Appendix II to "Measures of Order Zero"

Our previous appendix gave an account of Steel's proof of the existence of K, using second order set theory. We show here that with little extra work one can carry out a modified version of the proof in ZFC. This is due to Itay Neeman. In §5.2 of "Non Over-lapping Extenders" [NO] I purported to give a proof of the existence of K (assuming $\neg O^{\P}$) in ZFC. However, Neeman found an error in that proof and wound up replacing it by a simpler proof. I shall later send a corrected version of §5.2 of [NO] based on Neeman's work.

Assume $\neg O^\sharp$

Def $N = \langle J_\nu^E, F \rangle$ is a beaver for the weasel W iff $J_\nu^E = J_\nu^{E^W}$ and W is extendible by F (i.e. $\pi : W \longrightarrow_F W'$ exists.)

We use the following facts about beavers:

Lemma 1 Let $N = \langle J_\nu^E, F \rangle$ be a beaver for W. Let $\pi : W \longrightarrow_{E_\xi} W'$ where $\xi \geq \nu$. Then N is a beaver for W'.

Lemma 2 Let N, W be as above. Let W' be a simple iterate of W above κ, where F is on κ. Then N is a beaver for W'.

Lemma 3 Let N, W be as above. Let W' be a universal weasel s.t. $J_\nu^{E^W} = J_\nu^{E^{W'}}$. Then N is a beaver for W'.

Using $\neg O^1$ we get:

__Lemma 4__ Let N, W be as in Lemma 1. Let $\pi : W \underset{E}{\longrightarrow} W'$. Then W' is universal.

(This follows by the fact that $\beta^{+W} = \beta^{+W'} = \beta^{+}$ for stationarily many β.)

__Lemma 5__ Let X be massive in W. Let $N = \langle J_\nu, F \rangle$ be a beaver for W with F on κ s.t. $F \neq E_\nu^W$. Let $\pi : W \underset{F}{\longrightarrow} W'$. Let W'' be a simple iterate of W, W'. Set:

$$Y = \{x \in W \mid \pi_{WW''}(x) \in \text{rng}(\pi_{W'W''}\pi)\},$$

$$X' = \{\nu \in X \mid \nu = \pi_{WW''}(\nu) = \pi_{W'W''}\pi(\nu)\}.$$

Then $\kappa \notin Y \prec W$, $X' \subset Y$ and X' is massive in W, W', W''.

pf of Lemma 5.

$X' \subset Y \prec W$, X' massive (trivial).

We show $\kappa \notin Y$. Suppose not,

$\pi_{WW''}(\kappa) = \min(\mathrm{rng}(\pi_{WW''}) \setminus \kappa)$

$\pi_{W'W''}(\kappa) = \min(\mathrm{rng}(\pi_{W'W''}\pi) \setminus \kappa)$.

Hence $\pi_{WW''}(\kappa) \geq \pi_{W'W''}\pi(\kappa)$. But

$\pi_{WW''}(\kappa) \leq \pi_{W'W''}\pi(\kappa)$ by "Dodd-Jensen", since $\pi_{WW''}$ is the iteration map. Hence there is

$\tilde{\kappa} = \pi_{WW''}(\kappa) = \pi_{W'W''}\pi(\kappa)$. We

now note that

(1) $\mathcal{P}(\kappa) \cap W \subset h_W((\kappa+1) \cup X')$

(To see this, let $\sigma : \overline{W} \leftrightarrow h_W((\kappa+1) \cup X')$.
Then $\overline{W}$ is universal. Coiterate $W, \overline{W}$ to $\tilde{W}$. The coiteration is above κ. Hence $\mathcal{P}(\kappa) \cap W =$
$= \mathcal{P}(\kappa) \cap \overline{W} = \mathcal{P}(\kappa) \cap \tilde{W}$.)

