Σ_0 iterations

We now apply our methods to the Σ_0 iterations described at the end of §4.

Def Let $\sigma : \bar{M} \to M$ where $\bar{M}, M$ are premice. Let $\bar{\mathcal{I}}$ be a Σ_0 iteration of $\bar{M}$ of length θ, where $\bar{\mathcal{I}} = \langle\langle \bar{M}_i \rangle, \langle \bar{\nu}_i \mid i \in D \rangle, \langle \bar{\eta}_i \rangle, \langle \bar{\pi}_{ij} \rangle, T \rangle$. We say that $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ is the Σ_0-copy of $\bar{\mathcal{I}}$ onto $\bar{M}$ by σ with Σ_0-copying maps $\langle \sigma_i \rangle$ iff

(a) $\mathcal{I} = \langle\langle M_i \rangle, \langle \nu_i \mid \in D \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ is a Σ_0 iteration of M of length θ.

(b) $\sigma_i : \bar{M}_i \to_{\Sigma_0} M_i$ is a cardinal preserving map; $\sigma_0 = \sigma$; $\sigma_i \bar{\pi}_{ih} = \pi_{hi} \sigma_h$ if $h \leq_T i$,

(c) If $i = h+1$, $h \notin D$, then $\sigma_h(\bar{\eta}_h) = \eta_h$ and $\sigma_{i+1} = \sigma_h \upharpoonright \bar{M}_i$

(d) Let $i = h+1$, $h \in D$. Then;

(i) $\sigma_h(\bar{\nu}_h) = \nu_h$

(ii) Let $\zeta = T(i)$. Then $\sigma_\zeta(\bar{\eta}_h) = \eta_h$

(iii) Let $\bar{M}^* = \bar{M}_\zeta \| \bar{\eta}_h$, $M^* = M_\zeta \| \eta_h$, $\sigma^* = \sigma_\zeta \restriction \bar{M}^*$. If i is not simple in $\bar{\mathcal{Y}}$, then $\sigma^* : \bar{M}^* \longrightarrow_{\Sigma_0^{(n)}} M^*$ whenever $\omega\rho^n_{\bar{M}^*} > \bar{\kappa}_h$

(iv) Let $\bar{F} = E^{\bar{M}_h}_{\bar{\nu}_h}$, $F = E^{M_h}_{\nu_h}$. Then $\langle \sigma^*, \sigma_h \restriction \bar{\lambda}_h \rangle : \langle \bar{M}^*, \bar{F} \rangle \longrightarrow \langle M^*, F \rangle$

(v) $\sigma_i(\bar{\pi}_{\zeta i}(f)(\alpha)) = \pi_{\zeta i}\sigma^*(f)(\sigma_h(\alpha))$ whenever $\alpha < \bar{\lambda}_h$, $f \in \bar{M}^*$, $f : \bar{\kappa}_h \to \bar{M}^*$,

(i) $\sigma_i(\bar{\pi}_{\zeta i}(f)(\alpha)) = \pi_{\zeta i}\sigma^*(f)(\sigma_h(\alpha)$ whenever $\alpha < \bar{\lambda}_h$, i is not simple in $\bar{\mathcal{Y}}$, and $f \in \Gamma(\bar{\kappa}_h, \bar{M}^*)$, In place of Lemma 1 we have:

<u>Lemma 5</u> Let $\sigma : \bar{M} \to_{\Sigma_0} M$ be cardinal preserving. Let $\bar{I}$ be a normal Σ_0 iteration of $\bar{M}$ s.t. $I = \sigma(\bar{I})$, $\langle \sigma_i \rangle$ exist. If $i < |\bar{I}|$ is non simple in $\bar{I}$, then $\sigma_i : \bar{M}_i \to_{\Sigma_0^{(n)}} M_i$ whenever $\omega\rho^n_{\bar{M}_i} \geq \sup_{h \in Dni} \bar{\nu}_h$.

The proof is virtually the same (using the fact that if $i = h+1$ is non simple, $\xi = T(i)$, and $\bar{\eta}_h = ht(\bar{M}_\xi)$, then ξ is non simple.)

In place of Lemma 2.1 we then have:

<u>Lemma 6.1</u> Let S be a normal Σ_0 iteration strategy for M (above, beyond $\nu = \sigma(\bar{\nu})$) and let $\bar{S}$ be the
Let $\sigma : \bar{M} \to_{\Sigma_0} M$
be cardinal preserving
and let $\bar{S}$ be the derived strategy. Let $\bar{I}$ be a normal Σ_0 $\bar{S}$-iteration of $\bar{M}$ (above, beyond ν). Then $I = \sigma(\bar{I})$ exists and is a Σ_0 S-iteration of M.

