§9 Pseudo Projekta

Our intention is, under certain assumptions to produce a fine structural inner model with a Woodin cardinal. In this chapter we develop certain technical devices which we shall employ to that end. The most important of these is the notion of a sequence $\vec{\rho} = \langle \rho_i \mid i < \omega \rangle$ of pseudo projecta. If $M = \langle J_\alpha^A, B \rangle$ we shall write $M | \eta = \langle J_\eta^A, B \cap J_\eta^A \rangle$ for $\eta \leq \alpha$. (This is to be distinguished from $N \| \eta = \langle J_\eta^E, E_{\omega\eta} \rangle$, where $N = \langle J_\beta^E, E_{\omega\beta} \rangle$ is a premouse.)

Def Let $M = \langle J_\alpha^A, B \rangle$ be acceptable. $\vec{\rho} = \langle \rho_i \mid i < \omega \rangle$ is a good sequence of pseudo projecta for M iff

(a) $\omega\rho_i$ is p.r. closed if $i > 0$.

(b) $1 \leq \rho_{i+1} \leq \rho_i$; $\rho_i \leq \rho_M^i$ if $\rho_h = \rho_M^h$ for $h < i$,

(c) $J_{\rho_i}^A$ is cardinally absolute in M (i.e., if $\gamma \in J_{\rho_i}^A$ is a cardinal in $J_{\rho_i}^A$, then it is a cardinal in M).

(<u>Note</u> $\omega\rho_0 < \omega\rho_M^0 = On_M$ is possible. Also $\omega\rho_i$ need not be a cardinal in M when $\omega\rho_i \in M$. If not, however, M has a cardinal γ s.t. $\gamma < \omega\rho_i < \gamma^{+M}$.)

We shall generally write: '$\vec{\rho}$ <u>is good for</u> M' instead of '$\vec{\rho}$ is a good sequence of pseudo projecta'.

<u>Def</u> Let $\vec{\rho}$ be good for M.

$H_i = H_i(M, \vec{\rho}) =_{\text{Df}} |J_{\rho_i}^A|$ $(i < \omega)$.

$M \models \varphi(x_1, \ldots, x_n) \bmod (\vec{\rho})$ is defined exactly like $M \models \varphi(x_1, \ldots, x_n)$ with H_i in place of H_M^i (for Σ^*-fmlae).

A relation $R(x_1^{i_1}, \ldots, x_n^{i_n})$ is $\Sigma_1^{(n)}(M, \vec{\rho})$ (or $\Sigma_1^{(n)}(M) \bmod (\vec{\rho})$) iff it is M-definable $\bmod(\vec{\rho})$ by a $\Sigma_1^{(n)}$ formu

Similarly for $\underline{\Sigma}_1^{(n)}$, Σ^*, $\underline{\Sigma}^*$.

We then define:

<u>Def</u> $\sigma : M \longrightarrow_{\Sigma_i^{(n)}} M' \bmod (\vec{p}, \vec{p}')$ iff

iff the following hold:

(a) M, M' are acceptable.

(b) $\sigma'' H_i \subset H_i'$ for $i < \omega$, where
$H_i = H_i(M, \vec{p})$, $H_i' = H_i(M', \vec{p}')$.

(c) Let φ be $\Sigma_i^{(n)}$, $\varphi = \varphi(v_1^{i_1}, \ldots, v_p^{i_p})$, where $i_1, \ldots, i_p \leq n$. Then
$M \models \varphi[\vec{x}] \bmod \vec{p} \longleftrightarrow M' \models \varphi[\sigma(\vec{x})] \bmod$,
for all $x_1, \ldots, x_p \in M$ s.t. $x_l \in H_{i_l}$ $(l = 1, \ldots,$

<u>Def</u> $\sigma : M \longrightarrow_{\Sigma^*} M' \bmod (\vec{p}, \vec{p}')$ iff
iff $\sigma : M \longrightarrow_{\Sigma_0^{(n)}} M' \bmod (\vec{p}, \vec{p}')$ for $n < \omega$
(Hence $\sigma : M \longrightarrow_{\Sigma^*} M' \bmod (\vec{p}, \vec{p}')$ iff
iff $\sigma : M \longrightarrow_{\Sigma_1^{(n)}} M' \bmod (\vec{p}, \vec{p}')$ for $n < \omega$

We also define:

<u>Def</u> $\sigma : M \longrightarrow_{\Sigma_i^{(n)}} M' \bmod (\vec{p}')$ iff
iff $\sigma : M \longrightarrow_{\Sigma_i^{(n)}} M' \bmod (\vec{p}, \vec{p}')$,
where $p_i' =_{Df} p_M^i$ for $i < \omega$. Similarly for Σ^*.

<u>Lemma 1.1</u> Let $\sigma : M \to_{\Sigma^*} M'$. Let $\vec{\rho}$ be good for M and set:

$\rho'_i = \sigma(\rho_i)$ if $\rho_i < \rho^i_M$; $\rho'_i = \rho^i_{M'}$ if not.

Then $\sigma : M \to_{\Sigma^*} M' \bmod (\vec{\rho}, \vec{\rho}\,')$.

(Moreover $\sigma : M \to_{\Sigma_1^{(n)}} M' \bmod (\vec{\rho}, \vec{\rho}\,')$ if $\sigma : M \to_{\Sigma_1^{(n)}} M'$ is cardinal preserving)

proof

Clearly $\vec{\rho}\,'$ is good for M'. Now let R be $\Sigma_1^{(n)}(M, \vec{\rho})$. Then R is uniformly $\Sigma_1^{(n)}(M)$ in $u = u_n(M, \vec{\rho})$ = the finite set: $\langle H_i(\vec{\rho}) \mid i \leq n \wedge \rho_i < \rho^i_M \rangle$. But then $\sigma(u) = u_n(M', \vec{\rho}\,')$. If R' is $\Sigma_1^{(n)}(M', \vec{\rho}\,')$ then R' is $\Sigma_1^{(n)}(M')$ in $\sigma(u)$ by the same def. as R in u over M. Thus if σ is $\Sigma_1^{(n)}$-preserving, we have:

$$R\vec{x} \longleftrightarrow R'\sigma(\vec{x}).$$ QED(1.1)

Lemma 1.2 Let $\sigma, M, M', \vec{\rho}, \vec{\rho}'$ be as in Lemma 1.1. Let $\kappa = \mathrm{crit}(\sigma)$ where $\omega\rho_{i+1} \leq \kappa < \omega\rho_i$. Set: $\rho_i'' = \sup \sigma''\rho_i$ and $\rho_j'' = \rho_j'$ for $j \neq i$. Then

$\sigma : M \longrightarrow_{\Sigma^*} M' \bmod (\vec{\rho}, \vec{\rho}'')$.

To prove Lemma 1.2 we note that $\vec{\rho}''$ is still good for M' and prove that σ is $\Sigma_1^{(n)}$-preserving by induction on n.

Many of the basic lemmas on $\Sigma_\ell^{(n)}$ relations still hold modulo a good sequence of pseudo projecta. Fr. ins. (letting $\Sigma_\ell^{(n)} = \Sigma_\ell^{(n)}(M,\vec{\rho})$).

<u>Lemma 2.1</u> Let $n, \ell < \omega$. If $R(x^h, \vec{x})$ is $\Sigma_\ell^{(n)}$ and $k \geq h$, then $R(x^k, \vec{x})$ is $\Sigma_\ell^{(n)}$.

We again call $R(x^k, \vec{x})$ a specialization of $R(x^h, \vec{x})$ $(k \geq h)$ and note:

<u>Lemma 2.2</u> Let $n, \ell < \omega$. If $R(x^k, \vec{x})$ is $\Sigma_\ell^{(n)}$ and $k \geq h \geq n$, then R is a specialization of a relation $R(x^h, \vec{x})$.

<u>Lemma 2.3</u> Let $n, \ell < \omega$. $R(\vec{x}^{n+1}, \dots, \vec{x}^0)$ is $\Sigma_\ell^{(n+1)}(N)$ iff the relation

$$R_{\vec{x}} = \{ \vec{x}^{n+1} \mid R(\vec{x}^{n+1}, \vec{x}) \}$$

is uniformly $\Sigma_\ell(\langle H_{n+1}, \vec{Q}_{\vec{x}} \rangle)$, where each $Q_{i\vec{x}}$ has the form:

$$Q_{i\vec{x}} = \{ \vec{z}^{n+1} \mid Q_i(\vec{z}^{n+1}, \vec{x}) \}$$

and $Q_i(\vec{z}^{n+1}, \vec{x})$ is $\Sigma_1^{(n)}(N)$.

<u>Note</u> We do <u>not</u> claim that $\langle H_{n+1}, \vec{Q}_{\vec{x}} \rangle$ is amenable!

<u>Lemma 2.4</u> Let $p \leq n < \omega$, $1 \leq \ell < \omega$. Let $R(\vec{x}^n, \dots, \vec{x}^0)$ be $\Sigma_\ell^{(n)}(N)$. Let $\vec{F}^n, \dots, \vec{F}^0$ be s.t. each $F_j^i(\vec{z}^p, \dots, \vec{z}^0)$ is a $\Sigma_1^{(i)}$ map to H_i. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_\ell^{(n)}(N)$.

The proofs are exactly as before. It follows as before that if $R(x^{i_1},\dots,x^{i_p})$, $R(x^{j_1},\dots,x^{j_p})$ have the same graph, where $i_1,\dots,i_p,j_1,\dots,j_p \le n$ then one is $\Sigma_1^{(n)}$ iff the other is, since we can convert one to the other by composition with the identity functions $y^i = x^i$. In particular, a relation is $\Sigma_1^{(n)}$ iff the relation with the same graph and arguments of type 0 is $\Sigma_1^{(n)}$.