Let $X \in \mathbb{P}(\kappa) \cap W$, $X = t^W(\vec{\xi}, \vec{\eta})$ where $\vec{\xi} \leq \kappa$, $\vec{\eta} \in X'$. Then $\pi_{WW''}(X) = \pi_{W'W''}\pi(X) = t^{W''}(\vec{\xi}, \vec{\eta})$. $E^W_\nu \neq \emptyset$, since $\pi_{WW'}(\kappa) > \kappa$ and

$$X \in E^W_\nu \longleftrightarrow \kappa \in \pi_{WW'}(X)$$
$$\longleftrightarrow \kappa \in \pi_{W'W}\pi(X) \longleftrightarrow \kappa \in F$$

Hence $E^W_\nu = F$, Contr!

QED (Lemma 2)

Using this we get:

<u>Lemma 6</u> Let W, X be as above. The following are equivalent:

(a) There is a massive $X' \subset X$ s.t. $\kappa \notin h_W(\kappa \cup X')$.

(b) There is a beaver $\langle J^E_\nu, F \rangle$ for W with F on κ s.t. $F \neq E^W_\nu$.

prf. of Lemma 6.

(b) $\to$ (a). Let Y, X' be as above. Then $\kappa \notin Y \supset h_W(\kappa \cup X')$.

(a) $\to$ (b)

Assume w.l.o.g. $\kappa \notin h_W(\kappa \cup X)$. Let $\sigma : \overline{W} \xrightarrow{\sim} h_W(\kappa \cup X)$. Then $\kappa = \mathrm{crit}(\sigma)$ and $(J^E_{\kappa^+})^W = (J^E_{\kappa^+})^{\overline{W}}$, since $W, \overline{W}$ coiterate simply above κ. Set: $\nu = \kappa^{+W} = \kappa^{+\overline{W}}$.

Define F by: $Y \in F \leftrightarrow \kappa \in \sigma(Y)$.

Then $N = \langle J^{E^W}_\nu, F \rangle$ is a beaver for $\overline{W}$, since if $\pi : \overline{W} \to_F \overline{W}'$, then $\overline{W}'$ is well founded and there is $k : \overline{W}' \to_{\Sigma_1} W$ defined by: $k(\pi(f)(\kappa)) = \sigma(f)(\kappa)$.

It remains to show: $F \neq E^W_\nu$. Suppose not. Note that $k \restriction (\nu+1) = \mathrm{id}$; $k\pi = \sigma$.

But then $E_\nu^{\overline{W}'} = E_\nu^{W}$. If $E_\nu^{W} \neq \emptyset$ it follows that N is not a mouse & hence $F \neq E_\nu^{W}$. Otherwise $E_\nu^{W} = \emptyset \neq F$. QED (Lemma 6)

. . . .

Let W be the canonical ω_1-full weasel. We wish to define $\tilde{Y} \subset \omega \times On$, $\tilde{X} \subset On^2$ s.t., setting:

$$Y_\nu = \tilde{Y}``\{\nu\}, \quad X_\nu = \tilde{X}``\{\nu\}$$

we have:

(a) $Y_0 = W$, $X_0 = On$

(b) $Y_\alpha \prec W$ and $X_\alpha \subset Y_\alpha$ is massive in W.

(c) If $\alpha \notin Y_\alpha$, then $Y_{\alpha+1} = Y_\alpha$, $X_{\alpha+1} = X_\alpha$

(d) If $\alpha \in Y_\alpha$, define $\sigma = \sigma_\alpha$, $\overline{W} = \overline{W}_\alpha$ by: $\sigma : \overline{W} \overset{\sim}{\longleftrightarrow} Y$, where $\overline{W}$ is transitive. Set: $\overline{X} = \overline{X}_\alpha = \{\xi \in X_\alpha \mid \sigma(\xi) = \xi\}$. Then $\overline{X}$ is massive in $\overline{W}$, W. Set: $\mathcal{F} = \mathcal{F}_\alpha$ = the set of $\langle J_\nu^E, F \rangle$

s.t. $\langle J_\nu^E, F\rangle$ is a beaver for $\overline{W}$, F is on $\bar\alpha = \sigma^{-1}(\alpha)$, and $F \neq E_\nu^{\overline{W}}$.

If $\mathfrak{f} = \emptyset$, set: $Y_{\alpha+1} = Y_\alpha$, $X_{\alpha+1} = X_\alpha$.

(e) If $\mathfrak{f} \neq \emptyset$, then for each $F \in \mathfrak{f}$ let $\pi = \pi_F : \overline{W} \to_F \overline{W}' = \overline{W}_F$.

