§11 The Model K^c

In his notes The Core Model Iterability Problem Steel constructs an alternative inner model which he calls K^c below an inaccessible cardinal θ. As before, he constructs sequences $\langle M_\nu \mid \nu < \theta \rangle$, $\langle N_\nu \mid \nu < \theta \rangle$ which "converge" to $N = K^c = N_\theta$ in exactly the same sense as before. The construction is identical except in Case 2.1, where the criterion for admitting a model $M = N_{\xi+1} = \langle J_\alpha^E, F \rangle$ with a top extender is more liberal: We require only that $\langle J_\alpha^E, \emptyset \rangle = M_\xi$ and that $\langle J_\alpha^E, F \rangle$ be certifiable in the sense defined below.

<u>Def</u> Let $M = \langle J_\alpha^E, F\rangle$ be a pm s.t. $F \neq \emptyset$. Let $\kappa = \mathrm{crit}(F)$. $\langle N, G\rangle$ is a background <u>certificate</u> for M iff

(i) N is a transitive ZFC$^-$ model s.t. $V_\kappa \in N$

(ii) G is an extender of length $\geq \lambda = lh(F)$ at κ on N.

(iii) Let $\pi : N \longrightarrow_G N'$. Then $V_{\lambda+1} \subset N'$

(iv) $F(X) = G(X) \cap \lambda$ for $X \in \mathcal{P}(\kappa) \cap N \cap M$.

<u>Def</u> $M = \langle J_\alpha^E, F\rangle$ is <u>certifiable</u> iff iff for every $A \subset \kappa$ there is a certificate $\langle N, G\rangle$ s.t. $A \in N$.

Case 2.1 now requires only certifiabilit in this sense. (It is clear that our previous requirement implies certifiabilit since if $\pi : V \longrightarrow_G V'$ with $V_{\lambda+1} \subset V'$ then $\langle N, G\rangle$ is a certificate for $N = H_{\kappa^+}$ and $\mathcal{P}(\kappa) \subset N$.)

We note the following:

Fact 1 Let $M = \langle J_\lambda^E, F\rangle$ be certifiable, $\kappa = \mathrm{crit}(F)$. Then:

(a) κ is strongly inaccessible.

(b) If $\kappa^{+M} = \kappa^+$, then $\{\xi < \kappa \mid \xi^{+M} = \xi^+\}$ is stationary in κ.

proof.

(a) To see regularity, let $f : \lambda \to \kappa$ be a cofinal map witnessing the singularity of κ with $f \in N$, $\langle N, G\rangle$ a certificate. Let $\pi : N \to_G N'$. Then $\pi(f) : \lambda \to \pi(\kappa)$ cofinally. But $\pi(f) = f$, since $\pi(f)(\xi) = \pi(f(\xi)) = f(\xi)$ for $\xi < \lambda$. Contr! We now show: $2^\beta < \kappa$ for $\beta < \kappa$. Suppose not. Let $f : \kappa \xrightarrow{1-1} \mathbb{P}(\beta)$. ~~Let $f \in N$,~~ where N, π are as before. Then $\pi(f) : \pi(\kappa) \xrightarrow{1-1} \mathbb{P}(\beta)$ and $\mathbb{P}(\beta) \setminus \mathrm{rng}(f) = \pi(\mathbb{P}(\beta) \setminus \mathrm{rng}(f)) = \mathbb{P}(\beta) \setminus \mathrm{rng}(\pi(f))$. Contr! QED(a)

(b) Suppose not. Let $C \subset \kappa$ be cub. s.t. $\xi^{+M} < \xi^+$ for $\xi \in C$. Let $\langle N, G\rangle$ be a certificate with $J_\kappa^{E^M}, C \in N$. Let $J_{\lambda'}^{E'} = \pi(J_\kappa^E)$, $\pi : N \to_G N'$

Clearly $J_\lambda^E = h_{J_\lambda^E}(\lambda)$, where $\lambda = lh(F)$. But if $\pi : J^E_{\kappa+M} \to_F J^E_\alpha$, then $\pi(h_{J_\kappa^E}) = h_{J_\lambda^E}$. It follows that each $x \in J_\lambda^E$ has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f : \kappa \to \kappa$ is J_κ^E-definable, hence $f \in N$. Thus we can define $\sigma : J_\lambda^E \to_{\Sigma_0} J_{\lambda'}^{E'}$ by $\sigma(\bar\pi(f)(\alpha)) = \pi(f)(\alpha)$, where $f : \kappa \to \kappa$, $f \in N$. But then $\sigma \restriction \lambda = id$, by the usual proof. Hence $\sigma \restriction J_\lambda^E = id$ and $J_\lambda^E = J_{\lambda'}^{E'}$.

Now let $\kappa^{+M} = \kappa^+$. Then $\kappa^{+J_\lambda^{E'}} = \kappa^{+N}$, hence $\kappa \in G(A)$, where $A = \{\xi < \kappa \mid \xi^{+J_\kappa^E} = \xi^+\}$. But $G(C)$ is cub in $\pi(\kappa)$ and $C = \kappa \cap G(C)$ is cub in κ. Hence $\kappa \in G(A) \cap G(C) = G(A \cap C)$. Hence $A \cap C \neq \emptyset$. QED (Fact 1)

For the sake of brevity, let us now give a name to a property we have frequently used:

__Def__ Let M be a premouse. M is a __weak mouse__ (or __weakly iterable__) iff whenever $\sigma : Q \rightarrow_{\Sigma^*} M$ and Q is countable, then Q is countably iterable.

(We recall that if $A^\#$ exists for all $A \subset \omega_1$, then the coiteration of two countably iterable ~~basic~~ countable premice terminates below ω_1. Moreover, all the results of §7, 8 hold for such mice. Hence, by a Löwenheim-Skolem argument, they hold for all weak mice. (cf. the end of the appendix to §7).

We also note:

Fact 2 Let θ be inaccessible s.t. V_θ is closed under #. Let $M \in V_\theta$ be a weak mouse s.t. $\sup_M^\omega \le \nu$ for a ν s.t. $E_\nu^M \ne \emptyset$. Then M is uniquely ~~smoothly~~ iterable in V_θ

proof.

Let $\mathcal{I}$ be a smooth iteration of length $< \theta$. We claim that $\mathcal{I}$ can be continued. Let $\mathcal{I} \in H_\tau$ for some regular τ. Then $\mathcal{I}^\# \in H_\tau$.

Let $\sigma : \bar{H} \prec H_\tau$, where $\bar{H}$ is countable and transitive and $\sigma(\bar{\mathcal{I}}) = \mathcal{I}$. Then $\bar{\mathcal{I}}$ is a countable smooth iteration of $\bar{M} = \sigma^{-1}(M)$, where $\sigma \upharpoonright \bar{M} : \bar{M} \prec M$. Hence $\bar{M}$ is countably iterable. It follows mutatis mutandis from the results of §6 that $\bar{M}$ is countably uniquely smoothly iterable. We have two cases:

§6 Cor 6.

Case 1 $lh(\mathcal{I})$ is a successor ordinal. Then continuability says that a structure $\Gamma(\mathcal{I})$ is well founded, where $\Gamma(\mathcal{I})$ is definable from $\mathcal{I}$ in $H_{\bar{\tau}}$. But then $\Gamma(\bar{\mathcal{I}})$ is well founded by the iterability of $\bar{M}$. Hence $\Gamma(\mathcal{I}) = \pi(\Gamma(\bar{\mathcal{I}}))$ is well founded.

Case 2 $lh(\mathcal{I})$ is a limit.

We first show that $\mathcal{I}$ has a well founded (cofinal) branch. $\bar{\mathcal{I}}$ has a unique well founded branch $\bar{b}$ with limit model $M_{\bar{b}}$. Let $\delta < \omega_1$ s.t. $rn(M_{\bar{b}}) < \delta$ and δ is admissible in an $\bar{a} \subset On$ s.t. $\bar{a} \in \bar{H}$ codes $\bar{\mathcal{I}}$. Let $\bar{\mathcal{L}}$ be the infinitary language over $L_{\delta}[\bar{a}]$ with

Constants: $\underline{x}$ $(x \in L_{\delta}[\bar{a}])$, $\mathring{b}$, $\mathring{M}$

Axioms: ZFC^-; $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in L_{\delta}[\bar{a}]$; $\mathring{b}$ is a cofinal branch in $\underline{\bar{\mathcal{I}}}$; $\mathring{M} = M^{\bar{\mathcal{I}}}_{\mathring{b}}$ is transitive; $rn(\mathring{M}) = \underline{\gamma}$ (where $rn(M_{\bar{b}}) = \gamma$).

Clearly $\bar{\mathcal{L}}$ has a model $\mathfrak{A}$ with $\bar{b} = \mathring{b}^{\mathfrak{A}}$, $M_{\bar{b}} = \mathring{M}^{\mathfrak{A}}$. But if $\mathfrak{A}$ is any model (with $\underline{x}^{\mathfrak{A}} = x$ for $x \in L_{\delta}[\bar{a}]$),

Then $\mathring{b} = \bar{b}$, $\mathring{M}^{\Omega} = M_{\bar{b}}$. Hence by the completeness theorem for countable admissibles: $\bar{\mathcal{L}} \vdash \underline{\xi} \in \mathring{b} \longleftrightarrow \xi \in \bar{b}$.
Hence $\bar{b} \in L[\bar{a}]$. Clearly $\pi(\bar{a}^{\#}) = a^{\#}$ and we can use the indiscernables to extend $\pi \upharpoonright L_{\mathrm{On} \cap \bar{H}}[\bar{a}]$ to a map $\tilde{\pi} : L[\bar{a}] \prec L[a]$. Hence $b = \tilde{\pi}(\bar{b})$ is a cofinal well founded branch in $\mathcal{T} = \tilde{\pi}(\bar{\mathcal{T}})$.

