§3.6 Beavers

We now turn to the task of showing that the definition of K cannot break down for lack of uniqueness. Our central tool will be the following concept:

Def By a beaver we mean a structure $N = \langle J_\nu^E, U \rangle$ s.t.

(a) There is a largest cardinal κ in N

(b) U is a normal measure on κ in N

(c) There is a universal weasel W s.t.

(i) ν is a cardinal in W

(ii) $J_\nu^E = J_\nu^{E^W}$

(iii) There is $\pi : W \xrightarrow[U]{} W'$.

(Note It follows easily that N is amenable. If $\langle W, U \rangle$ is simply iterable*/, then N is a mouse.

*/ cf end of the previous section.

To see this, note that if $\pi : W \to_U W'$, then $E_\nu^{W'} = \emptyset$, since otherwise $\langle W|\nu, U\rangle$ would be an r-mouse. But then $N = \langle J_\nu^{E^W}, U\rangle$ is an iterable quasi premouse as defined at the end of §3.1

<u>Lemma 1</u> Let $N = \langle J_\nu^E, U\rangle$ be a beaver and let W be a weasel satisfying the conditions (i) – (iii) of (c). Let $\tilde{W}$ be a simple iterate of W above κ. Then $\tilde{W}$ is as in (c).

prf

Let W_i $(i \leq \theta \leq \infty)$ be the normal iteration from W to $\tilde{W}$ with indices ν_i, κ_i. Let $\sigma : W \to_U W'$. Define an iteration W_i' $(i \leq \theta)$ of W' with indices ν_i', κ_i' together with maps $\sigma^i : W_i \to_{\Sigma_1} W_i'$ by induction on $i \leq \theta$.

Let π_{ij}, π'_{ij} be the iteration maps for W_i, W_i' resp. We ensure that the following hold:

(1) $\sigma^i \pi_{hi} = \pi'_{hi} \sigma^h$ $(h \leq i)$

(2) $\kappa = \operatorname{crit}(\sigma^i)$, where U is on κ.

(3) $X \in U \longleftrightarrow \kappa \in \sigma^i(X)$ for $X \in P(\kappa) \cap W$

By (2), (3) we conclude:

(4) $\sigma^i : W_i \to_U W_i'$,

which proves the claim for $i = \theta$.

We define $\sigma^i, \kappa_i', \nu_i'$ inductively:

$$\sigma^0 = \sigma; \quad \kappa_i', \nu_i' = \sigma^i(\kappa_i, \nu_i);$$

$$\sigma^{i+1}(\pi_{i,i+1}(f)(\kappa_i)) = \pi'_{i,i+1}\sigma^i(f)(\kappa_i')$$

$$\sigma^\lambda \pi_{i\lambda} = \pi'_{i\lambda}\sigma^i \quad (i < \lambda, \text{ limit } \lambda).$$

(1) – (3) are verified by induction on i. The verification are straightforward. To prove (3) at $i+1$, let $x = \pi'_{i,i+1}(f)(\kappa_i') \in W'_{i+1}$, where $f = \sigma^i(g)(\kappa)$. Then

$$x = \pi'_{i,i+1}(\sigma^i(g)(\kappa))(\kappa_i') =$$

$$= (\pi'_{i,i+1}\sigma^i(g)(\kappa))(\kappa_i') =$$

$$= (\sigma^{i+1}\pi_{i,i+1}(g)(\kappa))(\kappa_i'),$$

where $\kappa_i' = \sigma^{i+1}(\kappa_i)$, as can be seen by taking $f = \mathrm{id}$ in the defn. of σ^{i+1}.

QED (Lemma 1)

<u>Lemma 2</u> Let $N = \langle J_\nu^E, U \rangle$ be a beaver. Let W be weakly universal and as in (c). Let W' be weakly universal with $W'|\nu = W|\nu$. Then W' is as in (c).

prf.

Coiterate W, W' to W^*. By Lemma 1 there is $\pi : W^* \xrightarrow[U]{} W^{**}$. Note that W^* is an iterate of W' above ν.

Let $X =$ the $\Sigma_0(W^{**})$-closure of $\mathrm{rng}(\pi\pi_{W'W^*}) \cup \{\kappa\}$. Set:

$$\sigma : W'' \stackrel{\sim}{\longleftrightarrow} X \quad , \quad \pi' = \sigma^{-1}\pi\pi_{W'W^*} .$$

Then W'' is a weasel and:

(1) $\pi' : W' \longrightarrow_{\Sigma_1} W''$

(2) $\pi' \upharpoonright \nu = \pi \upharpoonright \nu$, hence

(3) $\mathrm{crit}(\pi') = \kappa$ and

$Y \in U \longleftrightarrow \kappa \in \pi'(Y)$ for $Y \in \mathbb{P}(\kappa) \cap W'$.

(4) $W'' =$ the $\Sigma_0(W'')$-closure of $\mathrm{rng}(\pi') \cup \{\kappa\}$.

Hence $\pi' : W' \xrightarrow[U]{} W''$.

QED (Lemma 2)

We now strengthen Lemma 2 to:

<u>Lemma 3</u> Let $N = \langle J_\nu^E, U\rangle$ be a beaver. Let W' be almost universal s.t. $J_\nu^{E^{W'}} = J_\nu^E$. Then there is $\pi : W' \xrightarrow[U]{} W''$.

prf.

<u>Case 1</u> $E_\nu^{W'} = \emptyset$

There is a universal W satisfying (c) which, by Lemma 1, we may assume to have the property: $E_\nu^W = \emptyset$. The conclusion follows by Lemma 2.

<u>Case 2</u> Case 1 fails.

By Lemmas 1, 2 it suffices to prove the conclusion for some $\tilde{W}$ s.t. $\tilde{W}$ is universal and $\tilde{W}|\nu = W'|\nu$. Thus we may assume w.l.o.g. that there is an X which is massive in W'. Let $\sigma : W' \xrightarrow[E_\nu]{} W^*$.

By Case 1 there is $\pi : W^* \xrightarrow[u]{} W^{**}$.

Set: $X^* = \{\alpha \in X \mid \pi\sigma(\alpha) = \alpha\}$.

Then X^* is massive in W', W^*, W^{**}.

<u>Claim</u> If $\vec{\alpha} \in X^*$, $\vec{\nu} < \kappa$, and t is a term in the 1st order language of $W^* = J_\infty^{E^*}$, then

$$t(\vec{\alpha}, \vec{\nu}, \sigma(\kappa))_{W^*} \notin [\kappa, \sigma(\kappa)).$$

prf.

Suppose not. Then for $\vec{\nu} < \xi < \kappa$:

$$W^* \models \xi < t(\vec{\alpha}, \vec{\nu}, \sigma(\kappa)) < \sigma(\kappa)$$

hence:

$$W' \models \xi < t(\vec{\alpha}, \vec{\nu}, \kappa) < \kappa$$

for $\vec{\nu} < \xi < \kappa$. Contr! QED (Claim)

But then:

$$W^* \models \bigwedge \vec{\nu} < \kappa \;\; t(\vec{\alpha}, \vec{\nu}, \sigma(\kappa)) \notin [\kappa, \sigma(\kappa))$$

hence:

$$W^{**} \models \bigwedge \vec{\nu} < \pi(\kappa) \; t(\vec{\alpha}, \vec{\nu}, \pi\sigma(\kappa)) \notin$$
$$\notin [\pi(\kappa), \pi\sigma(\kappa)).$$

Set: $Y =$ the $\Sigma_1(W^{**})$-closure of
$$X^* \cup \kappa \cup \{\pi\sigma(\kappa)\}$$
$Y' =$ the $\Sigma_1(W^{**})$-closure of
$$X^* \cup \pi(\kappa) \cup \{\pi\sigma(\kappa)\}.$$

Let $\rho : \overline{W} \overset{\sim}{\longleftrightarrow} Y$, $\rho' : \overline{W}' \longleftrightarrow Y'$ where $\overline{W}, \overline{W}'$ are transitive. Set:
$$\overline{\pi} = (\rho'^{-1}\rho) : \overline{W} \xrightarrow[\Sigma_1]{} \overline{W}'.$$

By the above analysis:

(1) $\kappa = \mathrm{crit}(\overline{\pi})$, $\overline{\pi}(\kappa) = \pi(\kappa)$.

<u>Claim</u> $\overline{\pi} \restriction \mathbb{P}(\kappa) = \pi \restriction \mathbb{P}(\kappa)$.

pf.
Let $X \in \mathbb{P}(\kappa) \cap \overline{W} = \mathbb{P}(\kappa) \cap W^*$.
Let $\rho(X) = t(\vec{\alpha}, \vec{\nu}, \pi\sigma(\kappa))^{W^{**}}$.
Then $(\pi\sigma)^{-1}\rho(X) = t(\vec{\alpha}, \vec{\nu}, \kappa)^{W'}$,
where $\pi\sigma \restriction \kappa = \rho \restriction \kappa = \mathrm{id}$. Hence
$\pi\sigma(X) = t(\vec{\alpha}, \vec{\nu}, \pi\sigma(\kappa))_{W^{**}} = \rho(X)$.
Hence $\overline{\pi}(X) = \rho'^{-1}\rho(X) = \rho(X) \cap \overline{\pi}(\kappa) =$
$= \pi\sigma(X) \cap \pi(\kappa) = \pi(\sigma(X) \cap \kappa) = \pi(X)$.

QED (Claim)

But then:

(2) $Y \in U \longleftrightarrow \kappa \in \bar{\pi}(Y)$ for $Y \in \mathcal{P}(\kappa) \cap \bar{W}$,

(3) $\bar{W}' =$ the $\Sigma_0(\bar{W}')$ - closure of $\mathrm{rng}(\bar{\pi}) \cup \{\kappa\}$,

since $\bar{W}'$ is the Σ_0 - closure of $\mathrm{rng}(\bar{\pi}) \cup \pi(\kappa)$ and each $\nu \in \pi(\kappa)$ has the form $\nu = \pi(f)(\kappa) = \bar{\pi}(f)(\kappa)$ for an $f : \kappa \to \kappa$, $f \in \mathcal{P}(\kappa^2) \cap W \subset \bar{W}$.

