vec{f}(\vec{u})) \longleftrightarrow R(g(\vec{u}), \vec{f}(\vec{u}))$$

for $\vec{u} \in \mathrm{dom}(f_1) \times \dots \times \mathrm{dom}(f_r)$.

The proof is the same as in §1.3 Lemma 1.2.

Thus I is an equality relation for $\mathbb{D}$ and $\mathbb{D}$ satisfies extensionality. Until further notice assume:

(*) $\mathbb{D}$ is well founded.

Then there is a structure preserving $[\ ] : \mathbb{D} \xrightarrow{\sim} M$, where M is transitive and:

$$[x] = [y] \text{ iff } x\, I\, y$$
$$[x] \in [y] \text{ iff } x \in y.$$

Def $\tilde{\sigma} : \bar{M} \longrightarrow M$ is defined by: $\tilde{\sigma}(x) = [\langle 0, \{\langle x, 0\rangle\}\rangle]$

Our hope is that $\tilde{\sigma}$ is the desired extension of σ. We shall not be able to prove this, however, without a further assumption on $\bar{M}$. For the moment we content ourselves with proving: $\tilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ for $\bar{\nu} \leq \omega\rho^n_{\bar{M}}$.

Corresponding to §1.3 Lemma 1.3:

<u>Lemma 1.3</u> $\tilde{\sigma} \upharpoonright \bar{Q} = \sigma$

prf.

Set $\tilde{Q} = \bigcup \tilde{\sigma} '' \bar{Q}$ = The set of $[\langle a, f\rangle]$ s.t. $f \in \bar{Q}$. Then $\tilde{Q}$ is transitive and

$[\langle a,f\rangle] \in [\langle b,g\rangle] \longleftrightarrow \sigma(f)(a) \in \sigma(g)(b)$.

Define a map $\tilde{Q} \to Q$ by $[\langle a, f\rangle] \mapsto \sigma(f)(a)$. This is easily seen to be onto, hence the identity. Hence $[\langle a,f\rangle] = \sigma(f)(a)$ and

$\tilde{\sigma}(x) = [\langle 0, \{\langle x, 0\rangle\}\rangle] = \sigma(x)$ QED

We are now ready to prove:

<u>Lemma 2</u> Let $\bar{M}$ be $\bar{\nu}$-snave. Then $\tilde{\sigma} : \bar{M} \rightarrow_{\Sigma_0^{(m)}} M$ for $\bar{\nu} \le \omega\rho^n_{\bar{M}}$.

For the most part our proof will closely follow that of §1.3 Lemma 1. Just as there we define:

Def $\Gamma_n = \begin{cases} \{f \in \Gamma \mid \mathrm{rng}(f) \subset H^m_{\bar{M}}\} \text{ if } \omega\rho^{n+1}_{\bar{M}} \ge \bar{\nu} \\ \{\ '' \mid \ '' \in '' \} \text{ if} \\ \quad \text{if } \omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \le \omega\rho^n_{\bar{M}} \end{cases}$

Def $H_n = \{[\langle a, f\rangle] \mid f \in \Gamma_n \wedge \langle a, f\rangle \in D\}$ if Γ_n is defined.

As in §1.3 Lemma 2:

(1) H_n is transitive.

For $\bar\nu \leq \omega\rho^n_{\bar M}$ we interpret $\Sigma^{(n)}_m$ fmlae over M by letting x^n range over H_n.

Just as in §1.3 we get

(2) Let $\langle a_1, f_1\rangle, \ldots, \langle a_n, f_n\rangle \in D$ + let φ be either a $\Sigma^{(n)}_1$-fmla, where $\bar\nu \leq \omega\rho^{n+1}_{\bar M}$ or a $\Sigma^{(n)}_0$-fmla, where $\bar\nu \leq \omega\rho^n_{\bar M}$. Then:

$M \models \varphi[\overrightarrow{[\langle a, f\rangle]}] \longleftrightarrow \vec a \in \sigma(\{\vec u \mid M \models \varphi[\vec f(\vec u)]\})$

Since $\tilde\sigma \restriction H^n_{\bar M}$ is cofinal into H_n we conclude: for $\omega\rho^{n+1} < \bar\nu \leq \omega\rho^n$, we conclude:

(3) $\tilde\sigma : \bar M \longrightarrow_{\Sigma^{(n)}_1} M$ for $\omega\rho^n_{\bar M} \geq \bar\nu$.