Thus M is a mouse in V_θ. It follows as in §6 that M is uniquely iterable.

QED (Fact 2)

Similarly:
<u>Fact 3</u> Let θ be as above & let $M \in V_\theta$ be a weak mouse s.t. $\omega\rho^{\omega}_{M} < \mathrm{On}_M$ and no $\gamma \in M$ is Woodin in M. Then M is uniquely smoothly iterable in V_θ.

proof:
As before, using §6 Cor 6.1 in place of §6 Cor 6.

QED (Fact 3)

We again assume that V_Θ is closed under # and we construct a sequence M_ξ, N_ξ ($\xi < \bar{\Theta} \leq \Theta$) exactly as before, except that in Case 21 ($\xi = \zeta + 1$, $N_\xi = \langle M_\zeta, F \rangle$, $F \neq \emptyset$) we require only that N_ξ be certifiable (instead of requiring a "full background extender" F^* on V as before).

(We refer to this new sequence as the "$\mathcal{F}^c$-sequence".)

We must then show that each N_ξ is a weak mouse (hence that (a), (b) hold. In §10 in the case of the "full background sequence" we gave such a proof. This was a modification of Steel's original proof, and addressed the problem that our iterability requirement is somewhat stronger than Steel's. (Steel essentially requires that if $\sigma : Q \prec N_\xi$ and Q is countable, then Q is countably normally iterable

We believe — but have not yet checked — that our modification can be adapted to Steel's iterability proof for the K^c-sequence. (As before, we would show that if $\sigma : Q \xrightarrow[\Sigma^*]{} N_\xi \min(\vec{\rho}\,)$, then Q has a normal iteration strategy S s.t. if $\mathcal{T}$ is countable S-iteration, then

(a) If $lh(\mathcal{T}) = \delta + 1$, there is σ' s.t. $\sigma' : Q_\delta \xrightarrow[\Sigma^*]{} N_\gamma \min(\vec{\rho}\,')$ for a $\gamma \leq \xi$. Moreover, if $\pi_{0\delta}$ is total, then $\gamma = \xi$, $\sigma' \pi_{0\delta} = \sigma$ and $\rho_i' \leq \rho_i$ for $i < \omega$. Otherwise $\gamma < \xi$.

(b) If $lh(\mathcal{T})$ is a limit ordinal, then the branch $b = S(\mathcal{T})$ exists.)

(Similar our proof of the uniqueness of the extenders chosen in Case 2.1 should go through.)

From now on we shall assume without proof that:

<u>Lemma 1</u> N_ξ is defined (and unique) for $\xi < \Theta$ and (a), (b) hold.

(In particular, each N_ξ is a weak mouse

Exactly as before, we then define

$$N = K^c = J^E_\theta = \bigcup_{\xi<\theta} M_\xi \| \tilde{\mu}_\xi = \bigcup_{\xi<\theta} M_\xi \| \tilde{\kappa}_\xi ,$$

where $\tilde{\kappa}_\xi = \min\{\omega\rho^\omega_{M_\delta} \mid \xi \le \delta < \theta\}$ and $\tilde{\mu}_\xi = \tilde{\kappa}_\xi^{+M_\xi}$ ($= ht(M_\xi)$ if $\tilde{\kappa}_\xi = ht(M_\xi)$, otherwise $\tilde{\mu}_\xi =$ the maximal $\alpha \le ht(M_\xi)$ s.t. $\tilde{\kappa}_\xi$ is a cardinal in $M_\xi \| \alpha$).

We sometimes write N_θ or M_θ for K^c.

Exactly as in §10 (following the definition of M_ξ, N_ξ) we show:

(1) Let $\kappa = \omega\rho^\omega_{M_\xi} = \omega\rho^\omega_{M_{\xi+1}}$. Then Case 2.2 in the successor case of the definition holds and:

$$\kappa^{+M_{\xi+1}} = \kappa^{+N_{\xi+1}} = ht(M_\xi) + 1 .$$

(2) $ht(M_\lambda)$ is a limit ordinal if λ is

Def Let λ be a limit ordinal.

$\delta(\lambda) =_{Df} \sup\{\xi \mid \tilde{\mu}_\xi < \lambda\}$.

Lemma 2.1 Let λ be cardinally absolute in $N = N_\Theta$ (i.e. if $\tau < \lambda$ is a cardinal in J_λ^{EN}, then in N).

Then δ is a limit ordinal.

prf. Suppose not.

Then $\delta = \alpha + 1$. Let $\kappa = \tilde{\kappa}_\alpha$ and let $\xi \geq \alpha$ be minimal s.t. $\omega\rho^\omega_{M_\xi} = \kappa$. Then $\omega\rho^\omega_{M_{\xi+1}} = \kappa$. Hence Case 2.2 in the def. of $M_{\xi+1}$ holds and:

(*) $\kappa^{+M_{\xi+1}} = \kappa^{+N_{\xi+1}} = ht(M_\xi) + 1$.

If $\xi > \alpha$, then $\tilde{\mu}_{\alpha+1} = \kappa^{+M_{\alpha+1}} = \kappa^{+M_\alpha} = \tilde{\mu}_\alpha < \delta$. Contr! Thus $\xi = \alpha$. But then $M_{\alpha+1} \| \tilde{\mu}_{\alpha+1} = N \| \tilde{\mu}_{\alpha+1}$. But then $\tilde{\mu}_\alpha < \lambda$ is a cardinal in $N \| \lambda$, hence in N, hence in $N \| \tilde{\mu}_{\alpha+1}$, hence in $M_{\alpha+1} \| \tilde{\mu}_{\alpha+1}$. Contr! since

$ht(M_\alpha) \geq \lambda$ by $(*)$ and hence :

$\kappa < \tilde{\mu}_\alpha = \kappa^{+M_\alpha} < \lambda < \tilde{\mu}_{\alpha+1}$.

QED (Lemma 2.1)

<u>Lemma 2.2</u> Let λ, δ be as above. Then $M_\delta = N_\delta = \langle J_\lambda^{E^N}, \emptyset \rangle$.

prf.

$\tilde{\mu}_\xi = \mu_{\xi,\delta}$ for $\xi < \delta$, since $\omega\rho^\omega_{M_\gamma} \geq \tilde{\mu}_\gamma \geq$ $\geq \lambda$ for $\delta \leq \gamma$. Hence $N_\delta =$

$= \langle \bigcup_{\xi < \delta} J^{E^{N_\xi}}_{\tilde{\mu}_\xi}, \emptyset \rangle$ by Case 3 in the def. of N_ξ. But $\omega\rho^\omega_{N_\delta} \geq \lambda = ht(N_\delta)$

Hence $M_\delta = N_\delta$. QED (2.2)

Since $\tilde{\mu}_\delta \geq \lambda$ we also have :

<u>Cor 2.3</u> $\tilde{\mu}_\delta = \lambda$ (since $\omega\rho^\omega_{M_\delta} = \lambda$).

Moreover

<u>Cor 2.4</u> If λ is a limit cardinal in N, then $\tilde{\kappa}_\delta = \lambda$,

since otherwise $\tilde{\mu}_\delta = \tilde{\kappa}_\lambda^{+M_\lambda} < \lambda$.

Steels most important lemma on K^c is the "cheap covering lemma":

<u>Lemma 3</u> Let Θ be measurable with normal measure U. Assume that $K^c \models$ There is no Woodin cardinal. Then $\{\tau < \Theta \mid \tau^{+K^c} = \tau^+\} \in U$.

proof. Suppose not.

Let $\pi : V \longrightarrow_U V'$. Let $N = K^c$, $N' = \pi(N)$, $\tau = \Theta^{+N'}$. Then $\tau < \Theta^+ = \Theta^{+V'}$. Let $M = J_\tau^{E^{N'}}$, $M' = \pi(M)$. Set: $F = \pi \upharpoonright (\mathcal{P}(\Theta) \cap M)$. Then $F \in V'$, since if $X = \langle X_i \mid i < \Theta \rangle$ enumerates $\mathcal{P}(\Theta) \cap N'$, then $X, \pi(X) \in V'$ and $F = \{\langle \pi(X)_i, X_i \rangle \mid i < \Theta\}$. Clearly $\langle M', F \rangle$ satisfies all premouse conditions except the initial segment condition. We shorten it to satisfy that condition: Let $\lambda \leq \Theta' = \pi(\Theta)$ be least s.t.

$\pi(f)(\alpha) < \lambda$ whenever $f : \theta \to \theta$ in N' and $\alpha < \lambda$. (Note λ is definable in V'. To see this note that $\pi \upharpoonright M : M \xrightarrow[F]{} M'$, where $F \in V'$. Hence $\pi \upharpoonright M \in V'$.) Since θ' is inaccessible in V', it follows by a Löwenheim-Skolem argument that $\lambda < \theta'$. Let $\bar{F} = F|\lambda$, $\bar{\pi} : M \xrightarrow[\bar{F}]{} \bar{M}$. Then $\langle \bar{M}, \bar{F} \rangle$ is a premouse (~~satisfying~~ the initial segment condition by the minimality of $\lambda = \pi'(\theta)$). $\bar{M}$ is also basic. Hence so is $\langle \bar{M}, \bar{F} \rangle$, since otherwise there would be $\delta < \theta$ which is Woodin in $N = J_\theta^{E^{\bar{M}}}$. Contr! We also note:

(1) $\bar{M}$ is a proper segment of M'

proof.