Hence:

(4) $\bar{\pi} : \bar{W} \xrightarrow[U]{} \bar{W}'$.

Finally we notice that:

(6) $E_\nu^{\bar{W}} = E_\nu^{W}$,

since $X \in E_\nu^{W} \to \rho(X) = \pi\sigma(X) \in E_{\pi\sigma(\nu)}^{W^{**}}$

$\to X \in E_\nu^{\bar{W}}$.

QED (Lemma 3)

We aim to show that every beaver is a mouse (in fact, a strong one) and that if $\langle J_\nu^E, U\rangle$ is a beaver and W is universal with $J_\nu^{E^W} = J_\nu^E$, then $\langle W, U\rangle$ is iterable in the obvious sense. We also want to show that if $\langle J_\nu^E, U\rangle, \langle J_\nu^E, U'\rangle$ are beavers, then $U = U'$. (This will be the most important element in proving the uniqueness needed to define K). To these ends we define:

<u>Def</u> $N = \langle J_\nu^E, U, U'\rangle$ is a <u>double beaver</u> iff $\langle J_\nu^E, U\rangle, \langle J_\nu^E, U'\rangle$ are beavers.

<u>Def</u> Let $N = \langle J_\nu^E, U_0, U_1\rangle$ be a double beaver. $\langle N_i \mid i < \theta\rangle$ is an <u>iteration</u> of N with indices $\langle e_i, \alpha_i\rangle$ $(i+1 < \theta)$ and maps π_{ij} $(i \le j < \theta)$ iff the following hold:

(a) $\omega \alpha_i \leq On_{N_i}$

(b) $\nu_i \leq On_{N_i}$ or $\nu_i = \langle 0, h\rangle$ $(h = 0, 1)$

(c) The π_{ij} commute and are continuous at limits

(d) Set: $E^{N_i}_{\langle 0,h\rangle} = U^i_h$ $(h = 0, 1)$,

where $N_i = \langle J^{E^i}_{\nu_i}, U^i_0, U^i_1\rangle$. Then

either $E_{e_i} = \emptyset$ and $N_{i+1} = N_i | \alpha_i$

or E_{e_i} is a measure in $N_i | \alpha_i$ and

$\pi_{i,i+1} : (N_i | \alpha_i) \longrightarrow_{E_{e_i}} N_{i+1}$.

(e) $\{i \mid \omega \alpha_i \in N_i\}$ is finite.

<u>Def</u> Let W be a weasel s.t. $J^{E^W}_\nu = J^E_\nu$. The notion of an iteration of $\langle W, U_0, U_1\rangle$ with indices $\langle e_i, \alpha_i\rangle$ + maps π_{ij} is just as above. The i-th iterate is either a mouse, if we have truncated, or has the form $\langle W_i, U^i_0, U^i_1\rangle$ where

$$\pi_{0i} \restriction N : N \longrightarrow_{\Sigma_0} \langle J^{E^{W_i}}_{\nu'}, U^i_0, U^i_1\rangle$$

cofinally.

(i.e. $U_h^i = \bigcup_{x \in N} \pi_{0i}(x \cap U_h)$).

The notion of an iteration of a simple heaven $\langle J_\nu^E, U \rangle$ or a structure $\langle W, U \rangle$ ($J_\nu^{E^W} = J_\nu^E$) is of course defined exactly as above with (b) replaced by: (b′) $\nu_i \leq On_{N_i}$ or $\nu_i = \langle 0, 0 \rangle$.

The notions of normal, simple, and standard iteration are defined exactly as before.

<u>Note</u> In dealing with these structures we shall only need iterations of length $< \infty$.

Def Let $N = \langle J_\nu^E, U, U \rangle$ be a double beaver. Let $\mathcal{I} = \langle N_i \mid i < \theta \rangle$ be a simple normal iteration of N with indices e_i $(i+1 < \theta)$. $\mathcal{I}$ is <u>applicable</u> iff there is an almost universal weasel W s.t. $J_\nu^{E^W} = J_\nu^E$ and there is a simple iteration $\langle W_i, U_i, U_i \rangle$ $(i < \theta)$ with the same indices [hence $N_i = \langle J_{\nu_i}^{E^{W_i}}, U_i, U_i \rangle$, where $\nu_i = \pi_{0i}(\nu)$ and $\pi_{ij} \upharpoonright N_i$ are the iteration maps for $\mathcal{I}$, where π_{ij} $(i \leq j < \theta)$ are the iteration maps for $\langle W_i, U_i, U_i \rangle$ $(i < \theta)$].

If W is any weasel satisfying these conditions, we say that $\mathcal{I}$ is <u>applicable</u> <u>to</u> W and that the iteration $\langle W_i, U_i, U_i \rangle$ $(i < \theta)$ is the <u>application</u> of $\mathcal{I}$ to W.

The corresponding notion for single beavers is obvious.

Thus, saying that $N = \langle J_\nu^E, U\rangle$ is a beaver simply means that the two stage iteration $\pi : N \xrightarrow[U]{} N'$ is applicable to a universal W.

Pursuing this lead, we generalize lemmas 1-3 to the following lemma on applicability:

<u>Lemma 4</u> Let $N = \langle J_\nu^E, U, V\rangle$, where U, V are measures on κ. Let $\mathcal{I}$ be applicable to W. Let $\tilde{W}$ be a simple iterate of W above κ. Then $\mathcal{I}$ is applicable to $\tilde{W}$.

prf.

Let W^i $(i \leq \Gamma \leq \infty)$ be the normal iteration from W to $\tilde{W}$ with indices ν^i, κ^i (i.e. $E_{\nu^i}^{W^i}$ is a measure on κ^i). Let $\pi^{ij} = \pi_{W^i W^j}$ be the iteration maps.

Let $\langle W_i, U_i, V_i\rangle$ $(i < \theta)$ be the application of $\mathcal{Y} = \langle N_i \mid i < \theta\rangle$ to W with indices $e_i, \bar{\kappa}_i$ and iteration maps π_{ij}. (Assume w.l.o.g. that $E_{e_i}^{W_i} \neq \emptyset$ is a measure on $\bar{\kappa}_i$).

For each $h < \theta$ we now define an iteration W_h^i $(i \leq \Gamma)$ of W_h with indices ν_h^i, κ_h^i and iteration maps $\pi_h^{ij} = \pi_{W_h^i W_h^j}$. Simultaneously we define maps $\pi_{0h}^i : W^i \longrightarrow_{\Sigma_1} W_h^i$.

We define $\pi_{0h}^i, \kappa_h^i, \nu_h^i, W_h^i$ by ind. on i as follows:

$$\pi_{0h}^0 = \pi_{0h} \quad , \quad W_h^0 = W_h$$

$$\kappa_h^i, \nu_h^i = \pi_{0h}^i(\kappa^i, \nu^i)$$

(This defines W_h^{i+1}, $\pi_h^{i,i+1}$).

$$\pi_{0h}^{i+1}(\pi^{i,i+1}(f)(\kappa^i)) =$$

$$= \pi_h^{i,i+1}\pi_{0h}^i(f)(\kappa_h^i)$$

At limit λ, W_h^λ, $\pi_h^{i\lambda}$ $(i<\lambda)$ are then given and we define:

$$\pi_{0h}^\lambda \pi^{i\lambda} = \pi_h^{i\lambda}\pi_{0h}^i \qquad (i<\lambda),$$

This defines π_{0h}^λ.

But then we can extend the system of maps π^i_{0h} to maps

$$\pi^i_{hj} : W^i_h \xrightarrow[\Sigma_1]{} W^i_j \quad (h \leq j < \theta)$$

s.t.

(1) $\pi^i_{jk} \pi^i_{hj} = \pi^i_{hk}$; hence:

(2) $\pi^i_{hj}(\kappa^i_h) = \kappa^i_j$.

We define π^i_{hj} $(h \leq j < \theta)$ by ind. on i, verifying (1) at each stage:

$$\pi^0_{hj} = \overline{\pi}_{hj}$$

$$\pi^{i+1}_{hj}\left(\pi^{i,i+1}_h(f)(\kappa^i_h)\right) =$$

$$= \pi^{i,i+1}_j \pi^i_{hj}(f)(\kappa^i_j)$$

For limit λ set:

$$\pi^\lambda_{hj} \pi^{i\lambda}_h = \pi^{i\lambda}_j \pi^i_{hj}$$

We then have:

(3) $$\pi^j_{hk} \pi^{ij}_h = \pi^{ij}_k \pi^i_{hk}$$

$$\begin{tikzcd} W^i_k \arrow[r, "\pi^{ij}_k"] & W^j_k \\ W^i_h \arrow[u, "\pi^i_{hk}"] \arrow[r, "\pi^{ij}_h"'] & W^j_h \arrow[u, "\pi^j_{hk}"'] \end{tikzcd}$$

A straightforward induction on i shows:

(4) The system of maps π^i_{hk} $(h \leq k < \theta)$ is continuous at limits (i.e. $W^i_\lambda = \bigcup_{h<\lambda} \operatorname{rng}(\pi^i_{h\lambda})$ for limit λ).

Since $\bar{\kappa}_0 \leq \kappa^i_0$ for $i \leq \Gamma$ we have:

(5) $\kappa^i_h \geq \pi^i_{0h}(\bar{\kappa}_0) \geq \bar{\kappa}_h$ for $i \leq \Gamma$, $h < \theta$

(6) $\pi^i_{hk} \upharpoonright \bar{\kappa}_h = \mathrm{id}$

proof triv. for $i=0$. For $i>0$:

$\pi^i_{hk}(\xi) = \pi^i_{hk}\pi^{0i}_h(\xi) = \pi^{0i}_k \pi^0_{hk}(\xi) = \xi$

for $\xi < \bar{\kappa}_h$ by (5).   QED(6)

(7) $\bar{\kappa}_h = \operatorname{crit}(\pi^i_{hk})$ for $h < k$

prf.

We show $\pi^i_{hk}(\bar{\kappa}_h) > \bar{\kappa}_h$ by ind. on i. $i=0$ is trivial.