As before we define:

<u>Def</u> The good $\Sigma_1^{(n)}(N,\vec{\rho})$ functions comprise the smallest class s.t.

(a) Each partial $\Sigma_1^{(i)}(N,\vec{\rho})$ map $F(x_1^{j_1},\dots,x_p^{j_p})$ to H_i is good ($i, j_1,\dots,j_p \le n$).

(b) If $F(x_1^{j_1},\dots,x_p^{j_p})$ is good and $G_i(\vec{z})$ is a $\Sigma_1^{(n)}(N,\vec{\rho})$ to H^{j_i} ($i=1,\dots,p$) (the arguments of $G_i(\vec{z})$ being all of type $\le n$), then $F(\vec{G}(\vec{z}))$ is good.

As before:

<u>Lemma 2.5</u> Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_\ell^{(n)}$ ($n < \omega$, $1 \leq \ell < \omega$, $j_1, \ldots, j_p \leq n$). Let $F_i(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to H_{j_i} ($i = 1, \ldots, p$). Then $R(\vec{F}(\vec{z}))$ is $\Sigma_\ell^{(n)}$.

It is easily seen that good $\Sigma_1^{(n)}$ fcns are closed under composition.

<u>Lemma 2.6</u> Every good $\Sigma_1^{(n)}(N, \vec{p})$ function has a $\Sigma_1^{(n)}$ definition which is <u>functionally absolute</u> in the sense that it defines a good $\Sigma_1^{(n)}(N', \vec{p}')$ function whenever M' is an acceptable structure of the same type and $\vec{p}'$ is good for N'.

proof.

<u>Claim 1</u> Let $F(x_1^{i_1}, \ldots, x_p^{i_p})$ be a $\Sigma_1^{(i)}(N, \vec{p})$ map to H_i ($i \leq n$). Then F has a functionally absolute definition.

proof. Let:

$y^i = F(\vec{x}) \longleftrightarrow \bigvee z^i \, F'(z^i, y^i, \vec{x})$, where F' is $\Sigma_0^{(n)}$. Then:

$$y^i = F(\vec{x}) \longleftrightarrow \bigvee z^i (F'(z^i, y^i, \vec{x}) \wedge$$

$$\wedge \bigwedge \langle y, z\rangle <_{JE} \langle y^i, z^i\rangle \neg F'(z, y, \vec{x}))$$

This definition is $\Sigma_1^{(n)}$ and functionally absolute.

QED (Claim 1)

<u>Claim 2</u> Let $F(x_1^{i_1}, \ldots, x_p^{i_p})$ be good with a functionally absolute def. Let $G_h(z_1^{d_1}, \ldots, z_q^{d_q})$ be a $\Sigma_1^{(l_h)}(N, \vec{p})$ map to H^{i_h} $(h = 1, \ldots, p)$, where $l_1, \ldots, l_q \leq n$. Then $F(\vec{G}(\vec{z}))$ has a functionally absolute definition.

prf.

Let φ be the functionally absolute definition of F. By Claim 1 G_h has a functionally absolute definition ψ_h. By the proof of Lemma 2.4 there is a $\Sigma_1^{(n)}$ fml χ s.t. if φ defines a function F_φ and ψ_h defines a function F_{ψ_h}, then χ defines $F_\chi(\vec{z}) \simeq F_\varphi(F_{\psi_1}(\vec{z}), \ldots, F_{\psi_p}(\vec{z}))$.

But F_φ is always ~~good~~ by functional absoluteness. Hence so is F_χ.

QED (Lemma 2.6)

In the following suppose that:

(1) $\sigma : M \longrightarrow_{\Sigma^*} N \mod(\vec{\rho}\,')$.

We shall construct for N a <u>minimal</u> good sequence of pseudoprojecta $\vec{\rho} = \min(\vec{\rho}\,') = \min(\sigma, N, \vec{\rho}\,')$ s.t.

(a) $\sigma : M \longrightarrow_{\Sigma^*} N \mod(\vec{\rho})$

(b) $\sup \sigma'' \rho^i_M \leq \rho_i \leq \rho_i'$ $(i < \omega)$

(c) Let φ be $\Sigma_1^{(i)}$, $x \in M$, $z_1, \ldots, z_p \in$ $\in H_i(N, \vec{\rho})$. Then $N \models \varphi(\vec{z}, \sigma(x))$ holds $\mod(\vec{\rho})$ iff it holds $\mod(\vec{\rho}\,')$

$\vec{\rho}$ will have the additional properties:

(d) $\sigma : M \longrightarrow_{Q^*} N \mod(\vec{\rho})$

(i.e. if φ in $\Sigma_1^{(i)}$, $i < \omega$, then:

$M \models Q z^i \varphi(z^i, x) \longrightarrow N \models Q z^i \varphi(z^i, \sigma(x))$

(e) $\vec{\rho} = \min(\vec{\rho})$.

We first define:

<u>Def</u> Let $\sigma : M \xrightarrow{}_{\Sigma^*} N \mod(\vec{\rho}\,')$. Set:

$\wp_i(0) = \sup \sigma'' \wp^i_M$.

$\wp_i(n+1) =$ the supremum of the $F'' \wp_{i+1}(n)$

s.t. F is a $\Sigma_1^{(i)}(N, \vec{\rho}\,')$ function to $\wp_i'$ in parameters from $\mathrm{rng}(\sigma)$.

$\vec{\rho} = \min(\vec{\rho}\,') = \min(N, \sigma, \vec{\rho}\,')$ is then defined by: $\rho_i = \sup_n \rho_i(n)$.

<u>Def</u> $H_i(n) = H_i(N, \sigma, \vec{\rho}, n) =_{Df} J^{AN}_{\rho_i(n)}$.

$H_i =_{Df} H_i(N, \vec{\rho}\,) = J^{AN}_{\rho_i}$.

It is easily seen that:

(2) $H_i(0) = \bigcup \sigma'' H^i_M$

$H_i(n+1) = \bigcup \{F'' H_{i+1}(n) \mid F$ is $\Sigma_1^{(i)}(N, \vec{\rho}$ to $H_i(N, \vec{\rho}\,')$ in parameters from $\mathrm{rng}(\sigma)\}$.

$H_i = \bigcup_n H_i(n)$.

[<u>Note</u> $H_i(n+1) \supset H_i(n)$; hence $\rho_i(n+1) \geq \rho_i(n)$.

For $n > 0$ this is trivial given $\rho_i(n) \geq \rho_i(n-1)$

Now let $n=0$. Then each $\sigma(x)$ ($x \in H^i_M$) has the form $F(0)$, where F = (the constant fcn $\sigma(x)$) in $\Sigma_1(N)$ in $\sigma(x)$. Hence $H_i(0) \subset H_i(1)$ by the above definition.]

<u>Lemma 3.1</u> Let $\sigma : M \longrightarrow_{\Sigma^*} N \bmod (\vec{\rho}\,')$. Let $\vec{\rho} = \min(\vec{\rho}\,')$. Then $\vec{\rho}$ is a good sequence of pseudo projecta for N.

prf.

<u>Claim 1</u> $\rho_{i+1} \le \rho_i \le \rho_i' \le \rho^i_N$ (trivial).

<u>Claim 2</u> $\omega\rho_i$ is p.r. closed for $i>0$

prf. $\omega\rho^i_M$, $\omega\rho_i'$ are p.r. closed for $i>0$.

<u>Claim</u> $\omega\rho_i$ is closed under f, where f is a monotone p.r. function

Let $\nu < \omega\rho_i$. Then $\nu < F(\eta)$ for an $\eta < \omega\rho_{i+1}$, where F is a $\Sigma_1^{(i)}(N,\vec{\rho}\,')$ map to $\omega\rho_i$. But $f \circ F$ is a $\Sigma_1^{(i)}(N,\vec{\rho}\,')$ map to $\omega\rho_i$.

Hence $f(\nu) < f F(\gamma) < \omega\rho_i$. QED (Claim 2)

<u>Claim 3</u> H_i is cardinally absolute.

prf.

$H_i = \bigcup X$, where X = the set of $F(z)$ s.t. $z \in H_{i+1}$ and F is a $\Sigma_1^{(i)}(N, \vec{p}\,')$ map to $H_i' = H_i(N, \vec{p}\,')$. Moreover H_i' is cardinally absolute.

(1) Let $\alpha \in X$. Then $\bar{\bar{\alpha}}^N \in X$ and there is $f \in X$ s.t. $f : \bar{\bar{\alpha}}^N \xrightarrow{\text{onto}} \alpha$.

prf. Suppose not.

Define a Σ_1 map F by:

$F(\alpha) \simeq$ the $<_{JA}$-least pair $\langle \gamma, f \rangle$ s.t. $\gamma < \alpha$ and $f : \gamma \longleftrightarrow \alpha$.

Then $F''H_i' \subset H_i'$ and $F''X \subset X$.

Set $\alpha_0 = \alpha$, $\alpha_{i+1} = (F(\alpha_i))_0$ (where $(\langle x, y \rangle)_0 = x$). By induction on i, α_i is defined + $\alpha_i \in X$. But $\alpha_{i+1} < \alpha_i$ $(i < \omega)$. Contr!

