§ 3.7 Properties of K

We continue to assume $\neg O^{\#}$,

<u>Lemma 1</u> Let W be universal and let $\sigma : W \to_{\Sigma_1} K$. Then $\sigma = \mathrm{id}$.

proof. Suppose not.
Let $\kappa = \mathrm{crit}(\sigma)$. Set:
$$U = \{X \in \mathbb{P}(\kappa) \cap W \mid \kappa \in \sigma(X)\}.$$
Then $N = \langle J_\nu^{E^W}, U \rangle$ is a beaver, hence strong, where $\nu = \kappa^{+W}$.
Hence $E_\nu^K = U$. Since $J_\nu^W = J_\nu^N$ and N is a mouse, we have:
If $\pi : W \to_U W'$, then
$E_\nu^{W'} = E_\nu^{N'} = \emptyset$, where $\pi \restriction N : N \to_U N'$.
Clearly there is $\sigma : W' \to_{\Sigma_1} K$ s.t. $\sigma(\nu) = \nu$. Hence $E_\nu^K = \emptyset$.
Contradiction! QED (Lemma 1)

Cor 1.1 K is rigid

Cor 1.2 Let $\sigma : K \to_{\Sigma_1} W$. Let $Z \prec W$, $Z \subset \mathrm{rng}(\sigma)$ s.t. $\bar{K}$ is universal where $\bar{K} \simeq Z$ is transitive. Then $\bar{K} = K$ and $Z = \mathrm{rng}(\sigma)$.

prf.

Let $\pi : \bar{K} \overset{\sim}{\longleftrightarrow} Z$. Then

$\sigma^{-1}\pi : \bar{K} \to_{\Sigma_1} K$, $\bar{K}$ universal.

Hence $\sigma^{-1}\pi = \mathrm{id}$. QED (Cor 1.2)

We can improve Lemma 1 to:

Lemma 2 Let α be a cardinal in K s.t. $W' | \alpha = K | \alpha$. Let $\sigma : W \to_{\Sigma_1} W'$ where W is universal. Then $\sigma \upharpoonright \alpha = \mathrm{id}$.

The proof is exactly like that of Lemma 1.

Repeating the proof of Cor 1.2:

<u>Cor 2.1</u> Let W', α be as above and let $\sigma : W' \longrightarrow_{\Sigma_1} W$. Let $Z \prec W$, $Z \subset \mathrm{rng}(\sigma)$ s.t. $\overline{W}$ is universal where $\pi : \overline{W} \overset{\sim}{\longleftrightarrow} Z$. Then $J_\alpha^{E^{\overline{W}}} = J_\alpha^{E^K}$ and $\sigma \restriction \alpha = \pi \restriction \alpha$.

Now let $E = \langle E_\nu \mid \nu < \delta \rangle$ be s.t. $\langle J_\nu^E, E_\nu \rangle$ is a strong mouse for $\nu < \delta$. Define $K' = K[E]$ by setting $E_\nu^{K'} = E_\nu$ for $\nu < \delta$ and defining $E_\nu^{K'}$ as in the definition of K for $\nu \geq \delta$. (The uniqueness thm for strong mice + the sequence lemma again tell us that K' is defined). Repeating the above proofs we get versions of the above lemmas for $K[E]$:

<u>Lemma 3</u> Let $K' = K[E]$ be as above and let $\alpha \geq \delta$ be a cardinal in K' s.t. $W' | \alpha = K' | \alpha$. Let $\sigma : W \xrightarrow[\Sigma_1]{} W'$ where W is universal and $\sigma \upharpoonright \delta = \mathrm{id}$. Then $\sigma \upharpoonright \alpha = \mathrm{id}$.

<u>Lemma 3.1</u> Let K', W', α be as above and let $\sigma : W' \xrightarrow[\Sigma_1]{} W$. Let $Z \prec W$, $Z \subset \mathrm{rng}(\sigma)$ s.t. $\sigma''\delta \subset Z'$. Let $\pi : \overline{W} \longleftrightarrow Z$, where $\overline{W}$ is universal. Then $J_\alpha^{E^{\overline{W}}} = J_\alpha^{E^{K'}}$ and $\sigma \upharpoonright \alpha = \pi \upharpoonright \alpha$.

We are there in proving:

<u>Lemma 4</u> Let K' be a simple iterate of K by an iteration with normal indices $\langle \nu_i \mid i < \theta \rangle$ $(\theta < \infty)$. Set:

$\alpha = \mathrm{lub}_{i<\theta}\, \nu_i$; $E = \langle E_\xi^{K'} \mid \xi < \alpha \rangle$.