The proof is again virtually identical to the earlier one.

Exactly as before:

<u>Cor 6.2</u> Let $\sigma : \bar{M} \rightarrow_{\Sigma_0} M$ be cardinal preserving + let S be a normal Σ_0 iteration strategy for M (above, beyond $\nu = \sigma(\bar{\nu})$). The derived strategy $\bar{S}$ is then a normal Σ_0 iteration strategy for $\bar{M}$ (above, beyond $\bar{\nu}$).

The normal Σ_0 uniqueness property is defined as before + we again have that M is uniquely iterable iff M has an it. strategy + has the uniqueness property.

. . . .

<u>Def</u> Let $\sigma : \bar{M} \rightarrow M$, where $\bar{M}, M$ are premice. Let $\bar{\mathcal{J}}$ be a Σ_0 iteration of $\bar{M}$ of length θ. We say that $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ is a <u>full copy of</u> $\bar{\mathcal{J}}$ <u>onto</u> $\bar{M}$ by σ with Σ_0-<u>copying maps</u> $\langle \sigma_i \rangle$ iff (b), (c), (d) of the earlier definition hold together with:

(a) $\mathcal{J}$ is a generalized iteration of M of length θ.

(In other words, $\mathcal{I}$ employs $*$-ultraproducts everywhere and $\bar{\mathcal{I}}$ employs $*$-ultraproducts only in forming $\bar{M}_i$ for non simple $i = h+1$.). Then Lemmas 5, 6.1, 6.2 hold in an appropriate form for full copies. (I.e. Lemma 6.1 says that if S is a normal it. strategy for M, $\sigma: \bar{M} \to_{\Sigma_0} M$ is cardinal preserving, $\bar{S}$ is the derived Σ_0-strategy, and $\bar{\mathcal{I}}$ is a normal Σ_0 $\bar{S}$-iteration of $\bar{M}$, then the full copy $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ exists and is a normal S-iteration of M.)

(The "full copy" version of Cor 6.2 tells us among other things that if M is normally iterable, then M is normally Σ_0-iterable, since if S is a normal strategy for M, then the derived Σ_0 strategy $\bar{S}$ by $id: M \to M$ is a normal Σ_0 strategy for M.)

In place of Lemma 3 we have:

<u>Lemma 7</u> Let $\sigma : \bar{M} \xrightarrow[\Sigma_0]{} M$ be cardinal preserving. Let $\bar{\mathcal{I}}$ be a normal Σ_0-iteration of $\bar{M}$ s.t. that the Σ_0 copy $\mathcal{I} = \sigma(\bar{\mathcal{I}})$ exists with copying maps $\langle \sigma_i \rangle$. Then $\sigma_i : \bar{M}_i \xrightarrow[\Sigma^*]{} M_i$ whenever i is non simple in $\bar{\mathcal{I}}$.

This follows by induction on i from:

<u>Lemma 7.1</u> Let $\sigma, \bar{M}, M$ etc. be as above. If $i \in D$, $\xi = T(i+1)$, $\bar{M}^* = \bar{M}_\xi \| \bar{\eta}_i$, $M^* = M_\xi \| \eta_i$, $\sigma^* = \sigma_\xi \upharpoonright \bar{M}^*$, $\bar{F} = E^{\bar{M}_i}_{\bar{\nu}_i}$, $F = E^{M_i}_{\nu_i}$, then:

$\langle \sigma^*, \sigma_i \upharpoonright \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \xrightarrow{*} \langle M^*, F \rangle$.

(<u>Note</u> Lemma 7.1 does <u>not</u> hold for the full copy $\mathcal{I} = \sigma(\bar{\mathcal{I}})$.)

We prove this essentially as before. We again assume that $\bar{\mathcal{I}}$ is direct & define $\hat{n}_i, \hat{\tau}_i, \sigma_i, \hat{\eta}_i, \hat{\mu}_i, \dots, \hat{\bar{\eta}}_i$ as before. We then show:

Lemma 7.2 Let δ_i exist. Let $\bar{A} \subset \hat{\bar{\tau}}_i$, $A \subset \hat{\tau}_i$ s.t. A is $\Sigma_1(\bar{M}_i)$ in $\bar{p}$ and A is $\Sigma_1(M_i)$ in $p = \sigma_i(\bar{p})$ by the same definition. Then there is $\bar{q} \in \bar{M}_{\delta_i} \| \hat{\bar{\eta}}_i$ s.t. $\bar{A}$ is $\Sigma_1(\bar{M}_{\delta_i} \| \hat{\bar{\eta}}_i)$ in $\bar{q}$ and A is $\Sigma_1(M_{\delta_i} \| \hat{\eta}_i)$ in q by the same definition.