QED (1)

Now let α be a cardinal in H_i but not in N (hence not in H_i'). Then $\alpha \notin X$ by (1). But $\alpha < \beta$ for a $\beta \in X$. Hence $\bar{\bar{\beta}}^N > \alpha$ by (1). But then α is a cardinal in $J_\gamma^{A^N}$ for $\gamma = \bar{\bar{\beta}}^N$ and hence in N by acceptability. Contr! QED (Claim 3)

QED (Lemma 3.1)

<u>Lemma 3.2</u> Let $M, N, \sigma, \vec{\rho}', \vec{\rho}$ be as above. Let $\bar{B}(v', w)$ be $\Sigma_0^{(i)}(M)$ $(i < \omega)$. Let B' be $\Sigma_0^{(i)}(N, \vec{\rho}')$ and B be $\Sigma_0^{(i)}(N, \vec{\rho})$ by the same definition. Let $x \in M$. Then,

(a) $\langle H_i, B_{\sigma(x)} \rangle$ is amenable, where

$$B_{\sigma(x)} = \{z \in H_i \mid B(z, \sigma(x))\}$$

(b) $\bigwedge z \in H_i \; (B(z, \sigma(x)) \longleftrightarrow B'(z, \sigma(x)))$

prf. Ind. on i.

$i = 0$ is trivial. Let $i = h+1$. It suffices to prove (a), (b) for $\bar{B}$ which is $\Sigma_1^{(h)}(M)$.

We first prove (b). Let

$$\bar{B}(v, w) \longleftrightarrow \bigvee z^h \; \bar{D}(z^h, v, w),$$

where $\bar{D}$ is $\Sigma_0^{(h)}$. (Similarly for D', D)

Define a $\Sigma_1^{(h)}(N,\vec{p}\,')$ map to $\omega\rho_h'$ by:

$$\xi = F(v,w) \leftrightarrow (\bigvee z \in S_\xi\ D'(z,v,w) \wedge$$
$$\wedge \bigwedge \xi' < \xi \bigwedge z \in S_{\xi'}\ \neg D'(z,v,w)),$$

Then $F(v,\sigma(x)) < \omega\rho_h$ for all $v \in H_i$.

Hence for $v \in H_i$:

$$B'(v,\sigma(x)) \leftrightarrow \bigvee z \in H_h'\ D'(z,v,\sigma(x))$$
$$\leftrightarrow \bigvee z \in S_{F(v,\sigma(x))}\ D'(z,v,\sigma(x))$$
$$\leftrightarrow \bigvee z \in H_h\ D'(z,v,\sigma(x))$$
$$\leftrightarrow \quad '' \quad D(z,v,\sigma(x))$$

(by ind. hyp.)

$$\leftrightarrow B(v,\sigma(x)).\quad \text{QED (b)}$$

We now prove (a). Define a $\Sigma_1^{(i)}(N,\vec{p}\,')$ map to H_i' by:

$$y^i = F(u^i) \leftrightarrow y^i = u^i \cap \{z \mid B'(z,\sigma(x))\}$$

Let $w \in H_i$. Then $w \subset G(u)$, where $u \in H_{i+1}$ and G is a $\Sigma_1^{(i)}(N,\vec{p}\,')$ map in parameters from $\mathrm{rng}(\sigma)$. But then $F \circ G$ is such a map, and the $\Pi_1^{(i+1)}$ statement is:

$\bigwedge u^{i+1} \in \mathrm{dom}(G)\ u \in \mathrm{dom}(F \circ G)$

holds, since the corresponding statement holds in M. Hence $v = G(u) \wedge B_{\sigma(x)} =$
$= G(u) \wedge B'_{\sigma(x)} = FG(u) \in H_i$ and
hence $w \wedge B_{\sigma(x)} = w \wedge v \in H_i$.

QED (Lemma 3.2)

Since $\sigma : M \to_{\Sigma_0^{(i)}} N \bmod(\vec{\rho}\,')$, Lemma 3.1 gives us: $\sigma : M \to_{\Sigma_0^{(i)}} N \bmod(\vec{\rho})$.
Hence:

<u>Cor 3.3</u> $\sigma : M \to_{\Sigma^*} N \bmod(\vec{\rho})$

Another immediate corollary is:

<u>Cor 3.4</u> $\vec{\rho} = \min(N, \sigma, \vec{\rho})$.

Finally we prove:

<u>Cor 3.5</u> $\sigma : M \to_{Q^*} N \bmod(\vec{\rho})$.

proof.
Assume $M \models Qu^i \varphi(u^i, x)$, where φ is $\Sigma_1^{(i)}$

<u>Claim</u> $N \models Qu^i \varphi(u^i, \sigma(x)) \bmod(\vec{\rho})$.

Let $v \in H_i$. Then $v \subset w = G(\bar{w})$, where $w \in H_{i+1}$ and G is a

$\Sigma_1^{(i)}(N, \vec{p})$ map to H_i defined in parameters from $\mathrm{rng}(\sigma)$. Let $\varphi = \bigvee z^i \Psi(z^i, u^i, x)$, where Ψ is $\Sigma_0^{(i)}$. Define a $\Sigma_1^{(i)}(N, \vec{p})$ to H^i by:

$F(w) \simeq$ the N-least $\langle z, u \rangle \in H^i$ s.t.
$$w \subset u \wedge \Psi(z, u, \sigma(x)).$$

The $\Pi_1^{(i+1)}(N, \vec{p})$ statement:

$$\bigwedge a^{i+1} (a^{i+1} \in \mathrm{dom}(G) \rightarrow a^{i+1} \in \mathrm{dom}(F \circ G))$$

holds in N, since the corresponding statement holds in M (by $M \models Qu\varphi(u, x)$). Let $\langle z, u \rangle = FG(\bar{w}) = F(w)$. Then $v \subset w \subset u$ and $\Psi(z, u, \sigma(x))$. Hence $\bigvee u \supset v\, \varphi(u, \sigma(x))$ for all $v \in H_i$. Hence $N \models Qu\, \varphi(u, \sigma(x)) \bmod (\vec{p})$.

QED (Cor 3.5)

<u>Def</u> $\sigma : M \rightarrow_{\Sigma^*} N \; \min(\vec{p})$ iff

iff $\sigma : M \rightarrow_{\Sigma^*} N \bmod (\vec{p}) \;\wedge$

$\wedge\; \vec{p} = \min(N, \sigma, \vec{p})$.

(Similarly for $\Sigma_1^{(n)}$ etc.)

We now state a "copying lemma" for the relation $\sigma : \bar{M} \longrightarrow_{\Sigma^*} M \min(\vec{p})$ analogous to §3 Lemma 2. First, however, we weaken the relation $\langle \sigma, k \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{*} \langle M, F \rangle$ as follows:

Def $\langle \sigma, k \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{**} \langle M, F \rangle$ iff

(a) $\langle \sigma, k \rangle : \langle \bar{M}, \bar{F} \rangle \longrightarrow \langle M, F \rangle$

(b) Let $\bar{\alpha} < lh(\bar{F})$, $\alpha = k(\bar{\alpha})$. There are $\bar{G}, G, \bar{H}, H$ s.t., letting $\bar{\kappa} = crit(\bar{F})$, $\kappa = \sigma(\bar{\kappa})$:

(i) $\bar{G}, \bar{H}$ are $\Sigma_1(\bar{M})$ in a $\bar{q} \in \bar{M}$ and G, H are $\Sigma_1(M)$ in $q = \sigma(\bar{q})$ by the same definition.

(ii) $\bar{G} = \bar{F}_{\bar{\alpha}}$, $\bar{H} = \bar{M} \cap {}^{\bar{\kappa}}\mathcal{P}(\bar{\kappa})$

(iii) $G \subset F_\alpha$

(iv) $H \subset \{X \in {}^{\kappa}\mathcal{P}(\kappa) \mid \bigwedge i < \kappa\, (X_i \text{ or } \kappa \setminus X_i \in G)\}$

(Note $\xrightarrow{*}$ implies $\xrightarrow{**}$, since if $G = F_\alpha$, then we can take $H = M \cap {}^{\kappa}\mathcal{P}(\kappa)$.)

(Note Let $\bar{X} \in \bar{M} \cap {}^{\bar{\kappa}}\mathcal{P}(\bar{\kappa})$. If $X = \sigma(\bar{X})$, then $X \in H$; hence $\bigwedge i < \kappa\, (X_i \text{ or } \kappa \setminus X_i \in G)$.)

<u>Lemma 4</u> Let $\sigma : \overline{M} \longrightarrow_{\Sigma^*} M$ min $(\vec{p})$.

Suppose $\pi : M \longrightarrow_{\Sigma^*} M'$ s.t. $\kappa = \mathrm{crit}(\pi)$

Let F at κ, ν be defined by:

$F(X) = \pi(X) \cap \nu$ for $X \in \mathbb{P}(\kappa) \cap M$.

Let $\overline{F}$ at $\overline{\kappa}, \overline{\nu}$ w.r.t. $\overline{M}$ be weakly amenable s.t. $\langle \sigma, k \rangle : \langle \overline{M}, \overline{F} \rangle \xrightarrow{**} \langle M | \rho_0, F \rangle$

Then:

(a) There is $\overline{\pi} : \overline{M} \longrightarrow^*_{\overline{F}} \overline{M}'$

(b) There is $\sigma' : \overline{M}' \longrightarrow M'$ defined by $\sigma'(\overline{\pi}(f)(\alpha)) = \pi\sigma(f)(k(\alpha))$ for $f \in \Gamma^*(\overline{\kappa}, \overline{M})$, $\alpha < \overline{\nu}$.