Define $\sigma : \bar{M} \prec M'$ by:

$\sigma(\bar{\pi}(f)(\alpha)) = \pi(f)(\alpha)$ for $f : \theta \to \theta$ in M and $\alpha < \lambda$. Then $\sigma \upharpoonright \lambda = \lambda$.

Hence $J_\lambda^{E^{\bar{M}}} = J_\lambda^{E^{M'}}$. λ is easily seen to be a limit cardinal in M'.

Now let $\lambda < \bar{\xi} < ht(\bar{M})$ s.t. $\omega\rho^\omega_{\bar{M}\|\bar{\xi}} = \lambda$ (these pts are cofinal in $ht(\bar{M})$). Let $\xi = \sigma(\bar{\xi})$. Then $\sigma \upharpoonright \bar{M}\|\bar{\xi} : \bar{M}\|\bar{\xi} \xrightarrow[\Sigma^*]{} M'\|\xi$ with critical point λ, where $\bar{M}\|\bar{\xi}$ is sound and $M'\|\xi$ is a weak mouse. The results of §8 go thru for weak mice (since V_θ is closed under #). Hence $\bar{M}\|\bar{\xi}$ is a weak mouse and one of three alternatives holds:

(a) $\bar{M}\|\bar{\xi} = \mathrm{core}(M\|\xi)$ with core map $\sigma \upharpoonright \bar{M}\|\bar{\xi}$

(b) $\bar{M}\|\bar{\xi}$ is a proper segment of M'

(c) λ is not a limit.

We know (c) to be false. (a) is impossible since $M\|\xi \neq \bar{M}\|\bar{\xi}$ is sound. Hence (b) holds. QED(1)

But then $ht(\bar{M}) \leq \lambda^{+M'} = \lambda^{+N'}$, where $N' = \pi(N)$; hence $\bar{M}$ is a proper segment of N'.

Now let $\vec{N}' = \pi(\vec{N})$, where $\vec{N} = \langle N_\xi \mid \xi < \theta$
let $\vec{M}' = \pi(\vec{M})$. By Lemma 2.4,
if $\delta = \delta(\lambda) = \sup\{\xi \mid \tilde{\kappa}_\xi < \lambda\}$ in V',
then $M_\delta = N_\delta = J_\lambda^{E^{M'}} = J_\lambda^{E^{\bar{M}}}$ and
$\tilde{\kappa}_\delta = \lambda$. Now let $\bar{M} = J_{\bar{\alpha}}^{E^{\bar{M}}} = J_{\bar{\alpha}}^{E^{M'}}$.
λ is the largest cardinal in $\bar{M}$; hence
λ is cardinally absolute in N'. It
follows that if $\delta' = \delta(\bar{\alpha})$, then
$\delta' > \delta$ and $\bar{M} = M_{\delta'} = N_{\delta'}$. We
now observe that:

(2) $\langle \bar{M}, \bar{F} \rangle$ is certifiable.

prf.
Let $A \subset \theta$. Select $Q \prec H_{\theta^+}$ in V' s.t.
$V_\theta \in Q$ and $\bar{\bar{Q}} = \theta$, $A \in Q$. Let
$Q' = \pi(Q)$. It follows as before that
$G = \pi \restriction (\mathbb{P}(\theta) \cap Q) \in V'$. But then
$\langle Q, G \rangle$ is a background certificate
for $\langle \bar{M}, \bar{F} \rangle$ in V'. QED(2)

It follows that $N_{\delta'+1} = \langle \bar{M}, \bar{F} \rangle$. But
then $\omega\rho^\omega_{N_{\delta'+1}} < \lambda = \tilde{\kappa}_\delta \leq \omega\rho^\omega_{N_{\delta'+1}}$.
Contr! QED (Lemma 3)

Lemma 4 Let θ be Woodin. Then $L[N] \models \forall \delta \leq \theta$ δ is Woodin.

[Note In contrast to Steel we require that E_ν have length $\lambda = E_\nu(\kappa)$ where $\kappa = \text{crit}(E_\nu)$ whenever $E_\nu \neq \emptyset$. This makes it harder to find background extenders for the F in Case 2.1 of the definition of $\langle N_\xi \mid \xi < \theta \rangle$, $\langle M_\xi \mid \xi < \theta \rangle$. Steel uses shorter extenders and was able to prove Lemma 4 even for the model which used full background extenders (as in §10). We think it likely that Lemma 4 holds for the model in §10 but are unable to prove it. We prove it for our K^c model, but even here the proof is more convoluted than Steel's.]

The proof of Lemma 4 stretches over several sublemmas.

<u>Def</u> $X \subset \Theta$ is <u>Woodin</u> iff for each $A \subset \Theta$ there is $\tau \in X$ which is strong wrt A in V_Θ.

We leave it to the reader to prove:

<u>Fact 4</u> Every Woodin $X \subset \Theta$ is stationary. Moreover, Fodor's Theorem holds for Woodinness: Let $f: X \to \Theta$ be regressive. Then $V_\xi\, f^{-1}\{\xi\}$ is Woodin

<u>Fact 5</u> Let X be Woodin. Then $\{\tau \mid X \cap \tau$ is stationary in $\tau\}$ is Woodin

pf. Suppose not. Then $\bar{X} =$ $= \{\tau \mid X \cap \tau$ is not stat. in $\tau\}$ is Woodin. For $\tau \in \bar{X}$ choose cub $C_\tau \subset \tau$ s.t. $C_\tau \cap X = \emptyset$. Let $\bar{\tau} \in X$ be Woodin wrt. $\bar{X}$. Let F at $\bar{\tau}$ be $\bar{\tau}+1$-strong wrt. $\bar{X}$. Let $\pi: V \to_F V'$. Then $\tau \in \bar{X} \cap (\bar{\tau}+1) =$ $= \pi(\bar{X}) \cap (\bar{\tau}+1)$. Hence $\bar{\tau} \in$ $\in \pi(\bar{X}) \cap \pi(C_{\bar{\tau}}) = \pi(\bar{X} \cap C_{\bar{\tau}})$. Hence $\bar{X} \cap C_{\bar{\tau}} \neq \emptyset$. Contr! QED (Fact 5.)

Our cheap covering lemma now reads:

<u>Lemma 4.1</u> Assume that no $\delta < \Theta$ is Woodin in N. Then $\{\tau < \delta \mid \tau + N < \tau^+\}$ is not Woodin in Θ.

prf. Suppose not.
We assume Θ is Woodin, since otherwise there is nothing to prove. Let $\tau + N < \tau^+$ where τ is strong wrt. $\{\tau < \delta \mid \tau + N < \tau^+\}$. Let $\pi : V \to_F V'$, where F is at τ and $\tau + N' < \tau^+$, where $N' = \pi(N)$. We can then literally repeat the proof of Lemma 3. Contr! QED(4.1)

<u>Def</u> A <u>hamster</u> is a pair $\langle M, \tau \rangle$ s.t. $M = \langle J_\alpha^E, F \rangle$ is a basic premouse, $F \neq \emptyset$, $\tau \leq \lambda = F(\kappa)$ is a limit cardinal in M, where $\kappa = \mathrm{crit}(F)$, and τ generates F (i.e. $J_\alpha^E = \mathrm{Ult}(J_{\kappa+M}^E, F|\tau$

(Hence $\omega\rho^1_M \leq \tau$).

Def Let $\langle M, \tau \rangle$ be a hamiter. By an <u>iteration</u> of $\langle M, \tau \rangle$ we mean an iteration $\mathcal{I} = \langle \langle M_i \rangle, \langle \nu_i \mid i \in D \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ of M s.t. for all $i \in D$, if π_{0i} is total, then $\nu_i \notin (\tau_i, ht(M_i))$, where $\tau_i = \pi_{0i}(\tau)$. By the i-th <u>iterate</u> of $\langle M, \tau \rangle$ in $\mathcal{I}$ we mean $\langle M_i, \tau_i \rangle$ if π_{0i} is total and M_i otherwise.

Def Let $\langle M, \tau \rangle$, $\mathcal{I}$ be as above. The <u>hamiter indices</u> of the iteration are defined as follows: Set: $\lambda_i^* = \tau_i$ if π_{0i} is total and $\nu_i = ht(M_i)$; $\lambda_i^* \simeq \lambda_i$ otherwise. Set: $\nu_i^* \simeq (\lambda_i^*)^{+M_i}$. We call ν_i^* the i-th hamiter index.

Def Let $\langle M, \tau \rangle$, $\mathcal{I}$ be as above. $\mathcal{I}$ is a <u>normal</u> iteration of $\langle M, \tau \rangle$ iff

(a) $\mathcal{I}$ is standard (i.e. η_i is always maximal

(b) $T(i+1) =$ the least $\xi \in D$ s.t. $\kappa_i < \lambda_i^*$,

(c) $\lambda_i^* > \lambda_h^*$ for $h < i$

<u>Note</u> Set: $M_i^* = \langle (J_{\tau_i^+}^E)^{M_i}, E \mid \tau_i \rangle$ $\mathrm{ht}(M_i)$
if π_{0i} is total; $M_i^* = M_i$ if not.
Set: $\pi_{ij}^* = \pi_{ij} \restriction M_i^*$. Then:

(a) M_i^* is amenable.