If $\bar{\kappa}_h < \kappa^i_h$, then

$\pi^{i+1}_{hk}(\bar{\kappa}_h) = \pi^{i+1}_{hh}\,\pi^{i,i+1}_h(\bar{\kappa}_h) =$

$= \pi^{i,i+1}_k\,\pi^i_{hk}(\bar{\kappa}_h) > \bar{\kappa}_h$.

If $\bar{\kappa}_h = \kappa^i_h$, then

$\pi^{i+1}_{hk}(\bar{\kappa}_h) = \kappa^i_h > \bar{\kappa}_h$.

For limit λ we have $\bar{\kappa}_h = \pi^{i\lambda}_h(\tilde{\kappa})$,

for some $i < \lambda$. But then $\tilde{\kappa} \geq \bar{\kappa}_h$,

since otherwise $\tilde{\kappa} < \kappa^i_h$ and

$\bar{\kappa}_h = \pi^{i\lambda}_h(\tilde{\kappa}) = \tilde{\kappa} < \bar{\kappa}_h$. Hence:

$\pi^\lambda_{hk}(\bar{\kappa}_h) = \pi^\lambda_{hh}\,\pi^{i\lambda}_h(\tilde{\kappa}) = \pi^{i\lambda}_k\,\pi^i_{hh}(\tilde{\kappa}) \geq$

$\geq \pi^{i\lambda}_k\,\pi^i_{hk}(\bar{\kappa}_h) > \bar{\kappa}_h$.

QED (7)

By (5) we have

(8) $\mathcal{P}(\bar{\kappa}_h) \cap W_h = \mathcal{P}(\bar{\kappa}_h) \cap W^i_h \quad (i \leq \Gamma,\ h < \theta)$

But then

(9) Let $X \in \mathcal{P}(\bar{\kappa}_h) \cap W_h$, $h < \theta$, $i \leq \Gamma$.

Then $\bar{\kappa}_h \in \pi^i_{hh}(X) \longleftrightarrow X \in E^{W_h}_{e_h}$.

proof of (9),

Immediate for $i=0$. For $i>0$ we have $\varkappa_h^i > \varkappa_h^0 \geq \bar{\varkappa}_h$. Hence,

$\bar{\varkappa}_h \in \pi_{hk}^i(X) = \pi_{hk}^i(\bar{\varkappa}_h \cap \pi_h^{0i}(X)) \leftrightarrow$

$\leftrightarrow \bar{\varkappa}_h \in \pi_{hk}^i \pi_h^{0i}(X)$, since $\pi_{hh}^i(\bar{\varkappa}_h) > \bar{\varkappa}_h$

$\leftrightarrow \bar{\varkappa}_h \in \pi_k^{0i} \pi_{hh}^0(X)$

$\leftrightarrow \varkappa_h \in \pi_{hk}(X) \leftrightarrow X \in E_{e_h}^{W_h}$,

since $\varkappa_k^i \geq \varkappa_h^i > \bar{\varkappa}_h$ & hence $\pi_h^{0i}(\bar{\varkappa}_h) = \bar{\varkappa}_h$. QED(9)

(10) $W_{h+1}^i =$ the $\Sigma_0(W_h^i)$ - closure of $\operatorname{rng}(\pi_{h,h+1}^i) \cup \{\bar{\varkappa}_h\}$

prf. Ind. on i.
$i=0$ + $\lim(i)$ are straightforward.
For $i+1$ let $x = \pi_{h+1}^{i,i+1}(f)(\varkappa_{h+1}^i) \in W_{h+1}^{i+1}$ where $f = \pi_{h,h+1}^i(g)(\bar{\varkappa}_h)$. Then

$x = \pi_{h+1}^{i,i+1}(\pi_{h,h+1}^i(g)(\bar{\varkappa}_h))(\varkappa_{h+1}^i) =$

$= (\pi_{h+1}^{i,i+1}\pi_{h,h+1}^i(g)(\bar{\varkappa}_h))(\varkappa_{h+1}^i) =$

(since $\varkappa_{h+1}^i \geq \bar{\varkappa}_{h+1} > \bar{\varkappa}_h$)

$= (\pi_{h,h+1}^{i+1}\pi_h^{i,i+1}(\bar{\varkappa}_h))(\varkappa_{h+1}^i)$

But $\kappa^{i}_{h+1} = \pi^{i+1}_{h,h+1}(\kappa^{i}_{h})$ since

$$\pi^{i+1}_{h,h+1}(\kappa^{i}_{h}) = \pi^{i+1}_{h,h+1}\left(\pi^{i,i+1}_{h}(\mathrm{id})(\kappa^{i}_{h})\right) =$$

$$= \pi^{i,i+1}_{h+1}\,\pi^{i}_{h,h+1}(\mathrm{id})(\kappa^{i}_{h+1}) = \kappa^{i}_{h+1}.$$

QED (10)

By (7) – (10) we then have;

(11) $\pi^{i}_{h,h+1} : W^{i}_{h} \xrightarrow[E_{e_n}]{} W^{i}_{h+1}$.

Hence the iteration W^{i}_{h} $(h < \theta)$ is the application of $\mathcal{J}$ to W^{i}. This holds in particular for $W^{\Gamma} = \tilde{W}$. QED (Lemma 4)

<u>Lemma 5.</u> Let $N = \langle J_\nu^E, u, v \rangle$ be as above + let $\mathcal{Y}$ be applicable to the weakly universal W. Let W' be weakly universal s.t. $W|\nu = W'|\nu$. Then $\mathcal{Y}$ is applicable to W'.

prf.

Coiterate W, W' to W^*. $\mathcal{Y}$ is then applicable to W^*. Let $\langle W_i^*, u_i, v_i \rangle$ $(i < \theta)$ be the application of $\mathcal{Y}$ to W^* with indices e_i, κ_i and maps π_{ij}. Set:

$X_i =$ the $\Sigma_0(W_i^*)$ - closure of $\operatorname{rng}(\pi_{0i}\, \pi_{W'W^*}) \cup \{u_h \mid h < i\}$

$\sigma_i : W_i' \xrightarrow{\sim} X_i$, where W_i' is transitive; $\pi'_{ij} = \sigma_j^{-1}\, \pi_{ij}\, \sigma_i$ $(i \le j < \theta)$.

Clearly $W_0' = W'$. Moreover:

(1) $\pi'_{ij} : W_i' \to_{\Sigma_1} W_j'$

Let $\mathcal{Y} = \langle N_i \mid i < \theta \rangle$ where

$N_i = \langle J_{\nu_i}^{E^i}, u_i, v_i \rangle$.

Then $J^{E^i}_{\nu_i} = J^{E^{W^*_i}}_{\nu_i}$ and $J^{E^i}_{\nu_i} =$

$=$ the $\Sigma_1(J^{E^i}_{\nu_i})$-closure of

$$\mathrm{rng}(\pi_{N_0 N_i}) \cup \{\kappa_l \mid l < i\}.$$

Since $J^E_\nu \subset \mathrm{rng}(\pi_{W'W^*})$, we have $\mathrm{rng}(\pi_{N_0 N_i}) \cup \{\kappa_l \mid l < i\} \subset X_i$

and hence:

(2) $J^{E^i}_{\nu_i} \subset X_i$.

It follows trivially that:

(3) $\pi'_{ij} \upharpoonright J^{E^i}_{\nu_i} = \pi_{ij} \upharpoonright J^{E^i}_{\nu_i}$.

Hence:

(4) $\kappa_i = \mathrm{crit}(\pi'_{ij}) \quad (i < j)$

(5) $X \in E^{N_i}_{e_i} \longleftrightarrow \kappa_i \in \pi'_{ij}(X) \quad (i < j)$

for $X \in \mathbb{P}(\kappa_i) \cap J^{E^i}_{\nu_i}$.

Trivially:

(6) $W'_{i+1} =$ the $\Sigma_0(W'_i)$-closure

of $\mathrm{rng}(\pi'_{i,i+1}) \cup \{\kappa_i\}$

(7) $W'_\lambda = \bigcup_{i<\lambda} \mathrm{rng}(\pi'_{i\lambda})$;

Hence $W'_i \ (i < \theta)$ is the application of $\mathcal{I}$ to W'.

QED(1)

<u>Lemma 6</u> Let $N, \mathcal{I}$ be as above. Let W be weakly universal s.t. $J_\nu^{E^W} = J_\nu^E$. Then $\mathcal{I}$ is applicable to W.

proof.

<u>Case 1</u> $E_\nu^W = \emptyset$ as in Lemma 3

<u>Case 2</u> Case 1 fails.

By Lemmas 4, 5 it suffices to prove the conclusion for any weakly universal $\tilde{W}$ s.t. $\tilde{W}|\nu = W|\nu$. Hence we assume w.l.o.g. that some X is massive in W. Let $\sigma : W \xrightarrow[E_\nu]{} W^*$

Then $\mathcal{I}$ is applicable to W^*.

Let $\langle W_i^*, \mathcal{U}_i, \upsilon_i \rangle$ $(i < \theta)$ be the application of $\mathcal{I}$ to W^* with iteration maps π_{ij}. Set:

$$X^* = \{\alpha \in X \mid \pi_{0i}\sigma(\alpha) = \alpha \text{ for } i < \theta\}.$$

Then X^* is massive in W, W_i^* $(i < \theta)$

Set: $Y_i =$ the $\Sigma_1(W_i^*)$ - closure of

$$X^* \cup \pi_{0i}{}''\kappa \cup \{\kappa_h \mid h < i\} \cup \{\pi_{0i}\sigma(\kappa)\}$$

$$\rho_i : \bar{W}_i \stackrel{\sim}{\longleftrightarrow} Y_i \text{ where } \bar{W}_i \text{ transitive}$$

$$\bar{\pi}_{ij} = \rho_j^{-1}\pi_{ij}\rho_i \quad (i \leq j < \theta).$$

(1) $W^* \models \varphi(\vec{\alpha}, \vec{\tau}, \sigma(\kappa)) \longleftrightarrow$

$\longleftrightarrow W \models \varphi(\vec{\alpha}, \vec{\tau}, \kappa)$

for $\vec{\alpha} \in X^*$, $\vec{\tau} < \kappa$

Hence:

(2) $t(\vec{\alpha}, \vec{\tau}, \sigma(\kappa))_{W^*} \notin [\kappa, \sigma(\kappa))$

since otherwise $t(\vec{\alpha}, \vec{\tau}, \kappa)_W \in [\xi, \kappa)$

for all $\xi < \kappa$. Contr!