Coiterate $\overline{W}, \overline{W}'$ to $\overline{W}''$ and set:

$$\overline{Y}_F = \{x \in \overline{W} \mid \pi_{\overline{W}\,\overline{W}''}(x) \in \operatorname{rng}(\pi_{\overline{W}'\overline{W}''}\pi)\}.$$

(Hence $\bar\alpha \notin \overline{Y}_F \prec \overline{W}$). Set:

$$\overline{X}_F = \{\xi \in \overline{X} \mid \pi_{\overline{W}\,\overline{W}'}(\xi) = \pi_{\overline{W}'\overline{W}''}\pi(\xi) = \xi\},$$

Then $\overline{X}_F$ is massive in $\overline{W}, \overline{W}', \overline{W}''$ (hence in W, since $\overline{X}_F \subset \overline{X}$).

Set: $\overline{Y} = \bigcap_F \overline{Y}_F$. (Hence $\overline{Y} \prec \overline{W}$).

Set: $Y_{\alpha+1} = \sigma^{-1}{}''\overline{Y}$ and

$$X_{\alpha+1} = \bigcap\{\overline{X}_F \mid F \in \mathfrak{f}\}.$$

(f) If $\operatorname{lim}(\alpha)$, then

$$Y_\alpha = \bigcap_{\xi<\alpha} Y_\xi, \quad X_\alpha = \bigcap_{\xi<\alpha} X_\xi,$$

We first assume the existence of $\tilde{Y}, \tilde{X}$ as above and finish the proof of the existence of K just as Steel does. We first note:

<u>Lemma 7</u> Let $Z \subset X_\beta$ be massive in W. Then $Y_\beta \cap \beta \subset h_W(Z)$.

proof.
Suppose not. Let β be least s.t. $Z \subset X_\beta$, $Y_\beta \cap \beta \not\subset h_W(Z)$. Then $\beta = \alpha + 1$, $\alpha \in Y_\beta \subset Y_\alpha$, and $\alpha \notin h_W(Z)$. Let $\sigma : \bar{W} \overset{\sim}{\longleftrightarrow} Y_\alpha$, $\sigma(\bar{\alpha}) = \alpha$. Set $\bar{Z} = \{\xi \in Z \mid \sigma(\xi) = \xi\}$. Then $\bar{Z}$ is massive in $\bar{W}$. Since

$$\alpha \cap Y_\beta = \alpha \cap Y_\alpha \subset h_W(Z),$$

we have $\alpha \cap Y_\alpha = \alpha \cap h_W(Z) = \alpha \cap h_W((\alpha \cap Y_\alpha) \cup \bar{Z})$. Hence $\bar{\alpha} \in h_{\bar{W}}(\bar{\alpha} \cup \bar{Z})$. But $\bar{\alpha} \notin h_W(\bar{\alpha} \cup \bar{Z})$ since $\alpha \notin h_W(Z) \supset h_{\bar{W}}((Y_\alpha \cap \alpha) \cup \bar{Z})$.

Hence by Lemma 6, $f \neq \emptyset$, where f is as in (d), (e) above. Hence $\bar{\alpha} \notin \bar{Y}$ and $\alpha = \sigma(\bar{\alpha}) \notin Y_{\alpha+1} = \sigma^{-1}{}''Y$. Contradiction! QED (Lemma 7)

<u>Cor. 7.1</u> $\alpha \leq \beta \rightarrow \alpha \cap Y_\alpha = \alpha \cap Y_\beta$.

proof.
$\alpha \cap Y_\alpha \subset h_W(X_\beta) \subset Y_\beta$. QED

Exactly as in Steel's proof:

<u>Lemma 8</u> There is a cub class C s.t. $\alpha \in h_W(\alpha \cup Z)$ whenever $\alpha \in C$ and Z is massive in W.