(b) π_{ij}^* is total iff no $h+1$ s.t. $i \leq_T h+1 \leq_T j$ is a truncation pt. (i.e. $\eta_h = \mathrm{ht}(M_{T(h+1)})$ for such h),

(c) $\pi_{\xi, i+1}^* : M_\xi^* \| \eta_i \longrightarrow^*_{E^{M_i^*}} M_{i+1}^*$

for $i \in D$, $\xi = T(i+1)$.

Thus $\mathcal{Y}^* = \langle \langle M_i^* \rangle, \langle \nu_i^* \rangle, \langle \eta_i \rangle, \langle \pi_{ij}^* \rangle, T \rangle$ is an "iteration" of M^* in an obvious sense, with the difference that M_i^* need not always be a premouse. It is then obvious that $\mathcal{Y}$ is a normal iteration of $\langle M, \tau \rangle$ iff $\mathcal{Y}^*$ is normal in the usual sense. (However, we shall not need this in the sequel and hence do not verify (a)–(c).) Clearly

<u>Note</u> A normal iteration of $\langle M, \tau \rangle$ need not be a normal iteration of M.

<u>Lemma 4.2</u> Let $\mathcal{I}$ be a normal iteration of $\langle M, \tau \rangle$. Let $\xi = T(i+1)$. Then E_{ν_i} is Σ_1 amenable w.r.t. $M_\xi \| \eta_i$. (Hence $\pi_{\xi i} : M_\xi \| \eta_i \longrightarrow_{\Sigma^*} M_i$ whenever $i+1 \leq_T i$ and $\pi_{\xi i}$ is total.)

The proof is exactly like that of §4 Lemma 1. The details are left to the reader.

Our most important lemma on hamsters reads:

<u>Lemma 4.3</u> Assume that Θ is Woodin and no $\delta < \Theta$ is Woodin in N. Let $\langle M, \tau \rangle \in V_\Theta$ be a presolid normally iterable hamster in V_Θ. Then M is a mouse in V_Θ.

[<u>Note</u> $\langle M, \tau \rangle$ is <u>normally</u> <u>iterable</u> iff it possesses a normal iteration strategy in the usual sense.]

To prove this we need the notion of <u>coiteration</u> of a hamster $\langle M, \tau \rangle$ with a mouse N.

<u>Def</u> Let $\langle M, \tau \rangle$ be a hamster and N a premouse. By a <u>coiteration</u> of $\langle M, \tau \rangle$, N with <u>indices</u> $\langle \nu_i \rangle$ we mean a pair of iterations $\langle \mathcal{J}^M, \mathcal{J}^N \rangle$ of length $\theta \leq \infty$ s.t.

(a) $\mathcal{J}^M = \langle \langle M_i \rangle, \langle \nu_i^M \rangle, \ldots, T^M \rangle$ is a normal iteration of $\langle M, \tau \rangle$.

(b) $\mathcal{J}^N = \langle \langle N_i \rangle, \langle \nu_i^N \rangle, \ldots, T^N \rangle$ is a normal iteration of N.

(c) $\nu_i \simeq$ the least ν s.t. $E_\nu^{M_i} \neq E_\nu^{N_i}$ or $\nu_i = \tau_i \leq ht(N_i)$, where π_{0i}^M is total.

(d) $\nu_i^M = \begin{cases} ht(M_i) \text{ if } \pi_{0i}^M \text{ is total and } \nu_i = \tau_i, \\ \nu_i \text{ if not, and } E_{\nu_i}^{M_i} \neq \emptyset; \\ \text{otherwise undefined.} \end{cases}$

(e) $\nu_i^N = \begin{cases} \nu_i \text{ if } E_{\nu_i}^{N_i} \neq \emptyset; \\ \text{otherwise undefined.} \end{cases}$

As before, ν_i^M or ν_i^N must be defined if ν_i is defined. As $\langle M, \tau\rangle$, N are normally iterable, it is clear that the coiteration terminates.

<u>Lemma 4.3.1</u> Let $\langle M, \tau\rangle$ be a normally iterable hamster which is not a mouse. Let N be a mouse, where $M, N \in H_\delta$ and δ is regular. Then the coiteration of $\langle M, \tau\rangle$, N terminates below δ.

prf. Suppose not.

Let $\gamma > \delta$ be regular. Let $X \prec H_\gamma$ s.t. $\bar{\bar{X}} < \delta$ and $X \cap \delta$ is regular and $\mathcal{I}^M|\delta+1$, $\mathcal{I}^N|\delta+1 \in X$. Let $\sigma : \bar{H} \xrightarrow{\sim} X$ where $\bar{H}$ is transitive. Then, letting $\sigma(\bar{\delta}) = \delta$ we have $\bar{\delta} = \mathrm{crit}(\sigma)$,

$\mathcal{I}^M|\bar{\delta}, \mathcal{I}^N|\bar{\delta} = \sigma^{-1}(\mathcal{I}^M|\delta;\ \mathcal{I}^N|\delta)$.

But then:

$$(1)\ M_{\bar\delta}, \langle \pi^M_{i\bar\delta} \mid i < \bar\delta\rangle = \lim_{i \le j < \bar\delta} (M_i, \pi^M_{ij})$$

$$= \sigma^{-1}(M_\delta, \langle \pi^M_{i\delta} \mid i < \delta\rangle).$$

Similarly for $N_{\bar\delta}$, $\pi^m_{i\bar\delta}$.

But then:

(2) $\pi^M_{\bar\delta\delta} = \sigma \upharpoonright M_{\bar\delta}$, $\pi^N_{\bar\delta\delta} = \sigma \upharpoonright N_{\bar\delta}$,

since e.g. if $x \in M_{\bar\delta}$, $x = \pi_{i\bar\delta}(\bar x)$, then

$\pi^M_{\bar\delta\delta}(x) = \pi^M_{i\delta}(\bar x) = \sigma(\pi^M_{i\bar\delta}(\bar x)) = \sigma(x)$.

Now let $\bar\delta = T^M(j_M + 1)$, $j_M + 1 \leq_T \delta$ and $\bar\delta = T^N(j_N + 1)$, $j_N + 1 \leq_T \delta$. Then

(3) $\bar\delta = \mathrm{crit}(E^{M_{j_M}}_{\nu_{j_M}}) = \mathrm{crit}(E^{N_{j_N}}_{\nu_{j_N}})$ */ by (2)

(4) $\eta^M_{j_M} = \mathrm{ht}(M_{\bar\delta})$, $\eta^N_{j_N} = \mathrm{ht}(N_{\bar\delta})$,

since the maximal truncation pt. would have to be $< \bar\delta$ by (1). Hence.

(5) $(J^E_{\bar\delta+})^{M_{\bar\delta}} = (J^E_{\bar\delta+})^{M_\delta} = (J^E_{\bar\delta+})^{N_\delta} = (J^E_{\bar\delta+})^{N_{\bar\delta}}$;

~~since~~ $\nu_{\bar\delta} \geq \sup_{i<\bar\delta} \nu_i \geq \bar\delta$ if defined.

(6) $X \in (E^{M_{\bar\delta}}_{j_M})_\alpha \longleftrightarrow X \in (E^{N_{\bar\delta}}_{j_N})_\alpha$

for $\alpha < \min(\lambda^M_{j_M}, \lambda^M_{j_N})$

We consider several cases.

*/ $\mathrm{ht}(N_{\bar\delta}) > \bar\delta$, since otherwise $N_{\bar\delta}$ is a segment of $M_{\bar\delta}$ & the coiteration terminates at $\bar\delta$.

<u>Case 1</u> $\nu^M_{i_M} = \nu_{i_M}$ (hence π^M_{0i} is not total or $\nu_{i_M} < \tau_{i_M}$). We obtain a contradiction exactly as if we were coiterating two mice.

<u>Case 2</u> $\nu^M_i > \nu^M_i$ (hence π^M_{0i} is total, $\nu_{i_M} = \tau_{i_M}$ and $J^{M_{i_M}}_{\tau_{i_M}} = J^{N_{i_M}}_{\tau_{i_M}}$).

<u>Case 2.1</u> $i_N < i_M$. We get a contradiction exactly as before:

$E^N_{\nu_{i_N}} = E^{M_{i_M}}_{\nu^M_{i_M}} \mid \lambda_{i_N}$, where $\nu_{i_N} < \tau_i < \lambda^M_{i_M}$

Since $J^{E^M}_{\tau_{j_M}} = J^{E^{N_{i_M}}}_{\tau_{i_M}}$ and

$J^{E^{N_{i_M}}}_{\nu_{i_N}} = J^{E^{N_{i_M}}}_{\nu_{i_N}}$, we have:

$\langle J^{E^{M_{j_M}}}_{\nu_{i_N}}, (E^{M_{j_M}}_{\nu^M_{j_M}} \mid \lambda_{i_N}) \rangle$ is a premouse.

Hence $E^{M_{i_M}}_{\nu_{i_N}} \neq \emptyset$ by the initial segment condition.

But $E^{M_{i_M}}_{\nu_{i_N}} = E^{N_{i_M}}_{\nu_{i_N}} = \emptyset$. Contr!

QED (Case 2.1)

<u>Case 2.2</u> $i_M \leq i_N$.

We have:

$$\pi : (J^E_{\bar{\delta}^+})^{M_{i_M}} \longrightarrow_{F|\tau_{i_M}} J^E_\alpha ,$$

where $M_{i_M} = \langle J^E_\alpha , F \rangle$.

$$\pi' : (J^E_{\bar{\sigma}^+})^{M_{i_M}} \longrightarrow_{F'} J^{E'}_{\alpha'}$$

where $N_{i_N} \| \nu_{i_N} = \langle J^{E'}_{\alpha'} , F' \rangle$

Moreover $F'_\alpha = F_\alpha$ for $\alpha < \tau_{i_M}$.