Since $W^* \models \bigwedge \vec{\tau} < \kappa \; t(\vec{\alpha}, \vec{\tau}, \sigma(\kappa)) \notin [\kappa, \sigma(\kappa))$

for $\vec{\alpha} \in X^*$, we conclude

(3) $t(\vec{\alpha}, \vec{\tau}, \pi_{0i}\sigma(\kappa))_{W_i^*} \notin [\pi_{0i}(\kappa), \pi_{0i}\sigma(\kappa))$

for $\vec{\alpha} \in X^*$, $\vec{\tau} < \pi_{0i}(\kappa)$.

Clearly $\kappa \subset Y_0$ + $\bar{W}$ is universal.

Hence:

(4) $J_\nu^{E^{\bar{W}_0}} = J_\nu^E$, since

$J_\kappa^{E^{\bar{W}_0}} = J_\kappa^E$ + $\nu = \kappa^{+W}$; hence the coiteration of $\bar{W}_0, W$ would otherwise contain a truncation.

Set: $\nu^* = \sigma(\kappa)^{+W^*} = \sigma(\nu) = \rho_0(\nu)$.

(5) $\rho_0 \restriction \nu = \sigma \restriction \nu$

Prf.

(a) $\mathrm{rng}(\rho_0 \restriction \nu) \subset \mathrm{rng}(\sigma \restriction \nu)$

prf. Let $\xi \in \mathrm{rng}(\tilde{\rho}_0)$. Then $\xi = t(\vec{\alpha}, \vec{\tau}, \sigma(\kappa))_{W^*} < \nu^*$, where $\vec{\alpha} \in X^*$, $\vec{\tau} < \kappa$. Hence

$$\xi = \sigma(\underset{\overset{\wedge}{\nu}}{t(\vec{\alpha}, \vec{\tau}, \kappa)_W}) \in \mathrm{rng}(\sigma \restriction \nu)$$

(b) $\sigma(\xi) \le \rho_0(\xi)$ for $\xi < \nu$ by (a)

(c) $\mathrm{rng}(\sigma \restriction \nu) \subset \mathrm{rng}(\rho_0 \restriction \nu)$

Let $\xi < \nu$, $\rho_0(\xi) = t(\vec{\alpha}, \vec{\tau}, \sigma(\kappa))_{W^*} \ge \sigma(\xi)$, where $\vec{\alpha} \in X^*$, $\vec{\tau} < \kappa$. Then $\xi \le \sigma^{-1}\rho_0(\xi) = t(\vec{\alpha}, \vec{\tau}, \kappa)_W$. Thus the set of $t(\vec{\alpha}, \vec{\tau}, \kappa)_W < \nu$ is cofinal in ν. Since every $\xi < \nu$ has cardinality $\le \kappa$ in W, it follows easily that <u>every</u> $\xi < \nu$ has this form.

Hence, $\mathrm{rng}(\sigma \restriction \nu) \subset \mathrm{rng}(\rho_0 \restriction \nu)$, since $\sigma(t(\vec{\alpha}, \vec{\tau}, \kappa)_W) = t(\vec{\alpha}, \vec{\tau}, \sigma(\kappa))_{W^*}$.

QED(5)

Since $\sigma(\nu) = \rho_0(\nu) = \nu^*$, we conclude:

(6) $E_\nu^W = E_\nu^{\bar{W}_0}$ (hence $\bar{W}_0|\nu = W|\nu$).

Hence it will suffice to show that $\langle \bar{W}_i | i < \theta \rangle$ in the application of $\mathcal{Y}$ to $\bar{W}_0$.

Set: $\tilde{\kappa}_i = \pi_{0i}(\kappa)$, $\tilde{\nu}_i = \pi_{0i}(\nu)$, $E^i = E^{W_i^*}$

$$\kappa_i^* = \pi_{0i}\sigma(\kappa), \nu_i^* = \pi_{0i}\sigma(\nu).$$

Then each $\xi < \tilde{\nu}_i$ in $J_{\tilde{\nu}_i}^{E^i}$ - definable in parameters from $\pi_{0i}{}''\nu \cup \{\kappa_h | h < i\}$.

Note that since each $\xi < \nu$ has the form $t(\vec{\alpha}, \vec{\iota}, \kappa)_W$ $(\vec{\alpha} \in X^*, \vec{\iota} < \kappa)$, (1) gives us:

(7) $\bigwedge \vec{\iota} < \kappa \left(J_\nu^{E^0} \models \varphi(\vec{\xi}, \vec{\iota}) \longleftrightarrow J_{\nu^*}^{E^0} \models \varphi(\sigma(\vec{\xi}), \vec{\iota}\right.$

for all $\vec{\xi} < \nu$.

Hence in W_i^* we have:

(8) $\bigwedge \vec{\iota} < \tilde{\kappa}_i \left(J_{\tilde{\nu}_i}^{E^i} \models \varphi(\pi_{0i}(\vec{\xi}), \vec{\iota}) \longleftrightarrow \right.$

$\left. \longleftrightarrow J_{\nu_i^*}^{E^i} \models \varphi(\pi_{0i}\sigma(\vec{\xi}), \vec{\iota}) \right)$

for all $\vec{\xi} < \nu$.

In particular we can define

$\tilde{\sigma}_i : J^{E^i}_{\tilde{\nu}_i} \xrightarrow[\Sigma_1]{} J^{E^i}_{\nu_i^*}$ by:

(9) $\tilde{\sigma}_i(t(\pi_{0i}(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\tilde{\nu}_i}}) = t(\pi_{0i}\sigma(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\nu_i^*}}$

where $\vec{\xi} < \nu_i$, $\vec{h} < i$.

Clearly by (8)

(10) $\tilde{\sigma}_i \upharpoonright \tilde{\kappa}_i = id$, $\tilde{\sigma}_i(\tilde{\kappa}_i) = \kappa_i^* > \tilde{\kappa}_i$.

By a proof similar to (5) we now show:

(11) $\rho_i \upharpoonright \tilde{\nu}_i = \tilde{\sigma}_i \upharpoonright \tilde{\nu}_i$.

prf.

($rng(\rho_i \upharpoonright \tilde{\nu}_i) \subset rng(\tilde{\sigma}_i \upharpoonright \tilde{\nu}_i)$) Let

$\xi < \tilde{\nu}_i$, $\rho_i(\xi) = t(\vec{\alpha}, \pi_{0i}(\vec{\tau}), \vec{\kappa}_h, \kappa_i^*)_{W_i^*}$,

where $\vec{\alpha} \in X^*$, $\vec{\tau} < \kappa$. Define

$g \in J^{E^0}_\nu$ by: $g(\vec{\gamma}) = t(\vec{\alpha}, \vec{\tau}, \vec{\gamma}, \kappa)_W$

for $\vec{\gamma} < \kappa$. Then $\sigma(g) \in J^{E^0}_{\nu^*}$ and

$\sigma(g)(\vec{\gamma}) = t(\vec{\alpha}, \vec{\tau}, \vec{\gamma}, \kappa^*)_{W^*}$

for $\vec{\gamma} < \kappa^*$.

Hence $\rho_i(\xi) = \pi_{0i}\sigma(g)(\vec{\kappa}_h) = \tilde{\sigma}_i(\pi_{0i}(g)(\vec{\kappa}_h))$,

$(\mathrm{rng}(\tilde{\sigma}_i \restriction \tilde{\nu}_i) \subset \mathrm{rng}(\rho_i \restriction \tilde{\nu}))$

Let $\delta < \tilde{\nu}_i$. $\delta = t(\pi_{0i}(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\tilde{\nu}_i}}$.

Let $\xi_k = t_k(\vec{\alpha}, \vec{\tau}, \kappa)_W$ where $\vec{\alpha} \in X^*$, $\vec{\tau} < \kappa$. Then

$\sigma_i(\delta) = t(\pi_{0i}\sigma(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\tilde{\nu}_i}}$,

where $\pi_{0i}\sigma(\xi_h) = t_h(\vec{\alpha}, \pi_{0i}(\vec{\tau}), \kappa^*_i)_{W^*_i}$;

hence $\sigma_i(\delta) = s(\vec{\alpha}, \pi_{0i}(\vec{\tau}), \vec{\kappa}_h, \kappa^*_i)_{W^*_i}$

Hence $\sigma_i(\delta) \in X^* \cap \nu^* = \mathrm{rng}\, \rho_i \restriction \tilde{\nu}_i$.

QED (11)

Hence:

(12) $\overline{\pi}_{i1} \restriction \tilde{\nu}_i = \pi_{i1} \restriction \tilde{\nu}_i$, since, letting $\delta = t(\pi_{0i}(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\tilde{\nu}_i}}$,

$\overline{\pi}_{i1}(\delta) = \tilde{\sigma}_1^{-1}\pi_{i1}(t(\pi_{0i}\sigma(\vec{\xi}), \vec{\kappa}_h)_{J^{E^i}_{\nu^*_i}}) =$

$= \tilde{\sigma}_1^{-1}(t(\pi_{0j}\sigma(\vec{\xi}), \vec{\kappa}_h)_{J^{E^1}_{\nu^*_1}}) = t(\pi_{01}(\vec{\xi}), \kappa_h)_{J^{E^1}_{\nu^*_1}} = \pi_{i1}(\delta)$.

We can then conclude exactly as in Lemma 5 that $\overline{W}_i$ is the application of $\mathcal{I}$ to $\overline{W}_0$.