Hence:

<u>Lemma 9</u> There are arbitrarily large $\beta \in Y_\beta$.

proof. Suppose not.
Let $\beta \notin Y_\beta$ for $\beta \geq \alpha$. Set
$\theta_\beta = \min([\beta, \infty) \cap Y_\beta)$ $(\beta \geq \alpha)$,
Then $\beta < \theta_\beta$. Let $\gamma \in h_W(\gamma \cup Z)$

be s.t. $\gamma > \alpha$ and $\theta_\beta < \gamma$ for $\beta < \gamma$. Let $\gamma = t(\vec{\eta}, \vec{\nu})^W$, where $\vec{\eta} < \gamma$ and $\vec{\nu} \in X_\gamma$. Let $\vec{\eta} < \theta_\beta$ where $\beta < \gamma$. Then $\vec{\eta}, \vec{\nu}, \theta_\beta, \theta_\gamma \in Y_\beta$, $t(\vec{\eta}, \vec{\nu}) \in (\theta_\beta, \theta_\gamma]$ and $Y_\beta \prec W$. Hence there are $\vec{\mu} \in \theta_\beta \cap Y_\beta$ s.t. $t(\vec{\mu}, \vec{\nu}) \in (\theta_\beta, \theta_\gamma]$. But $\vec{\mu} \in \theta_\beta \cap Y_\beta \subset \alpha \cap Y_\alpha = \alpha \cap Y_\gamma$, where $\vec{\nu} \in Y_\gamma$, $\theta_\gamma \in Y_\gamma$. Hence $t(\vec{\mu}, \vec{\nu}) \in \theta_\gamma \cap Y_\gamma = \alpha \cap Y_\gamma$.

Contr! QED (Lemma 9)

Thus $\mathrm{otp}(\alpha \cap Y_\alpha)$ can be made arbitrarily large. By Lemma 7, if $\sigma : \bar{W} \xrightarrow{\sim} Y_\alpha$, $\bar{\alpha} = \sigma^{-1}{}''\alpha = \mathrm{otp}(\alpha \cap Y_\alpha)$, $\bar{X} = \{\xi \in X_\alpha \mid \sigma(\xi) = \xi\}$, then $\bar{X}$ is massive in $\bar{W}$ and $\bar{W}$ verifies the incompressibility of $J^{E^{\bar{W}}}_{\bar{\alpha}}$. Hence for every β

there is an incompressible J_β^E. The rest of the proof is as in Steel.

We now give the construction of $\tilde{Y}, \tilde{X}$, following Neeman.

For $\delta \in \text{Card}, \delta > \omega$ we define

$Y_\nu^\delta \prec_{\Sigma_1} W^\delta =_{\text{Df}} J_\delta^{E^W}$ and

$X_\nu^\delta \subset On \wedge Y_\nu^\delta = \delta$.

for $\nu \leq \theta_\delta \leq \delta$ as follows:

<u>Case 1</u> $\nu = 0$. $Y_0^\delta = J_\delta^{E^W}$, $X_0^\delta = \delta$.

<u>Case 2</u> $\nu = \alpha + 1 < \delta$

<u>Case 2.1</u> $\alpha \notin Y_\alpha^\delta$ or $\alpha + \omega = \delta$.

Set: $Y_{\alpha+1}^\delta = Y_\alpha^\delta$, $X_{\alpha+1}^\delta = X_\alpha^\delta$

<u>Case 2.2</u> Case 2.1 fails

Set $\sigma : \overline{W} \xrightarrow{\sim} Y_\alpha^\delta$, $\overline{W}$ transitive.

$\overline{W} = \overline{W}_\alpha^\delta$

Let $\sigma(\overline{\alpha}) = \alpha$

Let $\mathcal{F} = \mathcal{F}^\delta_\alpha$ = the set of F s.t. for $\nu = \bar{\alpha}^{+\bar{W}}$ (hence $\nu < \delta$), we have:

(a) $\langle J^{E^{\bar{W}}}_\nu, F\rangle$ is amenable, $F \neq E^{\bar{W}}_\nu$

(b) F is a normal UF on $\bar{\alpha}$ in $\langle J^{E^{\bar{W}}}_\nu, F\rangle$

(c) $\bar{W}$ is extendible by F

(i.e. the Σ_0 ultraproduct $\pi : \bar{W} \to_F \bar{W}'$ exists).

<u>Case 2.2.1</u> $\mathcal{F} = \emptyset$.