Thus we can define:

$$\sigma : J^E_\alpha \longrightarrow_{\Sigma_0} J^{E'}_{\alpha'}$$

by: $\sigma(\pi(f)(\alpha)) = \pi'(f)(\alpha)$

for $f \in M_{i_M}$, $f : \bar{\sigma} \to \bar{\sigma}^+$, $\alpha < \tau_{i_M}$.

Clearly $\sigma \upharpoonright \tau_{i_M} = \mathrm{id}$. We claim

<u>Claim</u> $\sigma : M_{i_M} \longrightarrow_{\Sigma_0} \langle J^{E'}_{\alpha'}, F' \rangle$.

[<u>Note</u> This will prove the theorem, since $\omega\rho^1_{M_{i_M}} \leq \tau_{i_M}$ + hence by §8 Lemma 4 M_{i_M} is a mouse. But then M is a mouse, since $\pi_{0 i_M} : M \longrightarrow_{\Sigma^*} M_{i_M}$. Contr!]

prf.

<u>Claim</u> $\sigma(x \cap F) = \sigma(x) \cap F'$ for $x \in M_{i_M}$.

Let $\gamma < \bar{\delta}^{+M_{i_M}}$, $\lim(\gamma)$ s.t. $x \subset J^E_{\pi(\gamma)}$. Let $f : \bar{\delta} \xrightarrow{\text{onto}} \mathbb{P}(\bar{\delta}) \cap J^E_\gamma$

Then $F \cap J^E_{\pi(\gamma)} = \{ \langle \pi(f)(\nu), f(\nu) \rangle \mid \nu < \bar{\delta} \}$

$F \cap J^{E'}_{\pi'(\gamma)} = \{ \langle \pi'(f)(\nu), f(\nu) \rangle \mid \nu < \bar{\delta} \}$.

But $\sigma\pi(\gamma) = \pi'(\gamma)$.

QED (Lemma 4.3.1)

<u>Lemma 4.3.2</u> Let $\langle M, \Sigma \rangle$ be a normally iterable presolid hamster which is not a mouse. Let Q be a mouse s.t. $\overline{\overline{M}} < ht(Q) = \delta$, where δ is regular. Then the coiteration of $\langle M, \Sigma \rangle$, Q has length $\delta + 1$. If M', Q' are the coiterates, then $ht(Q') = \delta$ and Q' is a simple iterate of Q and a segment of M'.

proof.

<u>Claim 1</u> The coiteration has length $\geq \delta + 1$.

proof. Suppose not.

Let M', Q' be the coiterates. Since M, Q are presolid, the usual proof shows that we cannot truncate on both sides and that if we truncate on the Q' side, then M' is a segment of Q'. This is impossible, since then M' would be a mouse. Hence so would M, since $\pi_{MM'} : M \xrightarrow[\Sigma^*]{} M'$. Contr!

Hence Q' is a simple iterate of Q.

Hence $ht(Q') \geq \delta > ht(M')$. Hence

M' is a proper segment of Q'. But then M' is a simple iterate of M since, if there were a truncation, M' would be unsound. But then it follows as before that M is a mouse. Contr!

QED (Claim 1)

Now let:

$\mathcal{J}^0 = \langle\langle M_i^0\rangle, \langle \nu_i^0\rangle, \langle \eta_i^0\rangle, \langle \pi_{ij}^0\rangle, T^0\rangle$

be the M-side and

$\mathcal{J}^1 = \langle\langle M_i^1\rangle, \langle \nu_i^1\rangle, \langle \eta_i^1\rangle, \langle \pi_{ij}^1\rangle, T^1\rangle$

be the Q-side of the coiteration.

<u>Claim 2</u> $ht(M_\delta^1) \leq \delta$

prf. Suppose not.

Clearly $ht(M_i^1) \leq \delta$ for $i < \delta$. Let $\alpha <_T \delta$ s.t. $\pi_{\alpha\delta}^1$ is total on M_α^1 and $\delta \in rng(\pi_{\alpha\delta}^1)$. Set: $\delta_i = \pi_{i\delta}^{-1}(\delta)$ for $\alpha \leq_{T^1} i \leq_{T^1} \delta$. Then $\delta_i < \delta$ for $i <_T \delta$. Set: $\tilde{Q}_i = (U_{\delta_i^+}^E)^{M_i^1}$ for $\alpha \leq_{T^1} i \leq_{T^1} \delta$, $\tilde{\pi}_{ij} = \pi_{ij}^1 \upharpoonright \tilde{Q}_i$ $(\alpha \leq_T i \leq_T j \leq_T \delta)$.

Then $\tilde{Q}_\lambda, \langle \tilde{\pi}_{i\lambda} \rangle = \lim_{i \leq_{T1} j <_{T1} \lambda} (\tilde{Q}_i, \tilde{\pi}_{ij})$ for all limit $\lambda \in (\alpha, \delta]_T$. Pick regular $\gamma > \delta$ s.t. $\mathcal{T}^0, \mathcal{T}^1 \in H_\gamma$ + let $X \prec H_\gamma$, $\sigma : \bar{H} \overset{\sim}{\longleftrightarrow} X$, $\bar{\delta} = \delta \cap X = \sigma^{-1}(\delta)$ be as in the proof of Lemma 4.3.1.

Exactly as before we get:

$\sigma^{-1}(\tilde{M}_\delta) = \tilde{M}_{\bar{\delta}}$; $\sigma \restriction \tilde{M}_{\bar{\delta}} = \tilde{\pi}_{\bar{\delta}\delta}$.

But then we can repeat the proof of Lemma 4.3.1 to get a contradiction. Contr! QED (Claim 2)

<u>Claim 3</u> $ht(M^1_\delta) = \delta$.

pf.

Otherwise $ht(M^1_\delta) = \alpha < \delta = ht(M)$. Hence a truncation occured on the branch below δ. Let $\alpha < \beta$, where $\beta <_{T0} \delta$ and $\beta <_{T1} \delta$. Then $\pi^1_{\beta\delta} = id$, $M^1_\delta = M^1_\beta$. But M^1_β is an initial segment of M^0_β. Hence the coiteration terminates below δ. Contr!

QED (Claim 3)

But clearly $J^{E^{M^1_\delta}}_{\nu_i} = J^{E^{M^1_i}}_{\nu_i} = J^{E^{M^0_i}}_{\nu_i} = $
$= J^{E^{M^0_\delta}}_{\nu_i}$ for $i < \delta$. Hence
M^1_δ is an initial segment of M^0_δ. .

QED (Lemma 4.3)

We are now ready to prove Lemma 4.3.
Suppose not. Let $\langle M, \tau \rangle$ be a counterexamp
Pick an inaccessible $\delta > \overline{\overline{M}}$ s.t. $A \cap \delta$
is stationary in δ, where $A = \{\tau < \theta \mid \tau^{+N} = \tau^+\}$.
(δ exists by Lemma 4.1.)

<u>Claim</u> $Q = \langle J^{E^N}_\delta, \emptyset \rangle$ is a mouse.

prf.
By Cor. 2.4. $Q = N_\gamma = M_\gamma$ for a γ s.t.
$\tilde{\kappa}_\gamma = \delta$ and $\gamma \geq \delta$. But then $N_{\gamma+1} =$
$= \langle J^{E^Q}_{\delta+1}, \emptyset \rangle$ is a weak mouse,
$\omega\rho^\omega_{N_{\gamma+1}} \leq \delta < ht(N_{\gamma+1})$ and no $\overline{\tau} \in N_{\gamma+1}$
is Woodin in $N_{\gamma+1}$. Hence $N_{\gamma+1}$ is a
mouse in V_θ by Fact 3. Hence so
is $Q = N_{\gamma+1} \| \delta$. QED (Claim)

We now coiterate $\langle M, \tau \rangle$, Q. Let $\mathcal{T}^0, \mathcal{T}^1$ as in the proof of Lemma 4.3 be the coiteration. Set: $M_i = M_i^0$; $Q_i = M_i^1$. The length of the coiteration is $\delta + 1$, Q_δ is a simple iterate of Q and $\mathrm{ht}(Q_\delta) = \delta$. Q_δ is a segment of M_δ. (It follows easily that Q_δ is a proper segment.) For $\xi < \delta$ we have $\pi^1_{0\delta}(\xi) < \delta$. Let C be the set of limit ordinals λ s.t. $\lambda T^1 \delta$ and $\pi^{1\,\prime\prime}_{0\delta} \lambda \subset \lambda$. Then C is cub in δ.

For $\lambda \in C$, $i \leq j$ in $[0, \lambda]_{T^1}$, we have: $\pi^1_{ij}(\xi) \leq \pi^1_{0\lambda}(\xi) < \lambda$ for $\xi < \lambda$.

Hence $\pi^1_{i\lambda}(\lambda) = \bigcup_{i \leq_{T^1} j \leq_{T^1} \lambda} \pi^{1\,\prime\prime}_{j\lambda} \lambda = \lambda$ for $\lambda \in C$. Let $C' =$ the set of $\lambda \in C$ s.t. $\lambda T^0 \delta$ and:

$$\lambda = \mathrm{crit}(\pi^0_{\lambda\delta}) = (\pi^0_{\lambda\delta})^{-1}(\delta),$$

where $\pi^0_{\lambda\delta}$ is total. Then C' is cub in δ.