QED (Lemma 6)

Suppose we are given a beaver $N = \langle J_\gamma^E, \mathcal{U}, \mathcal{V} \rangle$ and a pair of simple normal iterations $\bar{\mathcal{I}} = \langle \bar{N}_i \mid i \leq \bar{\theta} \rangle$, $\mathcal{I} = \langle N_i \mid i < \theta \rangle$ with indices $\bar{e}_i, \bar{\kappa}_i$ and e_i, κ_i resp. and iteration maps $\bar{\pi}_{ij}, \pi_{ij}$ resp. Assume, moreover, that $\kappa_0 =$ the largest cardinal in N (i.e. the first measure applied in $\mathcal{I}$ is either $\mathcal{U}$ or $\mathcal{V}$). It is natural to consider the iteration of a weasel W with $J_\gamma^{E^W} = J_\gamma^E$ obtained by first applying $\bar{\mathcal{I}}$ and then applying the "image" of $\mathcal{I}$ under $\bar{\pi}_{0\theta}$. This iteration, which we denote by $\bar{\mathcal{I}} * \mathcal{I}$, will also be simple and normal. It turns out that the $\bar{\theta}+i$-th stage can be obtained by first applying $\mathcal{I} | (i+1)$, then $\bar{\mathcal{I}}$.

The precise def. is:

<u>Def</u> Let N, $\bar{\mathcal{J}}$, $\mathcal{J}$ etc. be as above. Extend $\bar{\mathcal{J}}$ to an iteration $\langle \bar{N}_i \mid i < \tilde{\theta} \rangle$ $(\bar{\theta} < \tilde{\theta} \leq \bar{\theta} + \theta)$, simultaneously defining maps $\sigma_i : N_i \underset{\Sigma_1}{\longrightarrow} \bar{N}_{\bar{\theta}+i}$ as follows:

(a) $\sigma_0 = \bar{\pi}_{0\bar{\theta}}$

(b) If $\bar{N}_{\bar{\theta}+i}$, σ_i are defined, set: $\bar{e}_{\bar{\theta}+i} = \sigma_i(e_i)$. This defines $\bar{N}_{\bar{\theta}+i+1}$ <u>if</u> $\bar{N}_{\bar{\theta}+i}$ is extendible by $E_{\bar{e}_i}$. Otherwise $\bar{N}_{\bar{\theta}+i+1}$ is undefined.

(c) If $\bar{N}_{\bar{\theta}+i+1}$ is defined, define σ_{i+1} by:

$$\sigma_{i+1}(\pi_{i,i+1}(f)(\kappa_i)) = \bar{\pi}_{\bar{\theta}+i,\bar{\theta}+i+1}\,\sigma_i(f)(\bar{\kappa}_{\bar{\theta}+i}).$$

(d) If $\bar{N}_{\bar{\theta}+i}$, σ_i are defined for $i < \lambda$ where λ is a limit ordinal, set: $\bar{N}_{\bar{\theta}+\lambda} = \lim_{i<\lambda} \bar{N}_{\bar{\theta}+i}$ if this limit is well founded. Otherwise $\bar{N}_{\bar{\theta}+\lambda}$ is undefined.

(e) If $\bar{N}_{\bar{\theta}+\lambda}$ is defined, define σ_λ by: $\sigma_\lambda \pi_{i\lambda} = \bar{\pi}_{\bar{\theta}+i, \bar{\theta}+\lambda} \sigma_i$ $(i<\lambda)$,

. . . . .

Def If $\bar{N}_i$ is defined for $i < \bar{\theta}+\theta$, we set: $\bar{\mathcal{I}} * \mathcal{I} = \langle \bar{N}_i \mid i < \bar{\theta}+\theta \rangle$. Otherwise $\bar{\mathcal{I}} * \mathcal{I}$ is undefined.

It is obvious that $\bar{\mathcal{I}} * \mathcal{I}$ is a simple normal iteration of N.

Lemma 7 Let $N, \bar{\mathcal{I}}, \mathcal{I}$ be as above, where $\bar{\mathcal{I}}, \mathcal{I}$ are applicable. Then $\bar{\mathcal{I}} * \mathcal{I}$ exists and is applicable.

Proof of Lemma 7.

We turn the proof of Lemma 5 on its head. Let W be universal s.t. $J_\nu^{E^W} = J_\nu^E$. Let the iteration $\langle W^i, U^i, \upsilon^i \rangle$ be the application of $\mathcal{Y}$ to W with maps π^{ij} $(i \le j < \theta)$, Let $\langle \bar{W}_h, \bar{U}_h, \bar{\upsilon}_h \rangle$ be the application of $\bar{\mathcal{Y}}$ to W with maps $\bar{\pi}_{hk}$ $(h \le k \le \bar{\theta})$. For each $i < \theta$, let $\langle W^i_h, \bar{U}_h, \bar{\upsilon}_h \rangle$ be the application of $\bar{\mathcal{Y}}$ to W^i (since $J_\nu^{E^{W^i}} = J_\nu^E$) with maps π^i_{hk} $(h \le k \le \bar{\theta})$. Then $W^0_h = \bar{W}_h$, $\pi^0_{hk} = \bar{\pi}_{hk}$. Set:

$$e^i_h, \kappa^i_h = \pi_{0h}(e^i, \kappa^i),$$

where e^i, κ^i are the indices of $\mathcal{Y}$.

Define U^i_h, υ^i_h by:

$$U_h^i = \bigcup_{x \in N^i} \pi_{0h}^i(x \cap U^i)$$

$$V_h^i = \quad " \quad " \quad " \quad V^i \; .$$

We claim that $\langle W_h^i, U_h^i, V_h^i \rangle$ $(i < \theta)$ is a normal simple iteration of $\langle \bar{W}_h, \bar{U}_h, \bar{V}_h \rangle$ by the indices e_h^i $(i < \theta)$. (Note that $\langle W_0^i, U_0^i, V_0^i \rangle = \langle W^i, U^i, V^i \rangle$, $e_0^i = e^i$, so it holds for $h = 0$).

To this end we define commutative maps

$$\pi_h^{ij} : W_h^i \xrightarrow[\Sigma_1]{} W_h^j \quad (i \leq j < \theta)$$

by ind. on h and prove:

(a) $\kappa_h^i = \operatorname{crit}(\pi_h^{ij})$ $(i < j)$

(b) $X \in E_{e_h^i} \longleftrightarrow \kappa_h^i \in \pi_h^{ij}(X)$

for $X \in \mathbb{P}(\kappa_h^i) \cap W_h^i$, $i < j$,

where $E_{e_h^i} = \begin{cases} U_h^i & \text{if } e_h^i = \langle 0,0 \rangle \\ V_h^i & \text{if } e_h^i = \langle 0,1 \rangle \\ E_{\tau}^{W_h^i} & \text{if } e_h^i = \tau \in On \end{cases}$.

(c) W^i_h = the $\Sigma_0(W^i_h)$ - closure of
$rng(\pi^{0i}_h) \cup \{\kappa^l_h \mid l < i\}$.

This shows that $\langle W^i_h, U^i_h, U^i_h \rangle$ $(i < \theta)$ is the desired iteration with iteration maps π^{ij}_h.

We also get:

(d) $\pi^{ij}_h \pi^i_{l h} = \pi^j_{l h} \pi^{ij}_l$ $(i \leq j < \theta,\ l \leq h \leq \bar{\theta})$

Now set: $\bar{e}_{\bar{\theta}+i} = e^i_{\bar{\theta}}$;

$\langle \bar{W}_{\bar{\theta}+i}, \bar{U}_{\bar{\theta}+i}, \bar{U}_{\bar{\theta}+i} \rangle =$

$= \langle W^i_{\bar{\theta}}, U^i_{\bar{\theta}}, U^i_{\bar{\theta}} \rangle$;

$\sigma_i' = \pi^i_{0,\bar{\theta}}$;

$\bar{N}_{\bar{\theta}+i} = \langle J^{E^{\bar{W}_{\bar{\theta}+i}}}_{\bar{\nu}}, \bar{U}_{\bar{\theta}+i}, \bar{U}_{\bar{\theta}+i} \rangle$,

where $\bar{\nu} = \sigma_i(On \cap N_i)$.

Then $\langle \bar{W}_i, \bar{U}_i, \bar{U}_i \rangle$ $(i < \bar{\theta} + \theta)$ is the application of $\langle \bar{N}_i \mid i < \bar{\theta} + \theta \rangle$ to W with indices $\bar{e}_i$ $(i < \bar{\theta} + \theta)$.

Moreover, $\bar{e}_{\bar{\theta}+i} = \sigma'_i(e^i)$, where

(1) $\sigma'_0 = \bar{\bar{\pi}}_{0\theta}$

(2) $\sigma'_\lambda \pi^{i\lambda} = \bar{\bar{\pi}}_{\bar{\theta}+i,\bar{\theta}+\lambda}\sigma'_i$ (for $i \le \lambda$)

by (d).

(3) $\sigma'_{i+1}(\pi^{i,i+1}(f)(n^i)) =$

$= \pi_{\bar{\theta}+i,\bar{\theta}+i+1}\sigma'_i(f)(\bar{n}_{\bar{\theta}+i})$

To see (3), apply (d) and observe that:

$\sigma'_{i+1}(n^i) = \pi_{0\bar{\theta}}^{i+1}(n^i) = \pi^i_{i\bar{\theta}}(n^i) = n^i_{\bar{\theta}}$,

since $V^{W^i}_{n^i+1} = V^{W^{i+1}}_{n^i+1}$.

Thus $\langle \bar{N}_i \mid i < \bar{\theta}+\theta \rangle = \bar{y} * y$ with $\sigma_i = \sigma'_i \upharpoonright N_i$ as in the definition of $\bar{y} * y$, which proves the theorem.

It remains only to define the maps π_h^{ij} $(i \le j < \theta)$ and verify (a) — (d). We do this by ind. on h.

<u>Case 1</u> $h = 0$.

$\pi_0^{ij} = \pi^{ij}$. The verifications are trivial.

<u>Case 2</u> $h = l+1$. We define:

$\pi_h^{ij}(\pi_{lh}^{i}(f)(\bar{\kappa}_l)) = \pi_{lh}^{j}\pi_l^{ij}(f)(\bar{\kappa}_l)$.