Then $K' = K[E]$.

proof of Lemma 4

Let $K^* = K[E]$. Coiterate K, K^* to $\tilde{K}$ with coiteration indices $\langle \tilde{\nu}_i \mid i < \Gamma \rangle$ $(\Gamma \leq \infty)$. Then

(1) $\tilde{\nu}_i = \nu_i$ and $E^{K^*}_{\nu_i} = \emptyset$ for $i < \theta \leq \Gamma$.

Hence:

(2) $K' = K_\theta$ and $\pi_{K'K} \upharpoonright \alpha = \mathrm{id}$.

Pick a limit cardinal $\beta > \alpha$ s.t.

(3) $\sup \pi_{K\tilde{K}}{}''\beta = \sup \pi_{K^*\tilde{K}}{}''\beta = \beta$ and $\nu_i < \beta$ for $i < \Gamma \cap \beta$.

Pick W, X, W^*, X^* s.t.

(4) W is universal, X is massive in W, $J^{E^W}_\beta = J^{E^K}_\beta$.

(5) W^* is universal, X^* is massive in W^*, $J^{E^{W^*}}_\beta = J^{E^{K^*}}_\beta$.

Coiterate W, W^* to $\hat{W}$ with indices $\langle \hat{\nu}_i \mid i \leq \hat{\Gamma} \rangle$ $(\hat{\Gamma} \leq \infty)$. Then

(6) $\hat{\nu}_i \simeq \tilde{\nu}_i$ for $i < \beta$

and hence:

(7) $\pi_{K\tilde{K}} \upharpoonright \beta = \pi_{W\hat{W}} \upharpoonright \beta$, $\pi_{K^*\tilde{K}} \upharpoonright \beta = \pi_{W^*\hat{W}} \upharpoonright \beta$.

Set:

$Z = \{\alpha \in X \cap X^* \mid \pi_{W\hat{W}}(\alpha) = \pi_{W^*\hat{W}}(\alpha) = \alpha\}$.

Then Z is massive in $W, W^*, \hat{W}$.

Set:

(8) Y = the $\Sigma_1(\hat{W})$-closure of Z

(9) Y^* = " " - " " $Z \cup \alpha$.

Then $Y \prec \hat{W}$, $Y \subset \operatorname{rng}(\pi_{W\hat{W}})$. Set:

$\sigma : \bar{W} \stackrel{\sim}{\longleftrightarrow} Y$, $\bar{W}$ transitive.

Then $\bar{W}$ is universal + hence by Lemma 2.1 + (7):

(10) $\sigma \upharpoonright \beta = \pi_{W\hat{W}} \upharpoonright \beta = \pi_{K\tilde{K}} \upharpoonright \beta$

But $Y^* \prec \hat{W}$, $\alpha \subset Y \subset \operatorname{rng}(\pi_{W\hat{W}})$.

Setting: $\sigma^* : \bar{W}^* \stackrel{\sim}{\longleftrightarrow} Y$, $\bar{W}^*$ trans,

we conclude by Lemma 3.1 + (7):

(11) $\sigma^* \upharpoonright \beta = \pi_{W^*\hat{W}} \upharpoonright \beta = \pi_{K^*\tilde{K}} \upharpoonright \beta$.

We now show:

(12) $\pi_{K^*\tilde{K}} \upharpoonright \beta = \pi_{K'\tilde{K}} \upharpoonright \beta$

pf. of (12)

Note first that $Y \cap J_\beta^{E^{\tilde{\kappa}}} =$
$= \mathrm{rng}(\pi_{\kappa\tilde{\kappa}} \upharpoonright J_\beta^{E^\kappa})$ and
$Y^* \cap J_\beta^{E^{\tilde{\kappa}}} = \mathrm{rng}(\pi_{\kappa^*\tilde{\kappa}} \upharpoonright J_\beta^{E^{\kappa^*}})$
by (7) and (3). Since $J_\beta^{E^{\kappa'}} =$
$=$ the $\Sigma_1(J_\beta^{E^{\kappa'}})$-closure of
$\mathrm{rng}(\pi_{\kappa'\tilde{\kappa}} \upharpoonright \beta) \cup \alpha$, and $\pi_{\kappa'\kappa} \upharpoonright \alpha = \mathrm{id}$,
it suffices to show:

<u>Claim</u> $Y^* \cap J_\beta^{E^{\tilde{\kappa}}} =$ the $\Sigma_1(J_\beta^{E^{\tilde{\kappa}}})$-
-closure of $(Y \cap J_\beta^{E^{\tilde{\kappa}}}) \cup \alpha$.