(c) There is $\vec{p}\,'$ good for M' s.t.

(i) $\sigma' : \overline{M}' \longrightarrow_{\Sigma^*} M'$ min $(\vec{p}\,')$

(ii) $\rho_i' \leq \pi(\rho_i)$ and $\pi''\rho_i \subset \rho_i'$ for $i < \omega$.

This is our most important lemma on pseudo projecta.

The proof of Lemma 4 stretches over several sublemmas.

<u>Lemma 4.1</u> (a), (b) hold. Moreover:

$\sigma' : \bar{M}' \longrightarrow_{\Sigma_0^{(n)}} M' \bmod (\vec{\rho}^{\,*})$ for $\omega\rho^n_{\bar{M}} > \bar{\kappa}$,

where $\rho_i^* = \pi(\rho_i)$ if $\rho_i < \rho^i_M$; $\rho_i^* = \rho^i_{M'}$ if not.

proof. Let φ be $\Sigma_0^{(n)}$. Set $\mathbb{D} = \mathbb{D}^*(\bar{M}, \bar{F})$. Then:

$\mathbb{D} \models \varphi(\langle f_1, \alpha_1\rangle, \ldots, \langle f_p, \alpha_p\rangle) \longleftrightarrow$

$\longleftrightarrow \{\vec{\xi} < \bar{\kappa} \mid \bar{M} \models \varphi(\vec{f}(\vec{\xi}))\} \in \bar{F}_{\vec{\alpha}}$

$\longleftrightarrow \sigma(\qquad n \qquad) \in F_{k(\vec{\alpha})}$

$\longleftrightarrow \{\vec{\xi} < \kappa \mid M \models \varphi(\sigma(\vec{f})(\xi)) \bmod (\vec{\rho})\} \in F_{k(\vec{\alpha})}$

$\longleftrightarrow k(\vec{\alpha}) \in \pi(\qquad\qquad)$

$\longleftrightarrow M' \models \varphi(\pi\sigma(\vec{f})(\vec{\alpha})) \bmod (\vec{\rho}^{\,*})$.

Hence $\mathbb{D}$ is well founded, since $\langle f, \alpha\rangle \in_{\mathbb{D}} \langle g, \beta\rangle \longleftrightarrow \pi\sigma(f)(\alpha) \in \pi\sigma(g)(\beta)$,

But then $\bar{\pi} : \bar{M} \longrightarrow^*_{\bar{F}} \bar{M}'$ exists and σ' is defined with the above property.

QED (4.1)

Now set: $\rho''_i = \begin{cases} \sup \pi''\rho_i & \text{if } \omega\rho^{i+1}_{\bar{M}} \leq \bar{\kappa} < \omega\rho^i_{\bar{M}} \\ \rho_i^* & \text{if not} \end{cases}$

Then:

Lemma 4.2

(a) $\pi : M \xrightarrow[\Sigma^*]{} M' \mod(\vec{\rho}, \vec{\rho}'')$

(b) $\sigma' : \bar{M}' \xrightarrow[\Sigma_0^{(n)}]{} M' \mod(\vec{\rho}'')$ for $\omega\rho^n_{\bar{M}} > \bar{\kappa}$

proof.

(a) by Lemma 1.2 and $\omega\rho_{i+1} \le \kappa < \omega\rho_i$ iff

iff $\omega\rho^{i+1} \le \bar{\kappa} < \omega\rho^i$ in $\bar{M}$, since

σ is Σ^*-preserving $\mod(\vec{\rho}')$.

(b) then follows by Lemma 4.1. QED

(Note Lemmas 4.1 and 4.2 go through assuming only $\langle\sigma, k\rangle : \langle\bar{M}, \bar{F}\rangle \longrightarrow \langle M, F\rangle$ rather than $\xrightarrow{**}$.)

Lemma 4.3 $\sigma' : \bar{M}' \xrightarrow[\Sigma^*]{} M' \mod(\rho'')$

proof.

By induction on i we show that σ' is $\Sigma_1^{(i)}$-preserving $\mod(\rho'')$. For $\rho^{i+1}_{\bar{M}} > \bar{\kappa}$, this follows by Lemma 4.2. Now let $\rho^{n+1} \le \bar{\kappa} < \rho^n$ in $\bar{M}$. Let $\bar{R}(z, y)$ be $\Sigma_1^{(n)}(\bar{M}')$ + let $R(z, y)$ be $\Sigma_1^{(n)}(M', \vec{\rho}'')$ by the same definition. Let $\bar{x} \in \bar{M}'$, $x = \sigma'(\bar{x})$. Suppose that

$\bar{x} = \bar{\pi}(f)(\bar{\alpha})$ where $f \in \Gamma^*(\bar{\kappa}, \bar{M})$. Then $x = \pi\sigma(f)(\alpha)$, where $\alpha = k(\bar{\alpha})$. Let $f = \bar{p} \in \bar{M}$ or f be a good $\Sigma_1^{(n-1)}(\bar{M})$ function in $\bar{p}$ by a functionally absolute definition. Then $\sigma(f)$ has the same definition in $p = \sigma(\bar{p})$ over M mod$(\vec{p}\,)$ and $\pi\sigma(f)$ has the same def. in $\pi(p)$ mod $(\vec{p}\,'')$ over M'. Now let $\bar{G}, G, \bar{H}, H, \bar{q}, q$ be as given (for $\bar{\alpha}, \alpha$) by:

$$\langle \sigma, k \rangle : \langle \bar{M}, \bar{F} \rangle \xrightarrow{**} \langle M, F \rangle.$$

We prove:

<u>Claim 1</u> Let $\bar{M} = \langle J_{\bar{\beta}}^{\bar{A}}, \bar{B} \rangle$, $M = \langle J_\beta^A, B \rangle$

(a) $\bar{P} = \{ \xi < \bar{\kappa} \mid \bar{R}(\xi, \bar{x}) \}$ is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{\kappa}, \bar{p}, \bar{q}$

(b) $P = \{ \xi < \kappa \mid R(\xi, x) \}$ is $\Sigma_1^{(n)}(M, \vec{p}\,)$ in κ, p, q by the same definition.

To prove this, let:

$\bar{R}(w, x) \longleftrightarrow \bigvee z^n \bar{Q}'(z^n, w, x)$, where $\bar{Q}'$ is $\Sigma_0^{(n)}(\bar{M}')$. Let $\bar{Q}$ have the same definition over $\bar{M}$. Then:

(1) $\bar{R}(\xi, \bar{x}) \longleftrightarrow \bigvee u \in H^n_{\bar{M}} \underbrace{\bigvee z \in \pi(u)\, \bar{Q}'(z, \xi, \bar{x})}_{\Sigma_0^{(n)}}$

$\longleftrightarrow \bigvee u \in H^n_{\bar{M}} \{\delta < \bar{\kappa} \mid \bigvee z \in u\, \bar{Q}(z, \xi, f(\delta)\} \in \bar{F}_\alpha$

$\longleftrightarrow \bigvee u^n \bigvee v^n (v^n = \{\delta < \bar{\kappa} \mid \bigvee z \in u\, \bar{Q}(z, \xi, f(\delta))$
$\wedge\ v^n \in \bar{F}_\alpha)$

We know that:

(2) $\sigma'' H_n$ is cofinal in $H_n' = H_n(M', \vec{p}\,'')$.

(3) If A is $\Sigma_0^{(n)}(M, \vec{p})$, then $\langle H_n, A\rangle$ is amenable by Lemma 3.2; hence:

(4) If ψ is a $\Sigma_0^{(n)}$ fmla, then, for $\alpha < lh(F)$,
$M' \models \psi(\alpha, \pi\sigma(x))\ \mathrm{mod}(\vec{p}\,'') \longleftrightarrow$
$\longleftrightarrow \{\xi < \kappa \mid M \models \psi(\xi, \sigma(x))\} \in F_\alpha$.

Let Q' have the same def. as $\bar{Q}$ over $M'\ \mathrm{mod}(\vec{p}\,'')$ and Q the same def. over $M\ \mathrm{mod}(\vec{p})$. Then:

(5) $R(\xi, x) \longleftrightarrow \bigvee u \in H_n \bigvee z \in \sigma(u)\, Q'(z, \xi, x)$

$\longleftrightarrow$ " $\{\delta < \kappa \mid \bigvee z \in u\, Q(z, \xi, \sigma(f)(\delta)\} \in F_\alpha$

$\longleftrightarrow \bigvee u^n \bigvee v^n (v^n = \{\delta < \kappa \mid \bigvee z \in u^n Q(z, \xi, \sigma(f)(\delta))$
$\wedge\ v^n \in F_\alpha)$

Now let $\bar{P}, P, \bar{G}, G, \bar{H}, H$ be as given by $\xrightarrow{**}$ applied to $\bar{\alpha}$. We claim:

(6) $R(\xi, x) \leftrightarrow \bigvee u^n \bigvee v^n (v^n = \{\delta < \kappa \mid \bigvee z \in u^n\, Q(z, \xi, \sigma(f)(\delta)\}$
$\wedge\, v^n \in G)$

(<u>Note</u> Since $\bar{G} = F_\alpha$, this proves Claim 1)

($\leftarrow$) is trivial by (5). We prove ($\rightarrow$).

For $u \in H_n$, $\xi < \kappa$, set: $\theta(u, \xi) =_{\mathrm{df}}$
$=_{\mathrm{df}} \{\delta < \kappa \mid \bigvee z \in u\, Q(z, \xi, \sigma(f)(\delta)\}$.