Now let $\tau \in C'$ be a limit pt of C', where τ is a cardinal in V and $\tau^{+N} = \tau^+$. Then $\pi^1_{0\tau}(\tau) = \tau$ + hence $\pi^1_{0\tau}{}''\tau^+ \subset \tau^{+Q_\tau} \leq \tau^+$. Hence $\tau^{+Q_\tau} = \tau^+$. Clearly $\mathrm{crit}(\pi^1_{\tau\delta}) \geq \tau$.

Let $\tau = T^1(i+1)$, $i+1\ T^1\ \delta$. Then since $\pi^1_{\tau, i+1}$ is total and $\kappa_i = \mathrm{crit}(\pi^1_{\tau\delta}) \geq \tau$, we have

$$\tau^+ = \tau^{+Q_\tau} = \tau^{+Q_{i+1}} = \tau^{+Q_\delta}.$$

But the same argument yields $\tau^{+M_\tau} = \tau^{+M_\delta}$. Hence $\tau^{+M_\tau} = \tau^{+Q_\delta}$, since $Q_\delta = M_\tau \| \delta$,

But $\overline{\overline{\tau^{+M_\tau}}} \leq \overline{\overline{M}}_\tau = \tau$. Hence $\tau^{+Q_\delta} < \tau^+$. Contr!

QED (Lemma 4.3)

<u>Def</u> By a <u>good hamster</u> we mean a hamster $\langle M, \tau \rangle$

(a) $\langle M, \tau \rangle$ is presolid

(b) $\langle M, \tau \rangle$ is normally iterable by a strategy S

(c) If M' is a non simple S-iterate of $\langle M, \tau \rangle$, then S is a mouse.

<u>Lemma 4.4</u> Every good hamster is uniquely normally iterable.

proof (sketch).

One must redo the whole of §6, but this is straightforward. Lemmas 1-3 go through exactly as before. (In proving Lemmas 1, 2 it is convenient to use the indices ν_i^*, λ_i^* in place of ν_i, λ_i.) Lemma 4 and its corollaries go through as before. Hence so does the proof of Lemma 5: We again assume $M_b = Q$. If there is a truncation on b, then M_j is a mouse for sufficiently large $j \in b$ and we argue as before. If not, then Q a top extender; hence $\delta = ht(Q)+1$.

Contradiction! But then corollaries 6, 6.1, 6.2 follow as before. The conclusion follows by Corollary 6.

QED (Lemma 4.4)

Def $\langle M, \tau \rangle$ is a <u>countably good hamster</u> iff $\langle M, \tau \rangle$ is a countable hamster and $H_{\omega_1} \models \langle M, \tau \rangle$ is a good hamster,

Def $\langle M, \tau \rangle$ is a <u>weakly good hamster</u> iff whenever $\sigma : \bar{M} \prec M$, $\sigma(\bar{\tau}) = \tau$ and $\bar{M}$ is countable, then $\langle \bar{M}, \bar{\tau} \rangle$ is countably good.

<u>Lemma 4.5</u> Let $\langle M, \tau \rangle \in V_\theta$ be a weakly good hamster. Then M is a mouse in V_θ.

proof.

By Lemma 4.3 we need only show that $\langle M, \tau \rangle$ is normally iterable. But this follows from Lemma 4.4 by a repetition of the proof of Fact 2.

QED (Lemma 4.5)

Def Let $A \subset V_\theta$, $\lambda < \theta$. An extender $F \in V_\theta$ is <u>elementarily λ-strong</u> wrt. A in V iff $\pi : V \xrightarrow[F]{} V'$ implies;

$$\langle V_\lambda, A \cap V_\lambda \rangle \prec \langle \pi(V_\theta), \pi(A) \rangle.$$

Def $\tau < \theta$ is <u>elementarily strong</u> wrt A in V_θ iff whenever $\langle V_\lambda, A \cap V_\lambda \rangle \prec \langle \pi(V_\theta), \pi(A) \rangle$ then there is an F at τ which is elementarily λ-strong wrt A in V_θ.

<u>Fact</u> Let θ be Woodin, $A \subset V_\theta$. There are arbitrarily large $\tau < \theta$ which are elementary strong wrt. A in V_θ.

prf.
Let T = the satisfaction relation for $\langle V_\theta, A \rangle$. Let $\kappa < \theta$ be strong wrt. T. Let $\langle V_\lambda, A \cap V_\lambda \rangle \prec \langle V_\theta, A \rangle$. Let F at κ be λ-strong wrt T. Then F is elementarily λ-strong wrt. A.

QED

Def Let F be an extender at $\kappa < \eta$ which is elementarily η-strong wrt $N = K^c$. Let $\kappa < \lambda < \eta$, where λ is a limit cardinal in N. Set: $\tilde{F} = (F|\lambda)\upharpoonright N$ and let:
$\tilde{\pi} : (J^E_{\kappa^+})^N \longrightarrow_{\tilde{F}} J^{\tilde{E}}_{\tilde{\nu}}$. We set:
$Q_{\lambda, F} = \langle J^{\tilde{E}}_{\tilde{\nu}}, \tilde{F}' \rangle$, $\pi_{\lambda F} = \tilde{\pi}$, where $\tilde{F}' = \tilde{\pi} \upharpoonright \mathbb{P}(\kappa)$.

Note In order that $Q = Q_{\lambda F}$ be defined, we must show that $\mathfrak{A} = \mathrm{Ult}((J_{\kappa^+})^N, \tilde{F})$ is well founded. Let $\pi : V \longrightarrow_F V'$ we can define $k : \mathfrak{A} \longrightarrow_{\Sigma_\omega} \pi((J^E_{\kappa^+})^N)$ by:
$k([\langle \alpha, f \rangle]) = \pi(f)(\alpha)$ for $\alpha < \lambda$, $f : \kappa \to J^E_{\kappa^+}$ in N.
Thus $\mathfrak{A} = J^{\tilde{E}}_{\tilde{\nu}}$ is well founded. But $k \upharpoonright \lambda = \mathrm{id}$. Hence $J^{\tilde{E}}_\lambda = J^E_\lambda$ $(E = E^N)$. It is clear that Q is coherent and satisfies all premouse axioms except the initial segment condition. Q is trivially presolid. Moreover Q is basic, since otherwise some $\delta < \kappa$ would be Woodin in J^E_κ, hence in N, since κ is a cardinal in N.

Lemma 4.6 Let $Q = Q_{\lambda F}$ be as above. Then Q is a mouse in V_θ.

prf.

Claim 1 Q is a premouse.

We show that the initial segment condition is vacuously true. Suppose not. Let ν be least s.t. $\langle J_\nu^{\tilde{E}}, \tilde{F}' \| \nu \rangle$ satisfies the other premouse conditions (where $Q = \langle J_{\tilde{\nu}}^{\tilde{E}}, \tilde{F}' \rangle$). Then $\nu < \tilde{\nu}$ and $Q' = \langle J_\nu^{\tilde{E}}, \tilde{F}' \| \nu \rangle$ is a premouse. Then, letting λ' = the largest cardinal in $J_\nu^{\tilde{E}}$ we have $\tilde{F}' \| \nu = \tilde{F}' | \lambda'$. Hence $\lambda' < \lambda$, since $\tilde{F}' | \lambda = \tilde{F} | \lambda$ generates $\tilde{F}'$. Hence $\nu < \lambda'^{+Q} < \lambda$, $J_\nu^{\tilde{E}} = J_\nu^E$ $(E = E^N)$, and $\tilde{F}' \| \nu = \tilde{F} | \lambda'$. Set $F' = \tilde{F} | \lambda'$. There is $k : J_\nu^E \xrightarrow{}_{\Sigma_0} J_{\tilde{\nu}}^{\tilde{E}}$ defined by $k(\pi'(f)(\alpha)) = \tilde{\pi}(f)(\alpha)$, where $\pi' : (J_{\kappa^+}^E)^N \longrightarrow_{F'} J_\nu^E$.

Then $k \restriction \lambda' = \text{id}$. Hence $\pi'(f)(\alpha) = \tilde{\pi}(f)(\alpha)$ for $\alpha < \lambda'$, $f \in J_{\kappa}$ in N.

This holds in particular for $f = \langle \xi^+ | \xi < \kappa \rangle$. Hence λ' is a limit cardinal in N and $\lambda' < \nu < \lambda'^{+N}$ where ν is a limit ordinal. Hence there is γ s.t. $J^E_\lambda = M_\gamma$ and $\tilde{\pi}_\gamma = \lambda$. Moreover there is $\gamma' > \gamma$ s.t. $J^E_\nu = M_{\gamma'}$. But F' is certifiable, since $F' = (F|\lambda') \upharpoonright M_{\gamma'}$, where $\langle H_{\kappa^+}, F \rangle$ is a background certificate with $\mathbb{P}(\kappa) \subset H_{\kappa^+}$. Hence $M_{\gamma'+1} = Q'$. But then $\omega\rho^\omega_{M_{\gamma'+1}} < \lambda' \leq \tilde{\pi}_\gamma$. Contr!