As in §1.3 we also get:

(4) $\tilde\sigma : \bar M \longrightarrow_{Q^{(n)}} M$ for $\omega\rho^n_{\bar M} \geq \bar\nu$.

and:

(5) $\tilde{\sigma} : \overline{M} \longrightarrow_{\Sigma_2^{(n)}} M$ for $\omega\rho_{\overline{M}}^{n+1} \geq \overline{\nu}$.

In particular, since $\tilde{\sigma} : \overline{M} \longrightarrow_{Q} M$, we have:

(6) M is acceptable.

Using (3) it then follows easily that:

(7) $H_n = |J_{\rho^n}^B|$ where $M = J_\beta^B$

and $\omega\rho^n = On \cap H_n$.

Thus it remains only to prove:

<u>Claim</u> $\rho^n = \rho_M^n$ if $\overline{\nu} \leq \omega\rho_{\overline{M}}^{n+1}$

and $\rho^n \leq \rho_M^n$ if $\overline{\nu} \leq \omega\rho_{\overline{M}}^n$

The proof is by ind. on n. $n = 0$ is trivial, so assume $n > 0$.

To prove $\leq$ we let $A \subset \omega\rho^n$ be $\underline{\Sigma}_1^{(n-1)}(M)$ + imitate § 1.3 Lemma 2 (8) to show: $\langle H_n, A\rangle$ is amenable.

To prove $\geq$ (with $\overline{\nu} \leq \omega\rho_{\overline{M}}^{n+1}$) we imitate the proof of § 1.3 Lemma 2 (13). Let $\overline{A} \subset \omega\rho_{\overline{M}}^n$ be $\Sigma_1^{(n-1)}(\overline{M})$ in p. Let A be $\Sigma_1^{(n-1)}(M)$ in $\tilde{\sigma}(p)$ by the same definition.

Assume: $\bar{A} \notin \bar{M}$. <u>Claim</u> $A \cap \omega\rho^n \notin M$.
Suppose not. Let $A \cap \omega\rho^n = [\langle a, f\rangle]$.
The statement $A \cap \omega\rho^n = x$ is expressed by: $\bigwedge z^n (z^n \in A \longleftrightarrow z^n \in x)$
which is Π_1^n in $\tilde{\sigma}(p)$ — i.e.

$A \cap \omega\rho^n = x \longleftrightarrow M \models \varphi[[\langle a, f\rangle], \tilde{\sigma}(p)]$

where φ is Π_1^n. Hence

$a \in \sigma(\{u \mid \bar{M} \models \varphi[f(u), p]\}) = \sigma(\{u \mid \bar{A} = f(u)\})$,

by Łoś Lemma. Hence $\bigvee u\ \bar{A} = f(u) \in \bar{M}$.
Contr! QED (Lemma 2).

As a corollary of the proof:

<u>Lemma 2.1</u> $\tilde{\sigma} : \bar{M} \xrightarrow[\Sigma_2^{(n)}]{} M$ for $\bar{\nu} \leq \omega\rho_{\bar{M}}^{n+1}$

. . . .

If $\omega\rho_{\bar{M}}^{\omega} \geq \bar{\nu}$ $(\rho^\omega = \inf\{\rho^n \mid n < \omega\})$,
this lemma gives us all the information we want. Otherwise
$\omega\rho_{\bar{M}}^{n+1} < \bar{\nu} \leq \omega\rho_{\bar{M}}^{n}$ and we must
impose an additional condition
before we can show $\Sigma_1^{(n)}$ preservation

Unless specified otherwise, we shall in the following regard M^{np} as an abbreviation for $M^n\langle p, 0, \dots, 0\rangle = M^n\langle p^0, \dots, p^{n-1}\rangle$ ($p^0 = p$, $p^{i+1} = 0$).

<u>Def</u> $\bar{M}$ is $\bar{\nu}$-<u>clear</u> iff

(a) There is a largest cardinal $\bar{\beta} \in \bar{Q}$ (i.e. $\bar{\nu} = \bar{\beta}^{+\bar{Q}}$).

(b) If $\omega\rho_{\bar{M}}^{n+1} < \bar{\nu} \leq \omega\rho_{\bar{M}}^{n}$, then there is $\bar{p} \in \bar{M}$ s.t. $h_{\bar{M}^n\bar{p}}(\bar{\beta} \cup \{\bar{p}^n\})_n$ is cofinal in $\bar{\nu}$.