To see that this is a def. & that π_h^{ij} is Σ_1-preserving, note that $\bar{\kappa}_l \le \kappa_l^i$, hence $\pi_l^{ij} \upharpoonright \bar{\kappa}_l = \mathrm{id}$ by the ind. hyp.

Hence: $W_h^i \models \varphi(\pi_{lh}^i(x), \bar{\kappa}_l) \longleftrightarrow$

$\longleftrightarrow W_l^i \models \varphi(x, \xi) \bmod E_{\bar{e}_l}$

$\longleftrightarrow W_l^j \models \varphi(\pi_l^{ij}(x), \xi) \bmod E_{\bar{e}_l}$

$\longleftrightarrow W_h^j \models \varphi(\pi_{lh}^j \pi_l^{ij}(x), \bar{\kappa}_l)$

(d) at h is immediate from the def. & the ind. hyp. We verify (a) — (c):

To prove (a) we first show:

(1) $\pi_h^{ii'} \upharpoonright \kappa_h^i = \mathrm{id}$.

Let $\xi = \pi_{lh}^i(f)(\bar{\kappa}_l) < \kappa_h^i$. We may assume w.l.o.g. that $f : \bar{\kappa}_l \to \kappa_l^i$. Then $f = \pi_l^{ii'}(f) \upharpoonright \kappa_l^i$ by the ind. hyp. Since $(V_{\kappa_l^i+1})^{W_l^i} = (V_{\kappa_l^i+1})^{W_l^{i'}}$, we also have: $\pi_{lh}^i(f) = \pi_{lh}^{i'}(f)$.

Hence:

$\pi_h^{ii'}(\xi) = \pi_h^{ii'}(\pi_{lh}^i(f)(\bar{\kappa}_l)) =$

$= \pi_{lh}^{i'}(f)(\bar{\kappa}_l) = \pi_{lh}^i(f)(\bar{\kappa}_l) = \xi$.

QED(1)

(2) $\pi_h^{ii'}(\kappa_h^i) > \kappa_h^i$,

since $\pi_h^{ii'}(\kappa_h^i) = \pi_h^{ii'}\pi_{lh}^i(\kappa_l^i) =$

$= \pi_{lh}^{i'}\pi_l^{ii'}(\kappa_l^i) > \pi_{lh}^{i'}(\kappa_l^{i'}) = \kappa_h^{i'}$,

since $\pi_l^{ii'}(\kappa_l^i) > \kappa_l^{i'}$ by the ind. hyp. This proves (a).

To prove (b), let $X = \pi^{i}_{lh}(f)(\bar{\kappa}_l) < \kappa^i_l$.

Note that:

(3) $\pi^{i}_{lh}(e^i_l, \kappa^i_l) = e^i_h, \kappa^i_h$

(4) $\pi^{i}_{lh}(r \cap E^{W^i_l}_{e^i_l}) = \pi^{i}_{lh}(r) \cap E^{W^i_h}_{e^i_h}$,

where $r = \mathrm{rng}(f)$.

(5) $\pi^{ii}_l \upharpoonright \bar{\kappa}_l = \mathrm{id}$, since $\bar{\kappa}_l \leq \kappa^i_l$.

Hence:

$X \in E^{W^i_h}_{e^i_h} \longleftrightarrow f(\xi) \in E^{W^i_l}_{e^i_l} \mod E_{\bar{e}_l}$ by (4)

$\longleftrightarrow \kappa^i_l \in \pi^{ii}_l(f)(\xi) \mod E_{e_l}$

by ind. hyp. + (5)

$$\longleftrightarrow \kappa^i_h \in \underbrace{\pi^{i}_{lh}\, \pi^{ii}_l(f)(\bar{\kappa}_l)}_{= \,\pi^{ii}_h(X)} \text{ by (3)}$$

QED (b)

To prove (c), let $x = \pi^{i}_{lh}(f)(\bar{\kappa}_l)$,

where $f = t(\pi^{oi}_l(x), \kappa^{i_1}_l, \ldots, \kappa^{i_m}_l)_{W^i_l}$

Then $\pi^{i}_{lh}(f) = t(\pi^{oi}_h(\pi^{o}_{lh}(x)), \kappa^{i_1}_h, \ldots, \kappa^{i_m}_h)_{W^i_h}$.

Moreover $\bar{\kappa}_l \in \mathrm{rng}(\pi^{oi}_h)$ since

$\bar{\kappa}_\ell < \bar{\kappa}_h < \kappa_h^0 = \mathrm{crit}(\pi_h^{0i})$. QED (c)

<u>Case 3</u> $h = \lambda$, $\lim(\lambda)$

Def. $\pi_\lambda^{ii'}$ by: $\pi_\lambda^{ii'} \pi_{\ell\lambda}^{i} = \pi_{\ell\lambda}^{i'} \pi_\ell^{ii'}$ $(\ell < \lambda)$.
The verification are straightforward.

QED (Lemma 7)

As a corollary of the proof:

<u>Cor 7.1</u> Let $\bar{\mathcal{Y}} = \langle \bar{N}_i \mid i \leq \bar{\theta} \rangle$, $\mathcal{Y} = \langle N_i \mid i \leq \theta \rangle$ be normal simple iteration of N. Apply $\mathcal{Y}$ to W to get $\langle W', U_\theta \rangle$ with it. map π. Apply $\bar{\mathcal{Y}}$ to W' to get $\langle \bar{W}, \bar{U}_{\bar{\theta}} \rangle$ with map $\bar{\pi}$. Set $U' = \bigcup_{x \in N_\theta} \bar{\pi}(x \cap U_\theta)$. Then $\langle \bar{W}, U' \rangle$ is the result of applying $\bar{\mathcal{Y}} * \mathcal{Y}$ to W with it. map $\bar{\pi}\pi$.

<u>Lemma 8</u> Let $N = \langle J_\nu^E, U, U \rangle$ be a double beaver. Let $\mathcal{Y} = \langle N_i \mid i < \theta \rangle$ be applicable, where $\lim(\theta)$. Then

(a) $N_\theta = \lim_i N_i$ is well founded

(b) $\mathcal{Y}^* = \langle N_i \mid i \leq \theta \rangle$ is applicable.

proof of Lemma 8.

Let W be universal s.t. $J_\gamma^{E^W} = J_\gamma^E$.

Assume w.l.o.g. that $E_\gamma^W = \emptyset$ and that X is massive in W.

Let $\langle W_i, U_i, \nu_i \rangle$ $(i < \theta)$ be the application of $\mathcal{I}$ to W with iteration maps π_{ij}. We shall use the proof methods of Lemmas 5, 6. Set:

$M = \langle J_\tau^{E^M}, \emptyset \rangle$, where

$$J_\tau^{E^M} = \bigcup_i J_{\kappa_i}^{E^{W_i}} = \bigcup_i J_{\nu_i}^{E^{W_i}}$$

Then M is a strong mouse by the sequence lemma. Hence the coiteration of M, W must terminate. Coiterate M, W to M', W'. Then M' is a simple iterate of M. But W' is a simple iterate of W by the strongness of M.

(Let $\tilde{W} | \nu = M$ with $\tilde{W}$ universal. An initial part of the coiteration of $\tilde{W}, W$ yields $\tilde{W}', W'$. But no truncation can occur).

Set: $\kappa_i' = \pi_{MM'}(\kappa_i)$.

Coiterate W', W_i to W_i' $(i < \theta)$.

Since $W_i | \kappa_i = M | \kappa_i$, an initial part of this coiteration recapitulates the coiteration of $W' | \kappa_i'$, M, hence of $M' | \kappa_i'$, M and we have:

(1) $\pi_{W_i W_i'} \restriction \kappa_i = \pi_{MM'} \restriction \kappa_i$.

Since $M' | \kappa_i' = W' | \kappa_i'$ is not moved in this coiteration, we have:

(2) $\pi_{W' W_i'} \restriction \kappa_i' = \mathrm{id}$.

Now set: $\tilde{\kappa}_i = \pi_{0i}(\kappa)$, $\tilde{\nu}_i = \pi_{0i}(\nu)$

$\tilde{\kappa}_i' = \pi_{W_i W_i'}(\tilde{\kappa}_i)$, $\tilde{\nu}_i' = \pi_{W_i W_i'}(\tilde{\nu}_i)$.

(3) $\pi_{W W_i'} \restriction \kappa = \pi_{W_i W_i'} \pi_{0i} \restriction \kappa$ $(i < \theta)$

proof.

If $\kappa_h < \tilde{\kappa}_h$ for $h < i$, then $\pi_{0i} = \pi_{W W_i}$ is a simple normal iteration & hence $\pi_{W W_i'} = \pi_{W_i W_i'} \pi_{0i}$.

Otherwise let $i_0 < i$ be least s.t. $\kappa_{i_0} = \tilde{\kappa}_{i_0}$.

Then $\pi_{WW'} \upharpoonright \kappa = \pi_{0i_0} \upharpoonright \kappa = \pi_{0i} \upharpoonright \kappa$,

since $\pi_{i_0 i} \upharpoonright \kappa_{i_0} = \mathrm{id}$. But

$\pi_{W_i W_i'} \upharpoonright \kappa_i = \pi_{MM'} \upharpoonright \kappa_i$ and

$\pi_{MM'} \upharpoonright \kappa_{i_0} = \mathrm{id}$, since M is not moved in the first i_0 steps of the coiteration of W, M.

QED (3)

Set:

$$X^* = \{\alpha \in X \mid \alpha = \pi_{0i}(\alpha) = \pi_{WW_i'}(\alpha) \text{ for all } i < \theta\}.$$

Then X is massive in W_i, W_i'.