The proof is a virtual repeat of Lemma 3.2.

Cor 7.3 Let $\sigma : \bar{M} \longrightarrow_{\Sigma_0} M$ be cardinal preserving. Let $\bar{\mathcal{Y}}$ be a good Σ_0 iteration of $\bar{M}$ (above, beyond $\nu = \sigma(\bar{\nu})$). Then:

(a) If M has a good Σ_0 iteration strategy S (above, beyond ν) and $\bar{S}$ is the derived strategy, then $\bar{S}$ is a good Σ_0 strategy for $\bar{M}$ (above, beyond $\bar{\nu}$). Moreover, if $\bar{\mathcal{Y}}$ is an $\bar{S}$-iteration, then $\mathcal{Y} = \sigma(\bar{\mathcal{Y}})$ exists + is a good Σ_0 S-iteration,

(b) If $\mathcal{Y} = \sigma(\bar{\mathcal{Y}})$ exists with copying maps $\langle \sigma_i \rangle$, then:

(i) $\sigma_i : \bar{M}_i \longrightarrow_{\Sigma^*} M_i$ if i is non simple

(ii) Let $i \in D$, $\xi = T(i+1)$. Set: $\bar{M}^* = \bar{M}_\xi \| \bar{\eta}_i$, $M^* = M_\xi \| \eta_i$, $\sigma^* = \sigma_\xi \restriction \bar{M}^*$, $\bar{F} = E^{\bar{M}_i}_{\bar{\nu}_i}$, $F = E^{M_i}_{\nu_i}$. Then

$$\langle \sigma^*, \sigma_i \restriction \bar{\lambda}_i \rangle : \langle \bar{M}^*, \bar{F} \rangle \longrightarrow^* \langle M^*, F \rangle,$$

Cor 7.4 If $\sigma : \bar{M} \longrightarrow_{\Sigma_0} M$ is cardinal preserving where M has a Σ_0 iteration strategy and $\bar{M}$ has the Σ_0 uniqueness property, then $\bar{M}$ is uniquely iterable.

We have seen that a normal iteration strategy for M induces a normal Σ_0 iteration strategy for $\bar{M}$ if $\sigma : \bar{M} \to_{\Sigma_0} M$ is cardinal preserving. Our methods do not, however yield a proof of the same theorem for good Σ_0 iterations, since Lemma 7 appears not to hold for full copies. It is apparent however that we can get:

<u>Lemma 7.5</u> Let $\sigma : \bar{M} \to_{\Sigma_0} M$ be cardinal preserving. Let S be a good iteration strategy (for M). Let $\bar{S}$ be the derived Σ_0 strategy for $\bar{M}$ + let $\bar{\mathcal{J}} = \langle\langle \bar{M}_i \rangle, \ldots, T\rangle$ be a good Σ_0 $\bar{S}$-iteration of $\bar{M}$ of length $\theta + 1$ s.t. every $i \, T \, \theta$ is simple in $\bar{\mathcal{J}}$. Then the full copy $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ exists and is a good S iteration of M.

<u>Lemma 7.6</u> Let $\sigma, \bar{M}, M, S, \bar{S}$ be as above. Let $\bar{\mathcal{J}} = \langle\langle \bar{M}_i \rangle, \dots, T\rangle$ be a good Σ_0 $\bar{S}$-iteration of $\bar{M}$ with marking sequence $\langle \alpha_i \mid i \leq \Gamma \rangle$. If α_i is simple in $\bar{\mathcal{J}}$ for $i < \Gamma$, then the full copy $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ exists ~~and~~ is a good S-iteration of M.

<u>Lemma 7.7</u> Let $\sigma, \bar{M}, M, S, \bar{S}$ be as above. Let $\bar{\mathcal{J}} = \langle\langle \bar{M}_i \rangle, \dots, T\rangle$ be a good Σ_0 $\bar{S}$-iteration of $\bar{M}$ with marking sequence $\langle \alpha_i \mid i \leq \Gamma \rangle$. If there is $i < \Gamma$ s.t. α_i is simple in $\bar{\mathcal{J}}$ and $\alpha_i + 1$ is not, then the full copy $\mathcal{J} = \sigma(\bar{\mathcal{J}})$ exists.