QED (Claim 1)

Q is clearly presolid and basic. Moreover $\langle Q, \lambda \rangle$ is a hamster. In order to show that Q is a mouse it suffices by Lemma 4.5 to prove:

<u>Claim 2</u> $\langle Q, \lambda \rangle$ is a weakly good hamster.

proof. (sketch)

Let $\sigma : \bar{Q} \prec Q$, $\sigma(\bar{\lambda}) = \lambda$, where

$\bar{Q}$ is countable. We must produce a countable iteration strategy S for $\langle\bar{Q},\bar{\lambda}\rangle$ and show that, if $\bar{Q}'$ is a non simple S-iterate of $\langle\bar{Q},\bar{\lambda}\rangle$, then $\bar{Q}'$ is countably iterable. Since λ is a limit cardinal in N, there is $\gamma \geq \lambda$ s.t. $J_\lambda^E = M_\gamma$ and $\tilde{\kappa}_\gamma \geq \lambda$. We know $J_\lambda^E = J_\lambda^{\tilde{E}}$.

We claim:

(1) $J^E_{\lambda^{+Q}} = J^{\tilde{E}}_{\lambda^{+Q}}$ (hence $\lambda^{+Q} \leq \lambda^{+N}$).

proof.

Let $\lambda < \xi < \lambda^{+Q}$ s.t. $\omega\rho^\omega_{Q\|\xi} = \lambda$. Recall the map $k: J^{\tilde{E}}_{\tilde{\nu}} \longrightarrow \pi(J^E_{\kappa+N})$ defined above. Then:

$k \restriction (Q\|\xi) : Q\|\xi \prec Q^*$, where $Q^* = \pi(J^E_{\kappa+N}) \| k(\xi)$ is a sound mouse and $Q\|\xi$ is sound. By §8 Lemma 4 there are three possibilities:

(a) $Q\|\xi = \mathrm{core}(Q^*) = Q^*$, $k = \mathrm{id}$

(b) $Q\|\xi$ is a proper segment of Q^*

(c) λ is a successor cardinal in $Q\|\xi$.

(c) is impossible. If (b) holds, then $Q \| \xi$ is a segment of J_η^E, since $\eta > \lambda^{+N}$ and $\pi(E) \cap V_\eta = E \cap V_\eta$. Hence $Q \| \xi$ is a segment of $J_{\lambda^{+N}}^E$.

If (a) holds, then $k(\lambda) = \lambda$ + hence $k \upharpoonright \lambda^{+Q} = \mathrm{id}$. QED (1)

But then there is γ s.t. $J_{\lambda^{+Q}}^E = M_\gamma$.

Set: $N'_\xi = N_\xi$ for $\xi \leq \gamma$ and $N'_{\gamma+1} = Q$. We use the sequence $\vec{N}' = \langle N'_\xi \mid \xi \leq \gamma+1 \rangle$ the way we used $\vec{N}$ before. We have:

$$\sigma : \bar{Q} \xrightarrow[\Sigma^*]{} N'_{\gamma+1} \min(\vec{\rho}),$$

where $\vec{\rho} = \min(N'_{\gamma+1}, \sigma, \langle \rho^n_{N'_{\gamma+1}} \mid n < \omega \rangle)$

We construct a countable normal iteration strategy S for $\langle \bar{Q}, \bar{\lambda} \rangle$ s.t. if $\bar{Q}'$ is an S-iterate of $\bar{Q}$, then there is σ' s.t.

(a) $\sigma' : \overline{Q}' \xrightarrow[\Sigma^*]{} N'_\xi \ \min(\vec{p}')$,

(b) If $\overline{Q}'$ is a non simple iterate, then $\xi < \gamma+1$ (hence $\overline{Q}'$ is a countably iterable mouse since $N_\xi = N'_\xi$ is a weak mouse).

(c) If $\overline{Q}'$ is a simple iterate, then $\sigma' \pi_{\overline{Q}Q'} = \sigma$ and $\rho'_i \leq \rho_i$ for $i < \omega$.

This gives the desired result. We indicate roughly how we would modify the proof in §10 (if we were working in that setting rather than with background certificates). The first step is to define $S(\mathcal{I})$, where $\mathcal{I}$ is a countable normal iteration of $\langle \overline{Q}, \overline{\lambda} \rangle$ of length Γ. We again construct a coarse structural iteration $\langle \langle U_i \rangle, \langle F^*_i \rangle, \langle \tilde{\pi}_{ij} \rangle, T' \rangle$ and maps

$\sigma_i : \overline{Q}_i \xrightarrow[\Sigma^*]{} \tilde{Q}_i \ \min(\vec{p}^i)$ where $\tilde{Q}_i = \tilde{\pi}_{0i}(\vec{N}')_{\xi_i}$

for an $\xi_i \leq \tilde{\pi}_{0i}(\gamma+1)$. We have (a)-(c) holding as before. [We note that if $\nu \leq \lambda$ and $E^Q_\nu \neq \emptyset$, then in fact

$\nu \in M_\gamma$ and $\beta(Q,\nu) = \beta(M_\gamma, \nu)$. Hence there is exactly one $\eta < \gamma$ s.t. $Q \| \beta = M_\eta$ for $\beta = \beta(Q,\nu)$. Thus as long as we avoid points $\nu \in (\lambda, \tilde{\gamma})$, we can form the trace of ν in order to resurrect a background extender for E_ν. The same holds of $Q' = \pi_{U_i}(Q) = \pi_{U_i}(N')_{\pi(\gamma)}$ in U_i. Thus we can define the function involved in (a)–(c).] If δ_i is defined for $i < \Gamma$, we define $S(\mathcal{Y})$ exactly as before. Otherwise $S(\mathcal{Y})$ is undefined. We then verify as before that S has the desired property. QED (Lemma 4.6)

We are now ready to prove Lemma 4. Let $a \in L[N]$, $a \subset \Theta$. We must show that some $\kappa < \Theta$ is strong wrt a in N (assuming, of course, that no $\delta < \Theta$ is Woodin in N, hence $L[N]$). Let B = the set of λ s.t. for some $\kappa < \lambda$ there is an extender F at κ which is λ-strong wrt. a. It suffices to show:

<u>Claim 1</u> B is stationary in Θ.

(Hence, choosing $F = F_\lambda$, $\kappa_\lambda = \mathrm{crit}(F_\lambda)$ for each $\lambda \in B$, $\langle \kappa_\lambda \mid \lambda \in B \rangle$ is regressive. Hence there is κ s.t. $\{\lambda \mid \kappa_\lambda = \kappa\}$ is stationary in Θ. Hence κ is strong wrt. a in V_Θ.)

Now pick κ s.t. κ is elementarily strong wrt. a, E, where $N = L^E_\Theta$. It suffices by Lemma 4.1 to show:

<u>Claim 2</u> Let $\lambda > \kappa$ be inaccessible s.t. $\lambda^+ = \lambda^{+N}$. Then $\lambda \in B$.

From now on fix such a λ. Let $\lambda' > \lambda$ have the same properties. Let $\lambda' < \eta$ s.t. $\langle V_\eta, E \cap V_\eta, a \cap \eta \rangle \prec$
$\prec \langle V_\Theta, E, a \rangle$.

Let F at κ be elementarily η-strong wrt. E, a. Consider $Q = Q_{\lambda, F}$, $Q' = Q_{\lambda', F}$. Then Q, Q' are mice in V_θ. Let:
$Q = \langle J_\gamma^{\tilde{E}}, \tilde{F} \rangle$, $Q' = \langle J_{\gamma'}^{\tilde{E}'}, \tilde{F}' \rangle$. Then $\tilde{F}|\lambda = \bar{F}$, $\tilde{F}'|\lambda' = \bar{F}'$ where $\bar{F} = (F|\lambda) \upharpoonright N$, $\bar{F}' = (F|\lambda') \upharpoonright N$. There are $\pi : J_{\tau+N}^E \xrightarrow[\bar{F}]{} J_\gamma^{\tilde{E}}$, $\pi' : J_{\tau+N}^E \xrightarrow[\bar{F}']{} J_{\gamma'}^{\tilde{E}}$.
Clearly $J_\lambda^{\tilde{E}} = J_\lambda^E$, $J_{\lambda'}^{\tilde{E}'} = J_{\lambda'}^E$,
$\tilde{F} = \pi \upharpoonright \mathbb{P}(\kappa)$, $\tilde{F}' = \pi' \upharpoonright \mathbb{P}(\kappa)$,
We can define $k : J_\gamma^{\tilde{E}} \xrightarrow[\Sigma_\omega]{} J_{\gamma'}^{\tilde{E}'}$ by $k(\pi(f)(\alpha)) = \pi'(f)(\alpha)$ for $\alpha < \lambda$, $f : \kappa \longrightarrow J_{\kappa^+}^E$ in N. Note that $k \upharpoonright \lambda = \mathrm{id}$.

<u>Claim</u> $k : Q \xrightarrow[\Sigma_0]{} Q'$.

proof.
It suffices to show: $k(x \cap \tilde{F}) = k(x) \cap \tilde{F}'$,

Let $x \subset J_{\xi}^{\tilde{E}}$, $\xi = \pi(\bar{\xi}) < \nu$. Then $\bar{\xi} < \kappa + \omega$. Then $\tilde{F} \cap J_{\xi}^{\tilde{E}} = \tilde{F} \upharpoonright J_{\bar{\xi}}^{E}$. Let $f \in N$ s.t. $f : n \longleftrightarrow \mathcal{P}(\kappa) \cap J_{\bar{\xi}}^{E}$. Then $\tilde{F} | J_{\bar{\xi}}^{E} =$
$= \{\langle \pi(f)(i), f(i) \rangle \mid i < n\}$. Hence
$k(\tilde{F} | J_{\bar{\xi}}^{E}) = \{\langle \pi'(f)(i), f(i) \rangle \mid i < n\} =$
$= \tilde{F}' \upharpoonright J_{\bar{\xi}}^{E}$. Hence $k(x \cap \tilde{F}) =$
$k(x \cap (\tilde{F} \upharpoonright J_{\bar{\xi}}^{E})) = k(x) \cap (\tilde{F}' \cap J_{\bar{\xi}}^{E}) =$
$= k(x) \cap \tilde{F}'$. QED (Claim).