$(\supset)$ is obvious.

$(\subset)$ Let $x = t(\vec{\gamma}, \vec{\xi})_{\hat{W}} \in J_\beta^{E^{\hat{W}}}$,
where $\vec{\gamma} \in Z$, $\vec{\xi} < \alpha$. Pick
$\eta \in \mathrm{rng}(\pi_{\kappa\tilde{\kappa}}) \cap \beta$ s.t.
$x \in V_\eta^{\hat{W}}$. Define g on α^n
by $g(\vec{x}) \simeq t(\vec{\gamma}; \vec{x})_{\hat{W}} \cap V_\eta^{\hat{W}}$.
Then $g \in Y \cap J_\beta^{E^{\tilde{\kappa}}}$ and hence
$g(\vec{\xi}) = x$ is as desired. QED(12)

By (12) we then have:

(13) $\quad J_\beta^{E^{\tilde{K}}} = J_\beta^{E^{K'}}$.

But this holds for arbitrarily large β.

QED (Lemma 4)

<u>Corollary 5</u> Every weakly universal weasel is a simple iterate of K.

prf.

Let W be weakly universal. Coiterate K, W to W'. <u>Claim</u> $W = W'$.

Let $\langle K_i \mid i \leq \Gamma \rangle$, $\langle W_i \mid i \leq \Gamma \rangle$ be the coiteration ($\Gamma \leq \infty$) with indices ν_i. We show:

<u>Claim</u> $E_{\nu_i}^{W_i} = \emptyset$

Suppose not. Then $E_{\nu_i}^{K_i} \neq \emptyset$ by Lemma 4, since $J_{\nu_i}^{E^{K_i}} = J_{\nu_i}^{E^{W_i}}$ and $W_i \| \nu_i$ is strong. But Then $E_{\nu_i}^{K_i} = E_{\nu_i}^{W_i}$ by §3.6, Cor 9.3.

Contd! QED (Cor 5)

We now prove the full rigidity lemma:

<u>Corollary 6</u> Let $\sigma : K \to_{\Sigma_1} W$. Then W is a simple iterate of K and $\sigma = \pi_{KW}$.

proof.

We construct a normal iteration K_i ($i \leq \Gamma \leq \infty$) of K with indices ν_i, κ_i. Simultaneously we construct maps $\sigma_i : K_i \to_{\Sigma_1} W$ s.t.

(a) $\sigma_i \pi_{K_h K_i} = \sigma_h$ for $h \leq i$

(b) $\sigma_i \upharpoonright \nu_h = \mathrm{id}$ for $h < i$

We define K_i, σ_i and verify (a), (b) by ind. on i as follows:

$K_0 = K$, $\sigma_0 = \sigma$.

Now let K_i, σ_i be given. K_{i+1} is defined iff $\sigma_i \neq \mathrm{id}$.

In this case we set:

$$\kappa_i = \mathrm{crit}(\sigma_i)\ ,\quad \nu_i = \nu^{+K_i}\ .$$

Then $\kappa_i > \nu_h$ for $h < i$ by (b). Set: $U = \{X \in \mathcal{P}(\kappa_i) \cap K_i \mid \kappa_i \in \sigma_i(X)\}$. Then $\langle J_{\nu_i}^{E^{K_i}}, U\rangle$ is a beaver + hence $U = E_{\nu_i}^{K_i}$ by Lemma 4. Thus K_{i+1} is defined and we def. σ_{i+1} by:

$$\sigma_{i+1}(\pi_{K_i K_{i+1}}(f)(\kappa_i)) = \sigma_i(f)(\kappa_i).$$

The verification of (a), (b) is straightforward.

Now let K_i, σ_i be given for $i < \lambda$, $\mathrm{Lim}(\lambda)$, $\lambda \leq \infty$. We of course let $K_\lambda, \langle \pi_{K_i K_\lambda} \mid i < \lambda\rangle$ be the transitive direct limit of $\langle K_i\rangle, \langle \pi_{K_i K_j} \mid i \leq j < \lambda\rangle$ and define σ_λ by:

$$\sigma_\lambda \pi_{K_i K_\lambda} = \sigma_i \quad \text{for } i < \lambda.$$

(a) is immediate and (b) follows

by the fact that (a), (b) hold at $i < \lambda$. However, if $\lambda = \infty$, we must show that the above transitive direct limit exists - i.e. that:

(1) $\bigwedge \xi \bigvee i \ \pi_{\kappa_0 \kappa_i}(\xi) < \kappa_i$.