Set: $Y^\delta_{\alpha+1} = Y^\delta_\alpha$, $X^\delta_{\alpha+1} = X^\delta_\alpha$.

<u>Case 2.2.2</u> $\mathcal{F} \neq \emptyset$.

For each $F \in \mathcal{F}$ set:

$\pi = \pi^\delta_\alpha : \bar{W} \to_F \bar{W}' = \bar{W}_F$ Coiterate $\bar{W}, \bar{W}'$ (by a Σ_0 iteration). If $\bar{W}, \bar{W}'$ coiterate for each F, $\bar{W}, \bar{W}'$ coiterate simply to a common $\bar{W}'' = \bar{W}''_F$ s.t. $\mathrm{On} \cap \bar{W}'' = \delta$, we continue the construction. Otherwise $Y^\delta_{\alpha+1}, X^\delta_{\alpha+1}$ are undefined. Now let W'' be given for $F \in \mathcal{F}$.

Set: $\overline{Y}_F = \{x \in \overline{W} \mid \pi_{\overline{W}\overline{W}''}(x) \in \mathrm{rng}(\pi_{\overline{W}'\overline{W}''}\pi)\}$,

Hence $\overline{Y}_F \prec_{\Sigma_1} W$. Set:

$\overline{X} = \{\xi \in X^\delta_\alpha \mid \xi = \sigma(\xi)\}$ and:

$\overline{X}_F = \{\xi \in \overline{X} \mid \xi = \pi_{\overline{W}\overline{W}''}(\xi) = \pi_{\overline{W}'\overline{W}''}\pi(\xi)\}$

Then $\overline{X}_F \subset \mathrm{On} \cap \overline{Y}_F \subset \delta$. Set:

$$\overline{Y} = \bigcap_{F \in \mathfrak{F}} \overline{Y}_F \quad , \quad \overline{X}' = \bigcap_{F \in \mathfrak{F}} \overline{X}_F .$$

Then $\overline{Y} \prec_{\Sigma_1} \overline{W}$, $\overline{X}' \subset \mathrm{On} \cap \overline{Y} \subset \delta$,

At $\mathrm{otp}(\overline{X}') < \delta$, then $Y^\delta_{\alpha+1}$, $X^\delta_{\alpha+1}$ are undefined. Otherwise set:

$$Y^\delta_{\alpha+1} = \sigma^{-1}{}''\,\overline{Y} \quad , \quad X^\delta_{\alpha+1} = \overline{X} .$$

<u>Case 3</u> $\nu = \lambda$, $\mathrm{Lim}(\lambda)$.

Set: $Y = \bigcap_{\alpha < \lambda} Y^\delta_\alpha$, $X = \bigcap_{\alpha < \lambda} X^\delta_\alpha$.

Then $Y \prec Y^\delta_0$, $X \subset \mathrm{On} \cap Y \subset \delta$,

At $\mathrm{otp}(X) < \delta$, then $Y^\delta_{\alpha+1}$, $X^\delta_{\alpha+1}$ are undefined. Otherwise:

$$Y^\delta_{\alpha+1} = Y \quad , \quad X^\delta_{\alpha+1} = X .$$

Now set: $D_\nu = \{\delta \mid Y_\nu^\delta$ is defined$\}$.

We prove:

<u>Lemma 10</u> Let $\nu < \infty$.

(a) $\sup D_\nu = \infty$

(b) There is η s.t. whenever $\gamma, \delta \in D_\nu \setminus \eta$, $\gamma \le \delta$, then

$$Y_\nu^\gamma = J_\gamma^{E^W} \cap Y_\nu^\delta, \quad X_\nu^\gamma = \delta \cap X_\nu^\delta,$$

(c) Set $Y_\nu = \bigcup_{\delta \in D_\nu \setminus \eta} Y_\nu^\delta$, $X_\nu = \bigcup_{\delta \in D_\nu \setminus \eta} X_\nu^\delta$,

where η is as in (b). Then $Y_\nu \prec W$ and X_ν is massive in W.

(d) $D_\nu \setminus \eta$ is cub in ∞ for sufficiently large η.

prf. By ind. on ν.