But Q is sound above λ and $\omega\rho_Q^1 \leq \lambda$, since $\tilde{F} | \lambda$ generates $\tilde{F}$.

By §8 Lemma 4 it follows that one of three possibilities holds:

(a) $Q = \mathrm{core}_{\lambda}(Q')$ and k is the core map.

(b) Q is a proper segment of Q'.

(c) λ is not a limit cardinal in Q.

(c) is obviously false. Hence there remain two cases:

Case 1 (b) holds.

Then Q is an initial segment of $Q' \| \lambda' =$
$= J^E_{\lambda'}$, since $\lambda' > \lambda$ is a cardinal and
$\bar{\bar{Q}} = \lambda$. Hence $\tilde{F} = E_\nu$. Hence $N \models \tilde{F}$
is ω-complete. Hence $N \models F$ is ω-complete since $\bar{F} = \tilde{F} | \lambda$. Let
$\pi^* : L^E \longrightarrow_{\bar{F}} L^{E'}$. Then $\pi =$
$= \pi^* \upharpoonright J^E_{\kappa+N}$ and hence $J^E_\lambda \subset J^{\tilde{E}}_\nu \subset L^{E'}$,
where $J^E_\lambda = V^N_\lambda$. Hence $\bar{F}$ is
λ-strong. It remains to show:

<u>Claim</u> $\pi^*(a) \cap \lambda = a \cap \lambda$.

prf.
$\pi^*(a) \cap \lambda = \pi^*(a \cap \kappa) \cap \lambda = \pi(a \cap \kappa) \cap \lambda$.
Let $\pi'' : V \longrightarrow_F V'$. There
is $\sigma : J^{\tilde{E}}_\nu \longrightarrow \pi''(J^E_{\kappa+N})$ defined
by: $\sigma(\pi(f)(\alpha)) = \pi''(f)(\alpha)$ for
$\alpha < \lambda$, $f : \kappa \to J^E_{\kappa+}$ in N. But
$\sigma \upharpoonright \lambda = \mathrm{id}$. Hence $\pi(a \cap \kappa) \cap \lambda =$
$= \sigma\pi(a \cap \kappa) \cap \lambda = \pi''(a \cap \kappa) \cap \lambda = a \cap \lambda$,

since F is η-strong w.r.t. a and $\pi''(\kappa) > \lambda$

Hence $\pi^*(a) \cap \lambda = a \cap \lambda$. QED (Claim)

Thus $\bar{F} \in N$ witnesses Claim 2.

QED (Case 1)

<u>Case 2</u> (a) holds.

The map $k : Q \to_{\Sigma_0} Q'$ enables us to coiterate $\langle Q, Q', \lambda \rangle$ against Q' in a double rooted iteration. Let $\langle \mathcal{I}, \mathcal{I}' \rangle$ be the coiteration of length δ. Let

$\mathcal{I} = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{i_1 i} \rangle, T \rangle$

$\mathcal{I}' = \langle \langle Q_i' \rangle, \langle \nu_i' \rangle, \langle \eta_i' \rangle, \langle \pi_{i_1 i}' \rangle, T' \rangle$.

Let κ_i, κ_i' be the critical points.

Then Case 2.1 of the proof of §8 Lemma 4 holds, since otherwise (a) would fail. Hence $\delta \geq_T 0$ and Q_δ is a simple iterate of Q. Moreover $Q_\delta' = Q_\delta$ is a simple iterate of Q'. Then:

(1) $\kappa_i' < \lambda$ for some $i \leq_T \delta$.

proof.

Suppose not. Then

$\lambda^{+Q_\delta} = \lambda^{+Q'} = \lambda^+$ (since $\lambda^+ = \lambda^{+N \| \lambda'}$)

But $\kappa_i \geq \lambda$ for $i \leq_T \delta$. Hence $\lambda^{+Q_\delta} = \lambda^{+Q} < \lambda^+$, since $\overline{\overline{Q}} = \lambda$, Contr!

QED(1)

Let $i+1 \leq_T \delta$ be least s.t. $E^{Q'_i}_{\nu'_i} \neq \emptyset$. Let $\xi = T(i+1)$. Then $Q'_\xi = Q'$. Moreover $\kappa'_i = \text{crit}(\pi'_{0\delta})$ and $\kappa'_i < \lambda$. But

$\kappa'_i > \kappa$, since $\kappa = \text{crit}(F^*)$, where $F^* =$ the top extender of Q_δ (since $\kappa_i \geq \lambda > \kappa$ for $i <_T \delta$). Set:

$G = E^{Q'_i}_{\nu'_i}$.

(2) $G(a \cap \kappa_i) \cap \lambda = a \cap \lambda$.

Proof.

In Case 1 we saw: $\pi(a \cap \kappa) \cap \lambda = a \cap \lambda$

Similarly $\pi'(a \cap \kappa) \cap \lambda' = a \cap \lambda'$.

Thus $\pi'_{0\delta}(a \cap \kappa_i) = \pi'_{0\delta}(\tilde{F}(a \cap \kappa) \cap \kappa_i) =$

$= F^*(a \cap \kappa) \cap \pi'_{0\delta}(\kappa_i)$, since

$\tilde{F} = \pi \upharpoonright \mathbb{P}(\kappa)$, $\tilde{F}' = \pi' \upharpoonright \mathbb{P}(\kappa)$,

But $\alpha \in G(X) \leftrightarrow \alpha \in \pi'_{0\delta}(X)$ for $X \in \mathbb{P}(\kappa_i) \cap N$, $\alpha < \lambda$. Then

$G(a \cap \kappa_i) \cap \lambda = F^*(a \cap \kappa) \cap \lambda =$

$\pi_{0\delta}(\tilde{F}(a \cap \kappa)) \cap \lambda = \tilde{F}(a \cap \kappa) \cap \lambda = a \cap \lambda$,

since $\pi_{0\delta} \upharpoonright \lambda = \text{id}$. QED(2)

Set: $\bar{G} = G|\lambda$. We have shown:

(3) $\bar{G}(a \cap \kappa_i) = a \cap \lambda$.

We now prove:

(4) $\bar{G} \in Q'$.

proof.

Case A $\bar{G} \in Q'_i$.

If $Q'_i = Q'$, we are done. If not, let j be least s.t. $Q'_{j+1} \neq Q'_j$. Then $Q'_j = Q'$ and $\lambda_j > \lambda$ is a cardinal in Q'_i. Hence

$\bar{G} \in J^{E^{Q'_i}}_{\lambda_j} = J^{E^{Q'_j}}_{\lambda_j} \subset Q'$

Case B Case A fails.

Then G is the top extender of Q'_i. Hence π'_{0i} is not total, since otherwise $\kappa_i = \mathrm{crit}(G) = \pi'_{0i}(\kappa)$. Hence $\mathrm{crit}(\pi'_{0i}) \ngtr \kappa$, since $\kappa_i > \kappa$, But then, since $\lambda_j > \lambda$ for all j, we have $\kappa_i = \pi'_{0i}(\kappa) > \lambda$. Contr!

Let $j+1 \leq_T i$ be the last truncation point. Let $\zeta = T'(j+1)$. Then

$\pi'_{\zeta i} : Q'_\zeta \| \eta_\zeta \xrightarrow[\Sigma^*]{} Q'_i$ and $\mathrm{crit}(\pi'_{\zeta i}) = \kappa_j$.

But $\kappa_j \geq \lambda$, since otherwise $\kappa_j^+ Q'_\zeta = \kappa_j^+ J^{E^{Q'_\zeta}}_\lambda = \kappa_j^+ J^{E^{Q'_j}}_\lambda = \kappa_j^+ J^{E^{Q'_j}}_{\nu_j}$, so since λ is a cardinal in Q'_ζ, Q'_i.

But then $\eta_1 = ht(Q_3')$. Contradiction.

Hence, letting $\tilde{G}$ be the top extender of $Q_3' \| \eta_3$, we have $\tilde{G} \in Q_3'$ + hence $\bar{G} = \tilde{G} | \lambda \in Q'$ by the argument of Case A. QED(4).

We now show that $\bar{G}$ verifies Claim 2. Clearly $\bar{G} \in (J^E_{\lambda^+})^{Q'} = (J^E_{\lambda^+})^N \subset N$. But $\mathrm{Ult}_N((J^E_{\kappa^+})^N, \bar{G})$ is well founded, since $\mathrm{Ult}((J^E_{\kappa^+})^W, G)$ is. Hence $N \models \bar{G}$ is ω-complete. Hence there is $\pi^* : L[N] \xrightarrow[\bar{G}]{} L[N']$.

But $\pi^*(a) \cap \lambda = \pi^*(a \cap \kappa_i) \cap \lambda =$
$= \bar{G}(a \cap \kappa_i) = a \cap \lambda$. Moreover if $\tilde{E} \in \Theta$ canonically codes $E = E^N$, then $\pi^*(\hat{E}) \cap \lambda = \pi^*(\tilde{E} \cap \kappa_i) \cap \lambda = \bar{G}(\tilde{E} \cap \kappa_i)$
$= \hat{E} \cap \lambda$. Hence in $L[N]$ we have:

$$V_\lambda = J_\lambda^{\tilde{E}} = J_\lambda^{\pi(\hat{E})} \subset L[N'].$$

QED (Lemma 4)