Set:

$$Y_i' = \text{the } \Sigma_1(W') \text{ - closure of}$$
$$X^* \cup \pi_{WW'}{}''\kappa \cup \{\kappa_h' \mid h < i\} \quad (i \leq \theta)$$

For $\vec{\alpha} \in X^*$, $\vec{\xi} < \kappa$, $\vec{\eta} < \kappa_i$ we have:

$$W' \models \varphi(\vec{\alpha}, \pi_{WW'}(\vec{\xi}), \pi_{MM'}(\vec{\eta})) \longleftrightarrow$$
$$W_i' \models \varphi(\vec{\alpha}, \pi_{WW_i'}(\vec{\xi}), \pi_{W_i W_i'}(\vec{\eta})) \longleftrightarrow$$
$$W_i \models \varphi(\vec{\alpha}, \pi_{0i}(\vec{\xi}), \vec{\eta}) \text{ for } i < \theta$$

by (1), (2), (3). Hence for $i < \theta$:

(4) $Y_i' \simeq Y_i =_{Df}$ the $\Sigma_1(W_i)$ - closure of $X^* \cup \pi_{0i}''\kappa \cup \{\kappa_h \mid h < i\}$.

For $i \leq j \leq \theta$ set:

(5) $\bar\pi_i : \bar W_i \overset{\sim}{\longleftrightarrow} Y_i'$ with $\bar W_i$ transitive,

$\bar\pi_{ij} = \bar\pi_j^{-1}\,\bar\pi_i$.

Then $\bar W_i = J_\infty^{E^i}$ is a universal weasel.

By (4) there is:

(6) $\tilde\pi_i : \bar W_i \overset{\sim}{\longleftrightarrow} Y_i$ s.t.

$\bar\pi_{ij} = \tilde\pi_j^{-1}\,\pi_{ij}\,\tilde\pi_i$.

Clearly the $\bar\pi_{ij}$ commute and are continuous at limits. Set:

$\delta' =$ the least $\delta \in Y_0'$ s.t. $\delta \geq \tilde\kappa_0'$

$\delta_i =$ " " $\delta \in Y_i$ s.t. $\delta \geq \tilde\kappa_i$.

But then for $\vec\alpha \in X^*$ we have:

$$W \models \bigwedge \vec\xi < \kappa\,(t(\vec\alpha, \vec\xi) < \delta_0 \rightarrow t(\vec\alpha, \vec\xi) < \kappa),$$

Hence:

$$W_i \models \bigwedge \vec\xi < \tilde\kappa_i\,(t(\vec\alpha, \vec\xi) < \pi_{0i}(\delta_0) \rightarrow$$
$$\rightarrow t(\vec\alpha, \vec\xi) < \tilde\kappa_i),$$

where $\pi_{0i}(\delta_0) \in Y_i$ (since $\pi_{0i}{}''Y_0 \subset Y_i$ by the def of Y_i). Hence:

(7) $\pi_{0i}(\delta_0) = \delta_i \quad (i < \theta)$

Since $\kappa \subset Y_0$ we have:

(8) $J_\nu^{E^{\bar{W}_0}} = J_\nu^E$,

since the models coincide up to κ and $\nu = \kappa^{+W}$. (Thus $\neq$ would mean a truncation in the coiteration of $W, \bar{W}_0$). Clearly:

(9) $\tilde{\pi}_0^{-1}(\delta_0) = \bar{\pi}^{-1}(\delta') = \kappa$.

Set: $\nu_i^* = \delta_i^{+W_i}$; $\sigma = \sigma_0 = \bar{\pi} \upharpoonright J_\nu^E$.

Then:

(10) $\sigma : J_\nu^E \to_{\Sigma_1} J_{\nu_0^*}^E$.

Since in W:

$\bigwedge \vec{\xi} < \kappa \, (J_\nu^{E^0} \models \varphi(\vec{\xi}, \vec{\eta}) \longleftrightarrow J_{\nu_0^*}^{E^0} \models \varphi(\xi, \sigma(\eta$

for $\vec{\eta} < \nu$, we have in W_i:

$\bigwedge \vec{\xi} < \tilde{\kappa}_i \, (J_{\tilde{\nu}_i}^{E^i} \models \varphi(\vec{\xi}, \pi_{0i}(\vec{\eta})) \longleftrightarrow$

$\longleftrightarrow J_{\nu_i^*}^{E^i} \models \varphi(\vec{\xi}, \pi_{0i}\sigma(\vec{\eta}))$.

Hence we may define

$$\sigma_i : J^{E^i}_{\tilde{\nu}_i} \longrightarrow_{\Sigma_1} J^{E^i}_{\nu_i^*}$$

by $\sigma_i(t(\pi_{0i}(\vec{\eta}), \vec{\kappa}_h)_{J^{E^i}_{\tilde{\nu}_i}}) =$

$$= t(\pi_{0i}\sigma(\vec{\eta}), \vec{\kappa}_h)_{J^{E^i}_{\nu_i^*}}$$

for $\vec{\eta} < \nu$, $\vec{h} < i$.

Repeating (11) in the proof of Lemma 6 we get:

(11) $\sigma_i = \tilde{\pi}_i \upharpoonright J^{E^i}_{\tilde{\nu}_i}$.

Hence, as in (12) of Lemma 6:

(12) $\overline{\pi}_{ij} \upharpoonright \tilde{\nu}_i = \pi_{ij} \upharpoonright \tilde{\nu}_i$

It follows then exactly as in Lemma 5 that $\langle \overline{W}_i, U_i, \nu_i \rangle$ $(i < \theta)$ is the application of $\mathcal{Y}$ to $\overline{W}_0$

But $\overline{W}_\theta = \lim\limits_{i<\theta} \overline{W}_i$.

QED (Lemma 8)

<u>Cor 9.1</u> Let N be a double beaver. Then N is normally, simply iterable and any such iteration is applicable.

prf. Suppose not.

There is a least θ s.t. an iteration $\mathcal{J} = \langle N_i \mid i < \theta \rangle$ is either non applicable or non continuable.

Then $\theta = \delta + 1$ by Lemma 8.

Let W be universal s.t. $J_\nu^{E^N} = J_\nu^{E^W}$ ($N = \langle J_\nu^E, U_0, U_1 \rangle$) and let $\langle W', U_0', U_1' \rangle$ be the result of applying $\mathcal{J}$ to W. We claim that W' is extendible by $E_e^{\langle W', U_0', U_1' \rangle}$ for all indices e. Only $e = \langle 0, h \rangle$ ($h = 0, 1$) is problematic. We observe that by Lemma 7, $\mathcal{J} * \mathcal{J}'$ is applicable, where $\mathcal{J}'$ is the 1-step iteration of N with index $\langle 0, h \rangle$.

QED (Cor 9.1)

Cor 9.2 Let N be a double beaver. Then N is $*$-iterable.

proof

By Cor 9.1 there is a normal simple iterate N' of N all of whose measures are ω-complete. Hence N' is $*$-iterable. Hence so is N since $\pi : N \to_{\Sigma^*} N'$, where π is the iteration map.

QED (Cor 9.2)

By the bicephalus lemma we conclude:

Cor 9.3 Let $N = \langle J_\nu^E, U, U' \rangle$ be a double beaver. Then $U = U'$.

Thus we, in fact, need only consider single beavers. The previous lemmas translate straightforwardly into lemmas on single beavers.

Cor 9.4 Let $N = \langle J^E_\nu, U \rangle$ be a beaver. Let W be universal s.t. $J^{E^W}_\nu = J^E_\nu$. Then $\langle W, U \rangle$ is $\forall$-iterable.

prf.

Case 1 $E^W_\nu \neq \emptyset$.

Then $W|\nu$ is a beaver & hence $E^W_\nu = U$ by Cor 9.3. But W is iterable.

Case 2 Case 1 fails.

Let $\mathcal{I}$ be a normal simple it. of N to N' in which all measures are ω-complete. Let $\langle W', U' \rangle$ be the result of applying N' to W'. Let W'' be a result of iterating above ν' ($N' = \langle J^{E}_{\nu'}, U' \rangle$) so as to make all remaining measures ω-complete. Then all measures in $\langle W'', U' \rangle$ are ω-complete. The usual proofs show that $\langle W'', U' \rangle$ - and with it $\langle W, U \rangle$ - is iterable. QED (Cor 9.4)

The following lemma establishes the uniqueness property needed to define K:

<u>Lemma 10</u> Let $\langle J^{E}_{\nu}, U \rangle, \langle J^{E}_{\nu}, U' \rangle$ be strong s.t. $U, U' \neq \emptyset$. Then $U = U'$.

proof.

Suppose not. Let W be universal s.t. $W|\nu = \langle J^{E}_{\nu}, U \rangle$,

<u>Case 1</u> ν is a cardinal in W.

Then $\langle J_\nu^E, U\rangle$ is a beaver. If W' is universal s.t. $W'|\nu = \langle J_\nu^E, U'\rangle$ then a comparison of W, W' shows that ν is a card. in W' + hence $\langle J_\nu^E, U'\rangle$ is a beaver. Hence $U = U'$ by Cor 4.3.

<u>Case 2</u> Case 1 fails.

There are mice M, M' extending $\langle J_\nu^E, U\rangle, \langle J_\nu^E, U'\rangle$ resp. s.t. $\rho_M^\omega, \rho_{M'}^\omega \leq \kappa$, where U, U' are measures on κ, and ν is a cardinal in M, M'. Coiterate M, M' to $\tilde{M}, \tilde{M}'$.

<u>Claim</u> Both sides of the coiteration are simple.

Suppose not. Let $\tilde{M}$ be simple + $\tilde{M}'$ non simple. $\tilde{M}$ is not sound, since we iterated by $U = E_\nu^{\tilde{M}}$ at the first step. Hence $\tilde{M}$ is not a proper initial segment of $\tilde{M}'$.

Hence $\tilde{M} = \tilde{M}'$. Hence $\rho^\omega_{\tilde{M}'} \leq \kappa$. Hence κ is not a cardinal in M'.

Contr! QED (Claim)

But then $\rho^\omega_{\tilde{M}}, \rho^\omega_{\tilde{M}'} \leq \kappa$. Hence neither is a proper segment of the other and $\tilde{M} = \tilde{M}'$. Hence $\mathrm{core}(M) = \mathrm{core}(\tilde{M}) = \mathrm{core}(\tilde{M}') = \mathrm{core}(M')$.

Hence $\langle J^E_\nu, U\rangle = \mathrm{core}(M) | \nu =$
$\dot{=} \mathrm{core}(M') | \nu = \langle J^{E'}_\nu, U'\rangle$.