<u>Case 1</u> $\nu = 0$. Trivial.

<u>Case 2</u> $\nu = \alpha + 1$. Let η be as in (b), (d) for α.

Then $D = D_\alpha \setminus \eta \subset$ Card is cub and $Y^\delta_\alpha = Y_\alpha \cap J^{E^W}_\delta$, $X^\delta_\alpha = X_\alpha \cap \delta$ for $\delta \in D$. If $\alpha \notin Y_\alpha$, the conclusion is trivial, since $D_\alpha = D_{\alpha+1}$, $Y^\delta_\alpha = Y^\delta_{\alpha+1}$, $X^\delta_\alpha = X^\delta_{\alpha+1}$.

Now let $\alpha \in Y_\alpha$. Set: $\sigma : \bar{W} \overset{\sim}{\longleftrightarrow} Y_\alpha$. For $\delta \in D$ set: $\sigma_\delta : \bar{W}_\delta \overset{\sim}{\longleftrightarrow} Y^\delta_\alpha$. Then $\bar{W}_\delta = J^{E^{\bar{W}}}_\delta$ for $\delta \in D$. Set: $\mathfrak{f}$ = the set of F s.t. $\langle J^{E^{\bar{W}}}_\nu, F \rangle$ is a beaver for $\bar{W}$, F is on $\bar{\alpha} = \sigma^{-1}(\alpha)$, and $F \neq E^{\bar{W}}_\nu$. Let $\mathfrak{f}^\delta = \mathfrak{f}^\delta_\alpha$ be as in Case 2.2 of the def. of $Y^\delta_{\alpha+1}$.

Clearly $\mathfrak{f}^\delta = \mathfrak{f}$ for sufficiently large δ, so we assume w.l.o.g. that $\mathfrak{f}^\delta = \mathfrak{f}$ for $\delta \in D$.

If $\mathfrak{f} = \emptyset$, then $D \subset D_{\alpha+1}$

and $Y^\delta_{\alpha+1} = Y^\delta_\alpha$, $X^\delta_{\alpha+1} = X^\delta_\alpha$ for $\delta \in D$, so the conclusion is trivial.

Now let $\mathcal{F} \neq \emptyset$. Let $F \in \mathcal{F}$. Let $\pi : \bar{W} \to_F \bar{W}'$. Coiterate $\bar{W}', \bar{W}$ to $\bar{W}''$. Set:

$$D_F = \{\delta \in D \mid \pi_{\bar{W}\bar{W}''} {}``\delta,\ \pi_{\bar{W}'\bar{W}''}\pi {}``\delta \subset \delta\},$$

Note that if $\delta \in D_F^\circ$, then $\bar{W}''_\delta = J_\delta^{E^{\bar{W}''}}$ is the coiterate of $\bar{W}'_\delta, \bar{W}_\delta$, where $\bar{W}_\delta = J_\delta^{E^{\bar{W}}}$ and $\bar{W}'_\delta = J_\delta^{E^{\bar{W}'}}$. Moreover,

$$\pi_{\bar{W}_\delta \bar{W}''_\delta} = \pi_{\bar{W}\bar{W}''} \upharpoonright \bar{W}_\delta,$$

$$\pi_{\bar{W}'_\delta \bar{W}''_\delta} = \pi_{\bar{W}'\bar{W}''} \upharpoonright \bar{W}'_\delta, \text{ and}$$

$$\pi_\delta = \pi \upharpoonright \bar{W}_\delta, \text{ where } \pi_\delta : \bar{W}_\delta \to_F \bar{W}'_\delta.$$

[These facts use that δ is a cardinal $> \alpha$; hence that $f : \alpha \to \bar{W}_\delta$, $f \in \bar{W}$ implies $f \in \bar{W}_\delta$].