It suffices to take $\kappa_i > \sigma(\xi)$, since then:

$$\pi_{\kappa_0 \kappa_i}(\xi) \leq \sigma_i \pi_{\kappa_0 \kappa_i}(\xi) = \sigma(\xi) < \kappa_i.$$

This completes the construction and we need only note:

(2) $\sigma_\Gamma = \mathrm{id}$.

If $\Gamma < \infty$ this is immediate from the construction and if $\Gamma = \infty$ it follows from (b).

QED (Cor 6)

Finally we prove:

<u>Corollary 7</u> Let β be a singular cardinal. Then $\beta^+ = \beta^{+K}$.

proof.

Suppose not. Let τ = the least regular $\tau > \omega_1$ s.t. $cf(\beta^{+W}) < \tau$. Exactly as in §3.3.4 we generically collapse β^{+W} to τ and define $\sigma_\alpha : \overline{W}_\alpha \to K|\beta^+$, $\sigma_{\alpha\alpha'} : \overline{W}_\alpha \to \overline{W}_{\alpha'}$. We also write $\overline{K}_\alpha$ for $\overline{W}_\alpha$. It suffices to prove Lemma 1 in the proof of the weak covering lemma:

(*) There is a cub $C' \subset C$ s.t. $\mathcal{P}(\alpha) \cap K \not\subset \overline{K}_\alpha$ for $\alpha \in C'$ s.t. $cf(\alpha) > \omega$,

since the rest of the proof uses only the universality of $W = K$.

Suppose not. Let $S \subset C$ be a stationary set of pts of cofinality $> \omega$ s.t. $\mathcal{P}(\alpha) \cap K \subset \overline{K}_\alpha$ for $\alpha \in S$. We first note.

Let $\nu_\alpha = \alpha^{+\overline{K}_\alpha}$ for $\alpha \in S$. Then

(1) $\nu_\alpha = \alpha^{+K}$ and $J^{E^{\overline{K}_\alpha}}_{\nu_\alpha} = J^{E^K}_{\nu_\alpha}$.

proof.

$J^{E^{\overline{K}_\alpha}}_{\nu_\alpha} = J^{E^K}_{\nu_\alpha}$ since otherwise the coiteration of $\overline{K}_\alpha$, K would involve a truncation of the mouse $\overline{K}_\alpha$. But then $\nu_\alpha \leq \alpha^{+\overline{K}_\alpha}$, since α = the largest cardinal in $J^{E^{\overline{K}_\alpha}}_{\nu_\alpha}$. Hence $\nu_\alpha = \alpha^{+\overline{K}_\alpha}$, since $\mathcal{P}(\alpha) \cap K \subset \overline{K}_\alpha$. QED(1)

But then $K \| \beta$ is an extension of $Q_\alpha = J^{E^{\overline{K}_\alpha}}_{\nu_\alpha}$ for all $\beta \geq \tau$. For each regular $\beta \geq \tau$ the frequent extension lemma gives us an $\alpha = \alpha_\beta < \tau$ s.t. the canonical extension $\tilde{\sigma}_{\alpha\beta} : K \| \beta \to \tilde{K}_\beta$ of $\tilde{\sigma}_\alpha = \sigma_\alpha \upharpoonright Q_\alpha$ exists. It is clear

But then there is an α s.t. $\alpha = \alpha_\beta$ for arbitrarily large β. It is clear from the definitions that

(2) If $\alpha = \alpha_\beta = \alpha_{\beta'}$, $\beta \leq \beta'$, then $\tilde{K}_{\beta'}$ end extends $\tilde{K}_\beta$ and $\tilde{\sigma}_{\alpha\beta'} \supset \tilde{\sigma}_{\alpha\beta}$.

Hence, setting $\sigma = \bigcup_\beta \tilde{\sigma}_{\alpha\beta}$, $\tilde{K} = \bigcup_\beta \tilde{K}_\beta$ we have:

(3) $\sigma : K \longrightarrow_{\Sigma_1} \tilde{K}$.

Thus $\tilde{K}$ is a simple iterate of K. Since $\alpha = \mathrm{crit}(\sigma)$, we have:

(4) $E^K_{\nu_\alpha} \neq \emptyset$, $E^{\tilde{K}}_{\nu_\alpha} = \emptyset$.

But then α is a limit cardinal in K, since it carries a measure. Hence so is $\tau = \sigma_\alpha(\alpha)$. But then $\tau > \nu_\alpha = \alpha^{+K}$. But

(5) $\tilde{K} | \tau = K | \tau$

and hence:

$$E^{\tilde{K}}_{\nu_\alpha} = E^K_{\nu_\alpha} \neq \emptyset.$$

Contradiction! QED