Set: $\tilde{\theta}(u^n) = \langle \theta(u^n, \xi) \mid \xi < \kappa \rangle$. Then $\tilde{\theta}$ is $\Sigma_0^{(n)}(M, \vec{p})$ function in p, κ and is defined on all of H_n. Pick u s.t. $\theta(u, \xi) \in F_\alpha$. By the minimality of $\vec{p}$, we have $u \subset g(\zeta)$, where $\zeta < \omega\rho_{n+1}$ and g is a $\Sigma_1^{(n)}(M, \vec{p})$ map to H_n in parameters from $\mathrm{rng}(\sigma)$. Hence $\theta(g(\zeta), \xi) = \tilde{\theta}(g(\zeta))_\xi \in F_\alpha$. The function $\theta^*(\zeta) \simeq \tilde{\theta}(g(\zeta))$ is $\Sigma_1^{(n)}(M, \vec{p})$ in an $r = \sigma(\bar{r})$. Let $\bar{\theta}^*$ have the same $\Sigma_1^{(n)}(\bar{M})$ definition in $\bar{r}$. Then $\mathrm{dom}\,\theta^*$ is $\Sigma_1^{(n)}(M, \vec{p})$ in r

and $\operatorname{dom}\bar{\Theta}^*$ is $\Sigma_1^{(m)}(\bar{M})$ in $\bar{r}$ by the same definition. Since $\bar{H} = {}^{\omega}\mathcal{P}(\bar{\kappa}) \cap \bar{M}$, we have: $\bigwedge \xi^{n+1} (\xi^{n+1} \in \operatorname{dom}(\bar{\Theta}^*) \to \bar{\Theta}^*(\xi^{n+1}) \in \bar{H})$.

Hence, since this statement is Π^1_{m+1} in $\bar{r}, \bar{q}$, we have:

$\bigwedge \xi^{n+1} (\xi^{n+1} \in \operatorname{dom}(\Theta^*) \to \Theta^*(\xi^{n+1}) \in H)$.

In particular, for the ξ chosen above we have:

(7) $\bigwedge \varsigma < \kappa \, (\Theta^*(\xi)_\varsigma$ or $\kappa \setminus \Theta^*(\xi)_\varsigma \in G)$.

For our specific ς we know that $\Theta^*(\xi)_\varsigma = \Theta(g(\xi), \varsigma) \in F_\alpha$, hence $\kappa \setminus \Theta^*(\xi)_\varsigma \notin G \subset F_\alpha$. Hence

$\{\sigma < \kappa \mid \bigvee z \in u' \, Q(u, \varsigma, \sigma(f)(\sigma))\} =$

$= \Theta^*(\xi)_\varsigma \in G$, where $u' = g(\xi)$.

QED (Claim 1)

From this it follows easily that:

<u>Claim 2</u> $\{z \in \bar{M} | \bar{\kappa} \mid \bar{R}(z, \bar{x})\}$ is $\Sigma_1^{(m)}(\bar{M})$ in some $\bar{q}$ and $\{z \in M | \kappa \mid R(z, x)\}$ is $\Sigma_1^{(m)}(M, \vec{p})$ in $\bar{q} = \sigma(q)$ by the same def.

But $\rho^i = \pi(\rho^i) \leq \kappa$ for $i > n$ and it follows easily by induction on i that

<u>Claim 3</u> Let $i > n$. Let $\bar{R}$ be $\Sigma_1^{(i)}(\bar{M}')$ + R be $\Sigma_1^{(i)}(M')$ by the same def. Let $\bar{x}, x, f, \bar{q}, q$ be as in Claim 2. Then $\{w \in \bar{H}^i_{\bar{M}} \mid \bar{R}(w, \bar{x})\}$ is $\Sigma_1^{(i)}(\bar{M})$ in $\bar{q}$ and $\{w \in H_i \mid R(w, x)\}$ is $\Sigma_1^{(i)}(M, \vec{p})$ in q by the same definition.

Now let φ be $\Sigma_1^{(i)}$ $(i \geq n)$. Let $\bar{x} \in \bar{M}'$, $x = \sigma'(\bar{x})$. The statement $\bar{M}' \models \varphi(\bar{x})$ is $\Sigma_1^{(i)}(\bar{M})$ in parameter $\bar{q}$ and $M' \models \varphi(x)$ mod$(\vec{p})$ is $\Sigma_1^{(i)}(M, \vec{p})$ in $q = \sigma(\bar{q})$ by the same definition. Hence $\bar{M}' \models \varphi(\bar{x}) \longleftrightarrow M' \models \varphi(x)$ mod$(\vec{\rho}'')$, since $\sigma : \bar{M} \to_{\Sigma^*} M$ mod$(\vec{\rho})$.

QED (Lemma 4.3)

Now set: $\vec{\rho}' = \min(M', \sigma', \rho'')$.

<u>Lemma 4.4</u> Let $\vec{\rho}\,' = \min(M', \sigma', \vec{\rho}\,'')$.

(a) $\sigma' : \overline{M}' \xrightarrow[\Sigma^*]{} M' \min(\vec{\rho}\,')$

(b) $\rho'_i \leq \pi(\rho_i)$ $(i < \omega)$

(c) $\pi''\omega\rho_i \subset \omega\rho'_i$.

proof.

(a) is trivial, as is (b) since $\rho'_i \leq \rho''_i \leq \pi(\rho_i)$.

To prove (c) we show by induction on n that for all i:

$\pi''\omega\rho_i(n) \subset \omega\rho'_i(n)$.

For $n = 0$, if $\xi < \omega\rho_i(0)$, then $\xi < \sigma(\eta)$ for $\eta < \rho^i_{\overline{M}}$. Hence

$\pi(\xi) \leq \pi\sigma(\eta) = \sigma'\overline{\pi}(\eta) < \omega\rho'_i(0)$.

Now let $n = m+1$, $\xi < \omega\rho_i(n)$. Then $\xi < F(\eta)$ where $\eta < \omega\rho_{i+1}(m)$ and F is a $\Sigma_1^{(i)}(M, \vec{\rho})$ map in $\sigma(x)$ for an $x \in \overline{M}$. Let F' have the same (functionally absolute) $\Sigma_1^{(i)}(M', \vec{\rho}\,'')$ definition in $\pi\sigma(x) = \sigma'\overline{\pi}(x)$.

Then $\pi(\xi) \leq \pi(F(\eta)) \leq F'(\pi(\eta))$, where $\pi(\eta) < \rho'_{i+1}(m)$. Hence $\pi(\xi) < \rho'_i(m)$.

QED (Lemma 4.4)

This completes the proof of Lemma 4.
Some obvious corollaries of the proof are:

<u>Lemma 4.5</u> Let $\pi(\rho_n) = \rho'_n$ for $n < \omega$. Then $\pi : M \xrightarrow[\Sigma^*]{} M'$ mod (ρ, ρ').

proof. By Lemma 4.2, since $\rho'_n \leq \rho''_n \leq \pi(\rho_n)$; hence $\rho'_n = \rho''_n$. QED (4.5)

<u>Lemma 4.6</u> Let $\omega\rho_{n+1} \leq \bar{u} < \omega\rho_n$ in $\bar{M}$. Then $\rho'_n = \sup \pi''\rho_n$.

prf. $\sup \pi''\rho_n \leq \rho'_n \leq \rho''_n = \sup \pi''\rho_n$.

QED (4.6)

<u>Lemma 4.7</u> Let $\omega\rho^1_{\bar{M}} \leq \bar{u}$. Let $\bar{A} \subset \bar{M}|\bar{u}$ be $\Sigma_1(\bar{M}')$ in $\bar{p}$ and let $A \subset M|u$ be $\Sigma_1(M'|\rho'_0)$ in p by the same definition.
Then $\bar{A}$ is $\Sigma_1(\bar{M})$ in some $\bar{q}$ and A is $\Sigma_1(M|\rho_0)$ in $q = \sigma(\bar{q})$ by the same definition.

proof of Lemma 4.7

By Claim 2 in the proof Lemma 4.3, since in this case $\rho_0' = \rho_0'' = \sup \pi'' \rho_0$.

QED(4.7)

. . . . .

We now use Lemma 4 to prove:

Lemma 5 Let M be a smoothly iterable premouse. Then M is iterable.

The proof uses a construction similar to those in §5.

Suppose that M is a pm, $\mu < ht(M)$, $\mathcal{I} = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\eta_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ a direct normal iteration of $N = M\|\mu$ of length θ. We construct:

(A) A normal iteration $\mathcal{I}'$ of M of length $\bar\theta \le \theta$ with $\mathcal{I}' = \langle\langle M_i\rangle, \langle\nu_i'\rangle, \langle\eta_i'\rangle, \langle\pi_{ij}'\rangle, T'\rangle$, where $T' = T \cap \bar\theta^2$.

Set: $\mu_i = \begin{cases} \pi_{0i}'(\mu) & \text{if } \mu \in dom(\pi_{0i}'); \\ ht(M_i') & \text{if not.} \end{cases}$

(B) Maps σ_i + sequences $\vec\rho^{\,i}$ $(i < \bar\theta)$ s.t.

(i) $\sigma_i : N_i \longrightarrow_{\Sigma^*} M_i\|\mu_i$ $\quad \min(\vec\rho^{\,i})$

(ii) $\sigma_i\pi_{hi} = \pi_{hi}'\sigma_h$ for $h \le_T i$

If π_{hi}' is defined on $M_h\|\mu_h$, then:

(iii) $\rho_n^i \ge \pi_{hi}'(\rho_n^h)$ for $h \le_T i$, $n < \omega$

(iv) $\pi_{hi}''\rho_n^h \subset \rho_n^i$ " " " .