On the other hand, if $\delta \in D$, $\overline{W}_\delta$ is as above and $\pi_\delta : \overline{W}_\delta \to_F \overline{W}'_\delta$ and $\overline{W}_\delta, \overline{W}'_\delta$ coiterate to $\overline{W}''_\delta$ with $On \cap \overline{W}'' = \delta$, then it is easily seen that $W'_\delta = J_\delta^{E^{\overline{W}'}}$, $\overline{W}''_\delta = J_\delta^{E^{\overline{W}''}}$, $\pi_{\overline{W}_\delta \overline{W}''_\delta} = \pi_{\overline{W} W''} \restriction \overline{W}_\delta$, $\pi_{\overline{W}'_\delta \overline{W}''_\delta} = \pi_{\overline{W}' \overline{W}''} \restriction \overline{W}'_\delta$, $\pi_\delta = \pi \restriction \overline{W}_\delta$.

Hence $Y^\delta_{\alpha+1}, X^\delta_{\alpha+1}$ are undefined for $\delta \in D \setminus D_F$. Set:

$\overline{Y} = \{x \mid \pi_{\overline{W}\,\overline{W}''}(x) \in rng(\pi_{\overline{W}'\overline{W}''}\pi)\}$,

$\overline{Y}_\delta = \{x \mid \pi_{\overline{W}_\delta \overline{W}''_\delta}(x) \in rng(\pi_{\overline{W}'_\delta \overline{W}''_\delta}\pi_\delta)\}$

for $\delta \in D_F$. Then $\overline{Y}_\delta = \overline{Y} \cap J_\delta^{E^{\overline{W}}}$.

Set: $\overline{X} = \{\zeta \in X_\alpha \mid \sigma(\zeta) = \zeta\}$,

$\overline{X}_\delta = \{\zeta \in X^\delta_\alpha \mid \sigma(\zeta) = \zeta\}$ for $\delta \in D_F$

Then $\overline{X}_\delta = \overline{X} \cap \delta$. Finally set:

$\bar{X}_F = \{\xi \in \bar{X} \mid \pi_{\bar{W}\bar{W}''}(\xi) = \pi_{\bar{W}'\bar{W}''}\pi(\xi) = \xi\}$,

$\bar{X}_{\delta,F} = \{\xi \in \bar{X}_\delta \mid \pi_{\bar{W}_\delta \bar{W}_\delta''}(\xi) = \pi_{\bar{W}_\delta' \bar{W}_\delta''}\pi_\delta(\xi) = \xi\}$.

Then $\bar{X}_{\delta F} = \bar{X}_F \cap \delta$ for $\delta \in D_F$.

Set: $D_F' = \{\delta \in D_F \mid \mathrm{otp}(\bar{X}_{\delta,F}) = \delta\}$.

Clearly $Y^\delta_{\alpha+1}, X^\delta_{\alpha+1}$ are undefined for $\delta \in D \setminus D_F'$. Note that D_F' is cub in ∞. Set: $D' = \bigcap_F D_F'$. Then D' is cub in ∞ and $D' = \{\delta \in D_{\alpha+1} \mid \delta \geq \eta\}$. Finally, set: $\bar{Y} = \bigcap_F \bar{Y}_F$, $X = \bigcap_F \bar{X}_F$, $Y = \sigma^{-1}{}''\bar{Y}$. Then $Y \prec W$ and $\bar{X}'$ is massive in W. But clearly:

$Y = \bigcup_{\delta \in D'} Y^\delta_{\alpha+1}$, $X = \bigcup_{\delta \in D'} X^\delta_{\alpha+1}$.

Hence $Y = Y_{\alpha+1}$, $X = X_{\alpha+1}$.

QED (Case 2)

Case 3 $\nu = \lambda$, lim(λ).
For $\xi < \lambda$ pick η_ξ satisfying (b), (d) at ξ. Let $\eta > \sup_\xi \eta_\xi$.
Clearly $D_\lambda = \bigcap_{\xi<\lambda} D_\xi$. Hence
$$D_\lambda \setminus \eta = \bigcap_{\xi<\lambda} (D_\xi \setminus \eta)$$
is cub in ∞. Obviously, $Y_\lambda = \bigcap_{\xi<\lambda} Y_\xi$ and $X_\lambda = \bigcap_{\xi<\lambda} X_\xi$. Hence $Y_\lambda < W$ and X_λ is massive in W. QED (Lemma 10).

An examination of the above proof reveals, however:

Cor 10.1 (a) – (f) hold for the sequence X_ν, Y_ν $(\nu < \infty)$.

QED