[<u>Note</u> By (a), $\bar{\nu} = \bar{\beta}^{+\bar{M}}$ and $\omega\rho_{\bar{M}}^{n+1} < \bar{\nu}$ is equivalent to $\omega\rho_{\bar{M}}^{n+1} \leq \bar{\beta}$. Moreover $h_{\bar{M}^n,\bar{p}}(\bar{\beta} \cup \{\bar{p}^n\}) \cap \bar{\nu}$ is cofinal in $\bar{\nu}$ iff $\bar{\nu} \subset h_{\bar{M}^n,\bar{p}}(\bar{\beta} \cup \{\bar{p}^n\})$].

[<u>Note</u> If (a) holds and $\bar{M}$ is a mouse, then (b) holds].

Let us also observe that $[\langle a, f\rangle] = \tilde{\sigma}(f)(a)$ in M, where $\tilde{\sigma}(f)$ is defined in the obvious way (cf. §1.3 Cor 2.2).

<u>Lemma 3</u> Let $\bar{M}$ be $\bar{\nu}$-clear, where $\omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}}$. Then

(a) $\tilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally

(b) $\tilde{\sigma}(\bar{\nu}) = \nu$ (Taking $\tilde{\sigma}(On_{\bar{M}}) = On_M$)

(c) $\omega\rho^{n+1}_M < \nu \leq \omega\rho^n_M$

(d) M is ν-clear.

proof.

We first prove (b). Let $\xi = \tilde{\sigma}(f)(a) < \tilde{\sigma}(\bar{\nu})$ where $\langle a, f\rangle \in D$. We can assume w.l.o.g. that $\mathrm{rng}(f) \subset \bar{\nu}$. But it follows easily that $\mathrm{rng}(f)$ is not cofinal in $\bar{\nu}$. Hence, letting $\mathrm{rng}(f) \subset \eta < \bar{\nu}$, we have: $\tilde{\sigma}(f)(a) < \sigma(\eta) < \nu$. QED (b)

It is then obvious that $\beta^{+M} = \nu$, where $\tilde{\sigma}(\bar{\beta}) = \beta$. $\nu \leq \omega\rho^n_M$ by Lemma 2. To see that $\omega\rho^{n+1}_M \leq \beta < \nu$, we observe that $h_{M^n p}(\beta \cup \{p^n\}) \supset$
$\supset \tilde{\sigma}'' h_{\bar{M}^n, \bar{p}}(\bar{\beta} \cup \{\bar{p}^n\})$, where

$\bar{p}$ witnesses the clarity of $\bar{M}$, since $\tilde{\sigma}$ is $\Sigma_0^{(n)}$ - preserving. Hence $\nu \cap h_{M^{n,p}}(\beta \cup \{p^n\})$ is cofinal in ν, proving (c) and (d). To prove (a), observe that if H_n is as in the proof of Lemma 2 and A is defined by:

$$\tilde{\sigma} \upharpoonright H^n_{\bar{M}} : M^{n,\bar{p}} \xrightarrow[\Sigma_0]{} \langle H_n, A\rangle \text{ cofinally}$$

Then $A = A^{np}_M \cap H_n$ and $h_{\langle H_n, A\rangle}(\beta \cup \{p^n\}) \supset \tilde{\sigma}'' h_{\bar{M}^{n}p}(\bar{\beta} \cup \{\bar{p}^n\})$ & hence $\nu \cap h_{\langle H_n, A\rangle}(\beta \cap \{p^n\})$ is cofinal in ν. Hence $A \notin M$, Hence $\rho^n_M \leq \rho^n$ (ρ^n as in the proof of Lemma 2). But $\geq$ was proven in Lemma 2. QED (Lemma 3)

We now give a condition which can be used to prove $\Sigma_1^{(n)}$ preservation when $\bar{\nu}$-clarity fails.

<u>Def</u> $\bar{M}$ is <u>bound to</u> $\bar{\nu}$ (as witnessed by $\bar{p}$) iff either $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$ for all n or

There is n s.t. $\omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}}$

and $\bar{M}$ = the closure of $\bar{\nu} \cup \{\bar{p}\}$ under good $\Sigma_1^{(n)}$ fcns.

[<u>Note</u> The last condition is equivalent to: $H^n_{\bar{M}} \subset h_{\bar{M}^n, \bar{p}}(\bar{\nu} \cup \{\bar{p}^n\})$ and $H^h_{\bar{M}} \subset h_{\bar{M}^h, \bar{p}}(\omega\rho^h_{\bar{M}} \cup \{\bar{p}^h\})$ for $h < n$.]

<u>Lemma 4</u> Let $\bar{M}$ be bound to $\bar{\nu}$, where $\omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}}$. Then

(a) $\tilde{\sigma} : \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally

(b) $\omega\rho^{n+1}_M \leq \nu$

(c) M is bound to ν (in fact, M is the closure of $\nu \cup \{p\}$ under good $\Sigma_1^{(n)}$ fcns.