We construct $M_i, \langle\pi_{hi}' \mid h \le_T i\rangle, \sigma_i, \vec\rho^{\,i}$ as follows:

$M_0 = M$, $\sigma_0 = id$, $\rho_n^0 = \rho_M^n$.

Now let M_i etc. be given. Let $\xi = T(i+1)$. Set: $\nu_i' = \sigma_i(\nu_i)$, $\mu_i^* = \sigma_\xi(\eta_i)$ (where $\sigma_h(On \cap N_h) =_{df} \omega\mu_h$). Note that $\mu_i^* \leq \mu_\xi$ and $\mu_i^* = \mu_\xi$ if $\eta_i = ht(N_i)$. Clearly $\mu_i^* \leq \eta_i' =$ the maximal η s.t. $\nu_i' + M_\xi \| \eta$ $= \nu_i' + M_i \| \lambda_i$. Moreover $\eta_i' = \mu_i^*$ if $\eta_i < ht(N_i)$ and $\eta_i' \geq \mu_\xi$ otherwise. Set:

$$N^* = N_\xi \| \eta_i \ , \ M^* = M_\xi \| \mu_i^* \ , \ \sigma^* = \sigma_\xi \upharpoonright N^*.$$

Define $\langle \rho_n^* \mid n < \omega \rangle$ by:

$$\vec{\rho}^{\,*} = \begin{cases} \vec{\rho}^{\,\xi} & \text{if } \mu_i^* = \mu_\xi \\ \min(M^*, \sigma^*, \langle \rho_{M^*}^n \mid n < \omega \rangle) & \text{if not} \end{cases}$$

Then $\sigma^* : N^* \longrightarrow_{\Sigma^*} M^* \min(\vec{\rho}^{\,*})$.

$M_{i+1}, \sigma_{i+1}, \vec{\rho}^{\,i+1}$ will be defined iff the following two condition hold:

(*) $\tilde{\pi} : M_\xi \| \eta_i' \xrightarrow{*}_{E_{\nu_i'}^{M_i}} \tilde{M}$ exists.

(**) $\langle \sigma_i^*, \sigma_i \upharpoonright \lambda_i \rangle : \langle N^*, E_{\nu_i}^{N_i} \rangle \xrightarrow{**} \langle M^* \| \rho_0^*, E_{\nu_i'}^{M_i} \rangle$

Suppose now that (*), (**) hold. Set: $M_{i+1} = \tilde{M}$, $\pi'_{3,i+1} = \tilde{\pi}$, $\mu_{i+1} = \tilde{\pi}(\mu_i^*)$. Clearly:

$\tilde{\pi} \upharpoonright M^* : M^* \longrightarrow_{\Sigma^*} M_{i+1} \| \mu_{i+1}$. Since $\sigma^* : N^* \longrightarrow_{\Sigma^*} M^*$ $\min(\vec{p}^*)$, the proof of Lemma 4 then gives canonical $\sigma_{i+1}, \vec{p}^{\,i+1}$ s.t.

(a) $\sigma_{i+1} : N_{i+1} \longrightarrow_{\Sigma^*} M_{i+1} \| \mu_{i+1}$ $\min(\vec{p}^{\,i+1})$

(b) $\sigma_{i+1}(\pi_{3,i+1}(f)(\alpha)) = \tilde{\pi}\sigma^*(f)(\sigma_i(\alpha))$
(where $\sigma^*(f)$ is understood mod $(\vec{p}^*)$)

(c) $\tilde{\pi}(p_n^*) \geq p_n^{i+1}$, $\tilde{\pi}$ " $p_n^* \subset p_n^{i+1}$.

This completes the successor case of the construction.

Now let $M_i, \sigma_i, \vec{p}^{\,i}$ be defined for $i < \lambda$ where $\lim(\lambda)$. $M_\lambda, \sigma_\lambda, \vec{p}^{\,\lambda}$ will be defined iff:

(***) There is a transitive $\tilde{M}$ with:

$$\tilde{M}, \langle \tilde{\pi}_i \mid i \leq_T \lambda \rangle = \lim_{i \leq_T i' <_T \lambda} (M_i, \pi'_{ii'}).$$

Assume that (***) holds. We set: $M_\lambda = \tilde{M}$, $\pi'_{i\lambda} = \tilde{\pi}_i$. $\sigma : N_\lambda \to M_\lambda$ is defined by: $\sigma_\lambda \pi'_{i\lambda} = \pi'_{i\lambda} \sigma_i$ $(i \leq_T \lambda)$

To define ρ^λ_n we note that

$\rho^i_n \leq \pi_{hi}(\rho^h_n)$ for $h \leq_T i <_T \lambda$, whenever h is large enough that $\pi_{h\lambda}$ is total. It follows easily that $\rho^i_n = \pi_{hi}(\rho^h_n)$ $(h \leq_T i <_T \lambda)$ for sufficiently large $h \leq_T \lambda$. Thus we can define:

Def $\rho_n^\lambda = \pi_{h\lambda}(\rho_\lambda^h)$ for $h \leq_T \lambda$ large enough that $\pi_{hi}(\rho_n^h) = \rho_n^i$ for all i s.t. $h \leq_T i \leq_T \lambda$.

(Then $\pi_{i\lambda}(\rho_n^i) = \rho_n^\lambda$ for sufficiently large i.) It follows trivially that;

(1) $\pi'_{i\lambda}(\rho_n^i) \geq \rho_n^\lambda$; $\pi''_{i\lambda}\rho_n^i \subset \rho_n^\lambda$

for all $i \leq_T \lambda$.

(2) There is $i_0 \leq_T \lambda$ s.t. $\pi'_{i_0\lambda}(\rho_n^{i_0}) = \rho_n^\lambda$ for all $n < \omega$ (and $\pi'_{i_0\lambda}$ is total on M_{i_0}).

proof.

For $i \leq \lambda$ set: $v_i = \{n \mid \rho_n^i \neq \rho_k^i \text{ for all } k < n\}$.

Then v_i is finite. Moreover $v_i \subset v_j$ if π'_{ij} is total on M_j, since $i \leq_T j \leq_T \lambda$ + $\pi'_{i\lambda}$ is total on M_j, since

$\rho_n^i < \rho_k^i \to \rho_n^j \leq \pi_{ij}(\rho_n^i) \in \pi''_{ij}\rho_k^i \subset \rho_k^j$;

Since $v = v_\lambda$ is finite, there must be $i \leq_T \lambda$ s.t. $v_i = v$. Now choose j s.t. $i \leq_T j \leq_T \lambda$ and $\pi'_{j\lambda}(\rho_n^j) = \rho_n^\lambda$ for $n \in v$. Then $\pi'_{j\lambda}(\rho_n^j) = \rho_n^\lambda$ for all n, since $v_j = v$. QED (2)

By Lemma 4.5 we conclude:

(3) $\pi'_{i'i} : M_{i'} \xrightarrow[\Sigma^*]{} M_i \mod (\vec{\rho}^{i'}, \vec{\rho}^{i})$

for $i_0 \leq_T i' \leq_T i \leq_T \lambda$.

Since $\sigma_i : N_i \xrightarrow[\Sigma^*]{} M_i \mod (\rho^i)$ and $\sigma_\lambda \pi_{i\lambda} = \pi'_{i\lambda} \sigma_i$, it follows easily that

(4) $\sigma_\lambda : N_\lambda \xrightarrow[\Sigma^*]{} M_\lambda \mod (\vec{\rho}^\lambda)$.

Finally, we can repeat the proof of Lemma 4.4 to get:

(5) $\pi'_{i\lambda}{}'' \rho^i(n) \subset \rho^\lambda(n)$ for $i \leq_T \lambda$, $n < \omega$

Hence:

(6) $\vec{\rho}^\lambda = \min(\sigma_\lambda, M_\lambda, \vec{\rho}^\lambda)$.

prf.

Let $\nu < \rho_n^\lambda$. Claim $\forall m$ $\nu < \rho_m^\lambda(m)$.

Let $\nu = \pi_{i\lambda}(\bar{\nu})$, $i_0 \leq_T i \leq_T \lambda$.

Then $\bar{\nu} < \rho_m^i$. Hence $\bar{\nu} < \rho_m^i(m)$ for all $m < \omega$. Hence $\nu = \pi_{i\lambda}(\bar{\nu}) < \rho_m^\lambda(m)$ by (5). QED(6).

All further verifications are trivial.

This completes the construction.

If $\lambda < \theta$ is least s.t. M_λ is undefined, then either $\mathrm{Lim}(\lambda)$ and (***) fails or else $\lambda = i+1$ and (*) or (**) fails. We now simplify this by showing that (**) cannot fail. We in fact show:

<u>Lemma 5.1</u> Let M_i be defined. Let $\xi = T(i+1)$, $N = N_\xi \| \eta_i$, $M = (M_\xi \| \mu_\xi) \| \sigma_\xi(\eta_i)$, $\sigma = \sigma_\xi \upharpoonright N$. Set:

$$\vec{\rho}^* = \begin{cases} \vec{\rho}^\xi & \text{if } \eta_i = ht(N_i) \\ \min(M, \sigma, \langle \rho^n_M \mid n < \omega \rangle) & \text{if not,} \end{cases}$$

Then: letting $F = E^{N_i}_{\gamma_i}$, $F' = E^{M_i}_{\gamma'_i}$:

(a) $\langle \sigma, \sigma_i \upharpoonright \lambda_i \rangle : \langle N, F \rangle \xrightarrow{**} \langle M | \rho^*_0, F' \rangle$

(b) $\langle \sigma, \sigma_i \upharpoonright \lambda_i \rangle : \langle N, F \rangle \xrightarrow{*} \langle M, F' \rangle$.