QED (Lemma 10)

<u>Cor 10.1</u> K exists.

In order to develop fully the properties of K we shall need the following lemma:

<u>Lemma 11</u> Let $N = \langle J_\nu^E, U \rangle$ be a beaver. Then N is strong.

proof.

N is a mouse by Lemma 9.2. Let W be universal and let $J_\nu^{E^W} = J_\nu^E$. Assume $E_\nu^W = \emptyset$, since otherwise $E_\nu^W = U$ and there is nothing to prove. We also assume w.l.o.g. that there is $X \subset On$ which is massive in W. Let $\langle N_i \mid i \leq \bar{\theta} \rangle$, $\langle W_i \mid i \leq \bar{\theta} \rangle$ be the coiteration of N, W with indices ν_i, κ_i. Then the N-side of the coiteration is simple. But then so is the W-side, since otherwise there would be a least i with $\nu_i < \kappa_i^{+W_i}$. But there is a universal W' with $J_{\nu_i}^{E^{W'}} = J_{\nu_i}^{E^{N_i}}$, hence $\nu_i = \kappa^{+W'}$, since N_i is a beaver. Hence the coiteration of W', W_i involves a truncation. Contradiction! Let $\theta > \bar{\theta}$ be regular and continue the

iteration to $\langle N_i \mid i \leq \theta \rangle$, $\langle W_i \mid i \leq \theta \rangle$ with indices u_i, ν_i defined by:
$\nu_i = On \wedge N_i$ (hence $\nu_i = \pi_{W_{\bar\theta} W_i}(\nu_{\bar\theta})$)
for $i \geq \bar\theta$. Now let $\langle \tilde{W}_i, U_i \rangle$ $(i < \theta)$ be the application of $\langle N_i \mid i < \theta \rangle$ to W_θ with iteration maps $\tilde{\pi}_{ij}$.

Note that

(1) $\tilde{\pi}_{0i}(\theta) = \theta$ for $i < \theta$,

since θ is regular.

Coiterate $\tilde{W}_i$, W_θ to W_i^*.

<u>Claim</u> $\pi_{W_\theta W_i^*}(\theta) = \pi_{\tilde{W}_i W_i^*}(\theta) = \theta$.

proof of Claim.
Set $\tilde{N}_i = \tilde{W}_i \mid \theta^+ = \tilde{\pi}_{0i}(N_\theta)$ $(\theta^+ =_{\mathrm{Df}} \theta^{+\tilde{W}_i})$

(2) $\tilde{N}_i$ is a simple iterate of N

proof
Let W' be obtained by applying $\mathcal{J} = \langle N_i \mid i < \theta \rangle$ to W. Then W' coincides with W_θ up to $\nu_\theta =_{\mathrm{Df}} On \wedge N_\theta = \theta^{+W_\theta}$. Let W'' be obtained by applying

$\mathcal{I}_i = \langle N_h \mid h \le i \rangle$ to W' with iteration map π'. Then $\tilde{N}_i = \langle J^{E^{W''}}_{\nu'}, U'' \rangle$ where $\nu' = \pi'(\nu_\theta)$ and $U'' = \bigcup_{x \in U_\theta} \pi'(x \cap U_\theta)$. But by Cor 7.1 we have: $\langle W'', U'' \rangle$ is the result of applying $\mathcal{I}_i * \mathcal{I}$ to W and $\tilde{N}_i$ is the final iterate of $\mathcal{I}_i * \mathcal{I}$. QED(2)

Hence:

(3) $\tilde{N}_i, N_\theta$ have a common simple iterate.

Now let $\langle W^0_j \mid j \le \Gamma \rangle$, $\langle W^1_j \mid j \le \Gamma \rangle$ ($\Gamma \le \infty$) be the coiteration of $W^0 = W_\theta$, $W^1 = \tilde{W}_i$ with indices κ^*_i, ν^*_i. Then $W^*_i = W^0_\Gamma = W^1_\Gamma$ is a common simple iterate of W^0, W^1.

Let $\gamma \le \Gamma$ be maximal s.t. $\kappa^*_i < \theta$ for all $i < \gamma$. Set:

$N^0 = N_\theta$, $N^1 = \tilde{N}_i$, $N^h_i = \pi_{W^h_0 W^h_i}(N^h)$

for $i \leq \gamma$, $h = 0, 1$. Then $\langle N_i^h \mid i \leq \gamma \rangle$ $(h = 0, 1)$ is an initial part of the coiteration of N^0, N^1. Now set: $Q^h = N^h \mid \theta$; $Q_i^h = N_i^h \mid \theta$ $(h = 0, 1, i < \gamma)$. Then $\langle Q_i^h \mid i < \gamma \rangle$ is an initial part of the coiteration of Q^0, Q^1.

Clearly:

(2) $W^0 \mid \theta = W^1 \mid \theta$;

hence:

(3) $J_{\nu^*}^{E^{W^0}} = J_{\nu^*}^{E^{W^1_\gamma}}$ where

$\nu^* = \theta^{+W^0_\gamma} = \theta^{+W^1_\gamma}$,

since otherwise the coiteration of W^0_γ, W^1_γ would involve a truncation.

Since θ is regular and $\overline{\overline{f(\tau) \cap Q^h}} < \theta$ for $\tau < \theta$, the structure $\langle H_\theta, Q^0, Q^1 \rangle$ satisfies enough of ZFC that we can carry out the proof of §3.1 Lemma 1.4 to show that for at least one $h =$, we have:

(4) $\pi_{Q^h Q^h_i}{}'' \lambda \subset \lambda$ for $\lambda \leq i < \gamma$

for a cub set of $\lambda < \theta$;

hence:

(5) $\pi_{W^h W^h_\gamma}(\theta) = \theta$.

Now suppose that (4) fails for some $h = 0, 1$. Then by the proof of §3.1 Lemma 1.4 we have:

(6) There is a cub $C \subset \theta$ s.t. $\pi_{Q^h_i Q^h_j}(\kappa_i) = \kappa_j$ for $i \leq j$, $i, j \in C$;

hence:

(7) $\pi_{W^h_i W^h_\gamma}(\kappa_i) = \theta$ for $i \in C$.

In either case we have

(8) $E^{W^h_\gamma}_{\nu^*} \neq \emptyset$.

Hence by (3) and Cor 9.3 we conclude:

(9) $W^0_\gamma | \nu^* = W^1_\gamma | \nu^*$.

Hence $\nu_\gamma > \nu^*$ and

(10) $\pi_{W^h, W^h_\gamma}(\theta) = \pi_{W^h, W_\Gamma}(\theta)$.

But then (7) cannot hold, since we would then have: $N^{1-h}_\gamma = W^{1-h}_\gamma | \nu^*$ is a proper initial segment of N^h_γ, contradicting (3). Thus (5) holds for $h = 0, 1$ and hence by (10) we have: $\pi_{W^h, W^h_\Gamma}(\theta) = \theta$.

QED (Claim)

Recall that X is massive in W. Set:

(11) $X^* =$ the set of $\alpha \in X$ s.t.

$$\alpha = \pi_{W W^*_i}(\alpha) = \pi_{\tilde{W}_i W^*_i} \tilde{\pi}_{0i} \pi_{W W_\theta}(\alpha) \text{ for } i < \theta.$$

[<u>Note</u> We do <u>not</u> know that $\pi_{\tilde{W}_i W^*_i} \tilde{\pi}_{0i} \pi_{W W_\theta} = \pi_{W W^*_i}$].

Then X^* is massive in W, W_θ, $\tilde{W}_i$ and W_i^* for $i < \theta$. Set:

$$Y = \text{the } \Sigma_1(W_\theta)\text{-closure of } X^* \cup \kappa \cup \{\theta\}$$

(κ = the largest cardinal in $N = \langle J_\nu^E, u\rangle$)

(12) $Y \cap [\kappa, \theta) = \emptyset$

prf.

Suppose not. Let $\xi = t^{W_\theta}(\vec{\alpha}, \vec{\gamma}, \theta) \in [\kappa, \theta)$ where $\vec{\alpha} \in X^*$, $\vec{\gamma} < \kappa$.

Pick $i \in [\bar{\theta}, \theta)$ s.t. $\xi < \kappa_i$. Then

$$W_i^* \models \xi = t(\vec{\alpha}, \vec{\gamma}, \theta),$$

since $\pi_{\bar{W}_\theta W_i^*}(\xi, \vec{\alpha}, \vec{\gamma}, \theta) = \xi, \vec{\alpha}, \vec{\gamma}, \theta$.

But then

$$\tilde{W}_i \models \xi = t(\vec{\alpha}, \vec{\gamma}, \theta), \text{ since}$$

$$\pi_{\tilde{W}_i W_i^*}(\xi, \vec{\alpha}, \vec{\gamma}, \theta) = \xi, \vec{\alpha}, \vec{\gamma}, \theta.$$

Hence $\tilde{W}_i \models t(\vec{\alpha}, \vec{\gamma}, \theta) < \kappa_i$.

Hence $W_\theta \models t(\vec{\alpha}, \vec{\gamma}, \theta) < \kappa$,

since:

$\tilde{\pi}_{0i}(\kappa, \vec{\alpha}, \vec{\gamma}, \theta) = \kappa_i, \vec{\alpha}, \vec{\gamma}, \theta.$

Hence $\xi = t^{W_\theta}(\vec{\alpha}, \vec{\gamma}, \theta) < \kappa$. Contr!

QED(12).

Now let $\sigma : \hat{W} \overset{\sim}{\longleftrightarrow} Y$ where $\hat{W}$ is transitive. Then $\hat{W}$ is universal and $\hat{W}|\kappa = W|\kappa$, hence

(13) $\nu = \kappa^{+\hat{W}}$ and $J_\nu^{E^{\hat{W}}} = J_\nu^{E^W}$,

since otherwise the coiteration of $\hat{W}$, W would involve a truncation. Hence by Cor 9.3

(14) $\hat{W}|\nu = N$.

Thus N is strong.

QED(Lemma 11)