(The full copy $\mathcal{J}' = \sigma(\bar{\mathcal{J}} | \alpha_i + 2)$ exists by 7.5. Let $\langle \sigma_j \mid j \leq \alpha_i + 1 \rangle$ be the copying maps. Then $\sigma_{\alpha_i + 1} : \bar{M}_{\alpha_i + 1} \to_{\Sigma^*} M_{\alpha_i + 1}$ and the remainder of $\bar{\mathcal{J}}$ is a good iteration of $\bar{M}_{\alpha_i + 1}$

Σ_0 - uniqueness mice

For the moment we refer to these as Σ_0 mice.

By a Σ_0 uniqueness mouse we mean a uniquel Σ_0-iterable premouse (i.e. the uniqueness strategy is a good Σ_0 strategy). Clearly any non simple iterate of a Σ_0 mouse is a mouse.

The Dodd-Jensen lemma for Σ_0 mice reads:

Lemma 8. Let M be a Σ_0 mouse. If N is a Σ_0 iterate of M with iteration map π and $\sigma : M \xrightarrow[\Sigma_0]{} N$, then N is a simple Σ_0 iterate of M and $\sigma(\xi) \geq \pi(\xi)$ for $\xi \in M$.

proof.
We first show that N is not a non simple iterate of M. Suppose not. Then N is a mouse. Define a relation on mice by: $N' R N$ iff N' is a

nonsimple iterate of N. Let N be R-minimal for the property: There is a Σ_0 mouse M s.t. N is a non simple Σ_0 iterate of M and there is $\sigma : M \xrightarrow[\Sigma_0]{} N$. Then σ is cofinal in N, hence cardinal preserving (otherwise we could truncate N to $N' R N$ with the same property). Let

$\mathcal{Y} = \langle \langle M_i \mid i \leq \theta \rangle, \langle \nu_i \rangle, \langle \eta_i \rangle, \langle \pi_{ij} \rangle, T \rangle$

be a good Σ_0 iteration from M to N. Let k = the least $k \underset{T}{\leq} \theta$ which is non simple in $\mathcal{Y}$. Then the full copy $\mathcal{Y}'$ of $\mathcal{Y}|k+1$ onto N exists by Lemma 7.5. Let $\langle \sigma_i \mid i \leq k \rangle$ be the copying maps. Then $\sigma_k \pi_{0k} : M \xrightarrow[\Sigma_0]{} N_k$ where N_k is a non simple Σ_0 iterate of M and $N_k R N$. Contr!

We now show: $\sigma(\xi) \geq \pi(\xi)$ for $\xi \in M$. Suppose not. Define a relation R on pairs $\langle M, \xi\rangle$ s.t. M is a Σ_0 mouse and $\xi \in M$ by:

$\langle M', \xi'\rangle R \langle M, \xi\rangle$ iff M' is a simple Σ_0 iterate of M with map π s.t. $\pi(\xi) > \xi'$.

Then R is well founded. Let $\langle M, \xi\rangle$ be R-minimal with the property that M has a Σ_0 iterate N with iteration map π s.t. there is $\sigma : M \longrightarrow_{\Sigma_0} N$ with $\sigma(\xi) < \pi(\xi)$. Fix N, σ. Then σ is cofinal into N, since otherwise we could truncate N to a nonsimple Σ_0 iterate N' with $\sigma : M \longrightarrow_{\Sigma_0} N'$. Hence σ is cardinal preserving. Let $\mathcal{J} = \langle\langle M_i | i \leq \theta\rangle, \ldots, \langle \pi_{ij}\rangle, T\rangle$ be the iteration from M to M_θ with $\pi = \pi_{0\theta}$. Let $\mathcal{J}' = \langle\langle N_i | i \leq \theta\rangle, \ldots, \langle \pi'_{ij}\rangle, T\rangle$ be the copy onto N with copying maps $\langle \sigma_i | i \leq \theta\rangle$. Then we have:

$\sigma_\theta : N \xrightarrow[\Sigma_0]{} N_\theta$, $\sigma_\theta(\sigma(\xi)) < \sigma_\theta(\pi_{0\theta}(\xi)) =$
$= \pi'_{0\theta}(\sigma(\xi))$, where N_θ is a simple Σ_0 iterate of N with iteration map $\pi'_{0\theta}$ and $\langle N, \sigma(\xi)\rangle R \langle M, \xi\rangle$. Contr! QED (Lemma 8)

It follows as before that N cannot be both a simple and nonsimple Σ_0 iterate of M and that if it is simple, then the iteration map π is unique.