(a) expresses (**), but to prove it inductively we shall need to establish (b) as well. Both (a) and (b) are easily established if $F \in N_i$, so it reduces to the case that F is the top extender of N_i. The relevant lemma for this case will be established by an induction on the possibility of using the top extender. We imitate the methods of §5, which are in turn based upon §4 Lemma 1.

<u>Def</u> Let $E^{N_i}_{ht} \neq \emptyset$. (We write E^{Q}_{ht} instead of $E^{Q}_{ht(Q)}$.) Set:

$\bar{\kappa}_i = \text{crit}(E^{N_i}_{ht})$, $\bar{\tau}_i = \bar{\kappa}_i^{+N_i}$,

δ_i = the least δ s.t. $\delta = i$ or $\bar{\kappa}_i < \lambda_{\delta_i}$

$\bar{\eta}_i$ = the maximal $\eta \leq ht(N_{\delta_i})$ s.t. $\bar{\tau}_i$ is a cardinal in $N_i \| \eta$.

$\bar{\kappa}'_i, \bar{\tau}'_i$ have the same definition wrt $E^{M_i}_{\mu_i}$. Thus δ_i is least s.t. $\delta = i$ or $\bar{\kappa}'_i < \lambda'_\delta$. Set: $\bar{\mu}^*_i = \sigma_{\delta_i}(\bar{\eta}_i)$;

$\bar{\eta}'_i$ = the maximal $\eta \leq ht(M_{\delta_i})$ s.t. $\bar{\tau}'_i$ is a cardinal in $M_{\delta_i} \| \eta$

(Thus $\bar{\mu}^*_i \leq \mu_\delta$ and $\bar{\mu}^*_i = \bar{\eta}'_i$ if $\bar{\eta}_i < ht(N_{\delta_i})$. Otherwise $\bar{\mu}^*_i = \mu_\delta \leq \bar{\eta}'_i$.) Set:

$N'' = N_\delta \| \eta_i$; $M'' = M_\delta \| \bar{\mu}^*_i$, $\sigma'' = \sigma_\delta \upharpoonright N''$

$\langle \rho''_n \mid n < \omega \rangle$ is defined by:

$$\vec{\rho}\,'' = \begin{cases} \vec{\rho}^{\,\delta_i} & \text{if } \bar{\mu}^*_i = \mu_\delta \\ \min(N'', \sigma'', \langle \rho^n_{M''} \mid n < \omega \rangle) & \text{if not} \end{cases}$$

<u>Lemma 5.1.1</u> Let M_i be defined s.t. (a), (b) hold below i. Then

(+) Let $A \subset \bar{\tau}_i$ be $\Sigma_1(N_i)$ in p and $A' \subset \bar{\tau}_i'$ be $\Sigma_1(M_i \| \mu_i)$ in $p' = \sigma_i(p)$ by the same definition. Then A is $\Delta_1(N'')$ in some q and A' is $\Delta_1(M'')$ in $q' = \sigma''(q)$ by the same definition.

proof.

Suppose not. Let i be the least counter-example. Then $\delta_i < i$. It follows easily that i is not a limit ordinal. Let $i = h+1$. Set $\xi = T(i)$. Set:

$\kappa = \bar{\kappa}_i$, $\tau = \bar{\tau}_i$, $\delta = \delta_i$, $N^* = N_\xi \| \eta_h$,

$M^* = (M_\xi \| \mu_\xi) \| \eta_h'$, $\sigma^* = \sigma_\xi \upharpoonright N^*$.

(1) $\kappa < \kappa_h$ (hence $\pi_{\xi i} \upharpoonright \tau + N^* = \mathrm{id}$)

pf. Suppose not.

Let $\kappa' = \pi_{\xi i}^{-1}(\kappa) = \mathrm{crit}(E_{ht}^{M^*})$. Then $\kappa' \geq \kappa_h$, since otherwise $\kappa = \pi_{\xi i}(\kappa') = \kappa' <$

Hence $\kappa = \pi_{\xi i}(\kappa') \geq \pi_{\xi i}(\kappa_h) = \lambda_h$. . . . . .

Hence $\delta = i$. Contr! QED(1)

(2) $\delta \leq \xi$, since $\kappa < \kappa_h < \lambda_\xi$.

(3) $\omega\rho^1_{N_i} \leq \tau$

prf. of (3). Suppose not.
Let $A \subset \tau$ be $\Delta_1(N_i)$ in p and $A' \subset \tau'$ be $\Delta_1(M_i \| \mu_i)$ in $p' = \sigma_i(p)$ by the same def.
Then $A\xi \longleftrightarrow N_i \models \bigvee z\, \varphi_0(z, \xi, p)$

$\neg A\xi \longleftrightarrow N_i \models \bigvee z\, \varphi_1(z, \xi, p)$,

where φ_0, φ_1 are Σ_0. The same holds for A' with $M_i \| \mu_i$, p' in place of N_i, p.
Since $\tau < \omega\rho^1_N$, we have $A \in \mathbb{P}(\tau) \cap N_i \subset$
$\subset J_\delta^{E^{N_i}} = J_\delta^{E^{N_\delta}} \subset N''$. We have:

$$N_i \models \underbrace{\bigwedge \xi < \tau\ \bigvee z(\varphi_0(z, \xi, p) \vee \varphi_1(z, \xi, p))}_{\Pi^1_0 \text{ in } \tau, p.}$$

But then the same ~~fmla~~ holds in τ', p' in $(M_i \| \mu_i) \| \rho_0^{i}$, since $\sigma_i : N_i \xrightarrow[\Sigma^*]{} M_i \| \mu_i \bmod (\vec{\rho}^{\,i})$. At follows easily that:

$A'\xi \longleftrightarrow ((M_i \| \mu_i) \| \rho_0^{i}) \models \bigvee z\, \varphi_0(z, p', \xi)$

$\neg A'\xi \longleftrightarrow (\quad " \quad) \models \bigvee z\, \varphi_1(z, p', \xi)$

Thus A' is $\Delta_1((M_i \| \mu_i) \| \rho_0^{i})$ in p' by the same definition.

The formula $x = A$ is then $\Pi^1_0(N_i)$ in p, τ and $x = A'$ is $\Pi^1_0((M_i \| \mu_i) \| \rho^i_0)$ in p', τ by the same formula. Since $\sigma_i : N_i \longrightarrow_{\Sigma*} M_i \| \mu_i$ min(ρ^{i}), it follows that $\sigma_i(A) = A'$. But $\sigma_i \upharpoonright \lambda_\delta = \sigma_\delta \upharpoonright \lambda_\sigma$. Hence $\sigma_\delta(A) = \sigma_i(A)$ $= A'$. Thus A is $\Delta_1(N'')$ in the parameter A and A' is $\Delta_1(M'')$ in $A' = \sigma''(A)$ by the same def. Contr! QED(3)

Hence (+) holds.

(4) $\rho^1_{N*} \leq \tau$ (since $\pi_{\zeta i} : N^* \longrightarrow_{\Sigma*} N_i$, $\pi_{\zeta i}(\tau) = \tau$)

(5) Let $A \subset \kappa_n$ be $\Sigma_1(N_i)$ in p and $A' \subset \kappa'_n$ be $\Sigma_1(M_i \| \mu_i)$ in $p' = \sigma_i(p)$ by the same def. Then A is $\Sigma_1(N^*)$ in some q and A' is $\Sigma_1(M^*)$ in $q' = \sigma_\zeta(q)$ by the same def.

proof.

We first note that:

$\pi'_{\zeta i} \upharpoonright M^* : M^* \longrightarrow_{\Sigma_0} M_i \| \mu_i$ cofinally.

(To see this, note that there is a cofinal map $f : \tau_h' \longrightarrow ht(M^*)$ defined by:

$f(\alpha) =$ the least β s.t.

$$\bigwedge x \in \mathbb{P}(\kappa_h') \cap J_\alpha^E \bigvee y \in J_\beta^E \; y = E_{ht}(x).$$

If f' has the same def. over $M_i \| \mu_i$, then f' is cofinal in μ_i and $\pi_{\xi_i}(f(\alpha)) = f'(\alpha)$, since $\pi_{\xi_i} \upharpoonright \tau = \mathrm{id}$.)

Now let $A\varsigma \longleftrightarrow \bigvee z \, Bz\varsigma p$ and $A'\varsigma \longleftrightarrow \bigvee z \, B'z, \varsigma, p'$, where B is $\Sigma_0(N_i)$ and B' is $\Sigma_0(M_i \| \mu_i)$ by the same def. Let $F = E_{ht}^{N_i}$, $F' = E_{\mu_i}^{M_i}$.

By (3) we have: $\pi_{\xi_i} : N^* \longrightarrow_F N_i$.

Hence $p = \pi(f)(\alpha)$ for an $\alpha < \lambda_{h}'$, $f : \kappa_h \to N^*$, $f \in N^*$. Let $f' = \sigma^*(f)$ and $\alpha' = \sigma_i(\alpha)$. Then $p' = \pi'(f')(\alpha')$ (where $\pi = \pi_{\xi_i} \upharpoonright N^*$, $\pi' = \pi_{\xi_i}' \upharpoonright M^*$).