(<u>Note</u> We <u>cannot</u> infer $\omega\rho^{n+1}_M < \nu$).

proof of Lemma 4.

We first show (a). Let ρ^n, H_n be as in the proof of Lemma 1. It suffices to show: Claim $\rho^n = \rho^n_M$.

If $n=0$ this is trivial. Let $n>0$. Define A by:

$$\tilde{\sigma} \restriction H^n_{\bar{M}} : \bar{M}^{n,\bar{p}} \xrightarrow[\Sigma_0]{} \langle H_n, A \rangle \text{ cofinally}.$$

Then $\langle H_n, A \rangle \prec_{\Sigma_0} M^{n,p}$, where $p = \tilde{\sigma}(p)$. By the downward extension of imbeddings lemma there are $\hat{M}$, $\hat{p}$ s.t. $\hat{M}^{n,\hat{p}} = \langle H_n, A \rangle$ and $\hat{p} \in R^n_{\hat{M}}$. Moreover there is $\pi : \hat{M} \xrightarrow[\Sigma_0^{(n)}]{} M$ s.t. $\pi \restriction H_n = id$ and $\pi(\hat{p}) = p$. Since $\bar{p} \in R^n_{\bar{M}}$, there is $\hat{\sigma} : \bar{M} \xrightarrow[\Sigma_1^{(n)}]{} \hat{M}$ s.t. $\hat{\sigma} \restriction H^n_{\bar{M}} = \tilde{\sigma} \restriction H^n_{\bar{M}}$ and $\hat{\sigma}(\bar{p}) = \hat{p}$.

But then $\pi(\hat{\sigma}(f)(a)) = \tilde{\sigma}(f)(a)$ for $\langle a, f \rangle \in D$. Hence $\pi\hat{\sigma} = \tilde{\sigma}$ and π is onto. Hence $\pi = id$, $\hat{\sigma} = \tilde{\sigma}$ and $\hat{M} = M$. QED(a)

But then each $x \in H^n_M$ has the form $\tilde{\sigma}(f)(a)$, where $f \in H^n_{\bar{M}}$ and $a \in \sigma(\mathrm{dom}(f))$. If $f = h_{\bar{M}^n, \bar{p}}(i, \langle \xi, \bar{p}^n \rangle)$ for some $\xi < \bar{\nu}$, then
$\tilde{\sigma}(f)(a) = h_{M^n, p}(j, \langle\langle \xi, a \rangle, p^n \rangle)$
for some j. Hence $M^{n p} \subset h_{M^n, p}(\nu \cup \{p^n\})$ and (b), (c) hold.

QED (Lemma 4)

Finally, we note that if there is a $\sigma^*: \bar{M} \longrightarrow M^*$ which is $\Sigma_0^{(n)}$-preserving for $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$ and s.t. $\sigma^* \upharpoonright \bar{Q} = \sigma$, then the well foundedness of $\mathbb{D}$ is assured, since there is $k: \mathbb{D} \xrightarrow{\sim} M^*$ defined by $k(\langle a, f \rangle) = \sigma^*(f)(a)$. Pursuing this line of reasoning, we easily get the <u>interpolation lemma</u>:

<u>Lemma 5</u> Let $\sigma^*: \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M^*$ for $\omega\rho^n_{\bar{M}} \geq \bar{\nu}$ s.t. $\sigma^* \restriction \bar{Q} = \sigma$. Then

(a) The canonical completion $\tilde{\sigma}: \bar{M} \longrightarrow M$ of $\sigma: \bar{Q} \longrightarrow Q$ exists.

(b) There is a unique $\pi: M \longrightarrow_{\Sigma_0} M^*$ s.t. $\pi\tilde{\sigma} = \sigma^*$ and $\pi: M \longrightarrow_{\Sigma_1^{(n)}} M^*$ for $\omega\rho^{n+1}_{\bar{M}} \geq \bar{\nu}$. π is defined by: $\pi(\tilde{\sigma}(f)(a)) = \sigma^*(f)(a)$.

(c) If $\omega\rho^{n+1}_{\bar{M}} < \bar{\nu} \leq \omega\rho^n_{\bar{M}}$ and $\tilde{\sigma}: \bar{M} \longrightarrow_{\Sigma_0^{(n)}} M$ cofinally, then $\pi: M \longrightarrow_{\Sigma_0^{(n)}} M^*$.