By the ind. hyp. F_α is $\Sigma_1(N^*)$ in some r and F_α' is $\Sigma_1(M^*)$ in $r' = \sigma^*(r)$. Let $\bar{B}$ be $\Sigma_0(N^*)$ and $\bar{B}'$ be $\Sigma_0(M^*)$ by the same def as B.

Then:

$$A\xi \leftrightarrow \bigvee u \in N^* \bigvee z \in \pi(u)\, B(z,\xi,p)$$

$$\leftrightarrow \quad " \quad \underbrace{\{\eta < \kappa_h \mid \bigvee z \in u\, \bar{B}(z,\xi,f(\eta))\} \in F_\alpha}_{\Sigma_1(N^*) \text{ in } \kappa_h, f, r}$$

By the cofinality of π' and the fact that $\alpha' \in \pi'(X) \leftrightarrow X \in F_{\alpha'}$, we can do the same analysis to get:

$$A'\xi \leftrightarrow \bigvee u \in M^* \underbrace{\{\eta < \kappa'_h \mid \bigvee z \in u\, \bar{B}'(z,\xi,f'(\eta))\} \in F_{\alpha'}}_{\Sigma_1(M^*) \text{ in } \kappa'_h, f', r'}$$

QED (5).

(6) $\bar{\xi} > \delta$

prf. Suppose not. Then $\bar{\xi} = \delta$. Hence $\eta_h \leq \bar{\eta}_i$, since $\tau < \kappa_h$. If $\eta_h = \bar{\eta}_i$, then $N^* = N''$, $M^* = M''$, $\sigma^* = \sigma''$ and it is immediate from (5) that (+) holds. Contr. Hence $\eta_h < \bar{\eta}_i$. Hence $\sigma''(N^*) = M^*$. But by (5), if A, A' are as in (+), then A is $\Sigma_1(N^*)$ in a q and A' is $\Sigma_1(M^*)$ in q' by the same def. Hence $\sigma''(A) = A'$. Hence (+) holds. Contr! QED (6)

(7) $N^* = N_\xi$ (i.e. $\eta_h = ht(N_\xi)$).

prf.

If not, $\tau^{+N_\xi} > \omega\eta_h = On \cap M^*$ by (4).

But $\tau < \lambda_\delta$, where $\lambda_\delta < \lambda_\xi$ is a limit cardinal in N_ξ. Contr. QED (7)

Thus $\pi_{\xi i} : N_\xi \to_{\Sigma^*} N$, $\pi_{\xi i}(\kappa) = \kappa$.

Hence $\kappa = \bar\kappa_\xi$, $\tau = \bar\tau_\xi$, $\delta = \delta_\xi$, $\bar\eta_\xi = \bar\eta_i$.

Since (+) holds at ξ, it follows from (5) that (+) holds at i. Contr!

QED (Lemma 5.1.1)

<u>Def</u> i is <u>bold</u> iff δ_i is defined and whenever $A \subset \tau$ is $\Delta_1(N_i)$ in p and $A' \subset \tau'$ is $\Delta_1(M_i \| \mu_i)$ in $p' = \sigma_i(p)$ by the same definition, then $A \in N''$ and $A' = \sigma''(A)$

The following pendant to Lemma 5.1 is obtained by a very slight modification of its proof:

<u>Lemma 5.1.2</u> Let $\delta = \delta_i$ be defined s.t. (a), (b) hold below i and i is

(++) not bold. Let $A \subset \overline{\overline{\tau}}_i$ be $\Sigma_1(N_i)$ in p and $A' \subset \overline{\overline{\tau}}_i'$ be $\Sigma_1((M_i \| \mu_i) | \rho_0^i)$ in $p' = \sigma_i(p)$ by the same def.

Then A is $\Sigma_1(N'')$ in some q and A' is $\Sigma_1(M'' | \rho_0'')$ in $q' = \sigma(q)$ by the same definition.

prf.

Suppose not. Let i be least counterexample. Then $\delta < i = h+1$. Define ξ etc. as before.

(1) $\kappa < \kappa_h$. (proof. Exactly as before)

(2) $\delta \leq \xi$ (")

(3) $\omega\rho^1_{N_i} \leq \tau$.

This is proven as before, but is somewhat easier.

(4) $\rho^1_{N^*} \leq \tau$ (as before)

A literal repetition of our previous proof gives;

(5) (as before).

But by (4) and Lemma 4.7:

(5.1) Let $A \subset u_h$ be $\Sigma_1(N_i)$ in p and $A' \subset u'_h$ be $\Sigma_1((M_i \| \mu_i) | \rho_0^i)$ in $p' = \sigma_i(p)$ by the same def. Then A is $\Sigma_1(N^*)$ in some q and A' is $\Sigma_1(M^* | \rho_0^*)$ in $q' = \sigma_\xi(q)$ by the same def.

Hence:

(6) $\xi > \delta$

pf. Suppose not. Then $\xi = \delta$ and $\eta_h \leq \bar{\eta}_i$ as before. At $\eta_h = \bar{\eta}_i$ it follows as before from (5.1) that (++) holds. At $\eta_h < \bar{\eta}_i$ it follows as before from (5) that i is bold. Contr!

QED(6)

(7) $N^* = N_\xi$ (pf. as before).

The conclusion then follows exactly as before, using (5.1) in place of (5). QED (Lemma 5.1.2)

We now prove Lemma 5.1 by induction on i. Let $\xi, N, M, \sigma, \vec{\rho}^*, F, F'$ be as in Lemma 5.1. Let $\alpha < \lambda_i$, $\alpha' = \sigma_i(\alpha)$.

<u>Case 1</u> $F \in N_i$.

Then $F_\alpha \in N$, since if $\xi < i$, then $F_\alpha \in J_{\lambda_\xi}^{E^{N_i}} = J_{\lambda_\xi}^{E^N} \subset N$. Thus $\sigma(F_\alpha) = \sigma_i(F_\alpha) = F'_{\alpha'}$. This establishes (b).

It also establishes:

$$\langle \sigma, \sigma_i \restriction \lambda_i \rangle : \langle N, F \rangle \longrightarrow^* \langle M | \rho_0^*, F' \rangle,$$

since $F'_{\alpha'} = \sigma(F_\alpha) \in M | \rho_0^*$. Hence (a) holds.

<u>Case 2</u> $F \notin N_i$

Then F is the top extender and $\xi = \delta_i$, $\kappa_i = \bar{\kappa}_i$, $\tau_i = \bar{\tau}_i$. We note that F_α is $\Delta_1(N_i)$ in α, since:

$$X \in F_\alpha \longleftrightarrow \bigvee Y (\alpha \in Y = F(X))$$

$$X \notin F_\alpha \longleftrightarrow \bigvee Y (\alpha \notin Y = F(X))$$

where $X \in \mathbb{P}(\kappa_i) \cap N_i$.

$F'_{\alpha'}$ is $\Delta_1(M_i \| \mu_i)$ in $\alpha' = \sigma_i(\alpha)$ by the same definition. Hence by Lemma 5.1.1. F_α is $\Sigma_1(N)$ in some q and $F'_{\alpha'}$ is $\Sigma_1(M)$ in $q' = \sigma(q)$ by the same def. Hence (b) holds.

We now prove (a):

<u>Case 2.1</u> i is bold.

Then $F_\alpha \in N$, $\sigma(F_\alpha) = F'_{\alpha'}$ and the conclusion follows as in Case 1.

<u>Case 2.2</u> Case 2.1 fails.

Set $F_\alpha = \bar{G}$ and let G be $\Sigma_1((M_i \| \mu) \| \rho_0$ in α' by the above Σ_1 def of F_α.

Define $\bar{H} \subset {}^{n_i}\mathbb{P}(\kappa_i)$ by:

$$X \in \bar{H} \longleftrightarrow \bigvee \gamma \in N_i \bigwedge j < \kappa_i \bigvee Y \in J_\gamma^{E^{N_i}} \; Y = F(X_i).$$

Then $\bar{H} = (N_i \cap {}^{n_i}\mathbb{P}(\kappa_i))$ is $\Sigma_1(N_i)$.

Let H be $\Sigma_1((M_i \| \mu_i) \| \rho'_0)$ by the same def. Then clearly:

$$X \in H \longrightarrow \bigwedge j < \kappa'_i (X_j \text{ or } \kappa'_i \setminus X_j \in G).$$

By Lemma 5.1.1 $\bar{G}, \bar{H}$ are $\Sigma_1(N)$

in some q and G, H are $\Sigma_1(M|\rho_0^*)$ in $q' = \sigma(q)$ by the same def.
Hence $\bar{G}, G, \bar{H}, H$ verify (a).

QED (Lemma 5.1)

<u>Remarks</u>

(A) The same proof goes through for $\sigma : N \xrightarrow[\Sigma^*]{} M \| \mu$ min $(\vec{\rho})$ where $\mu \leq ht(M)$ with: $\sigma_0 = \sigma$, $\mu_0 = \mu$, $\vec{\rho}^0 = \vec{\rho}$. If $\mu = ht(M)$ and $\sigma = id$, then in fact $\mathcal{J}' = \mathcal{J}$ and $\sigma_i = id$ for $i < \Theta$.

(B) By Lemma 5.1, M_i can only be undefined for an $i<\theta$ if there is a failure of well foundedness. This cannot occur if $\mathcal{I}'$ obeys a normal iteration strategy for M. Let S be such a strategy. Define a normal iteration strategy $\bar{S}$ for N as follows: Let $\bar{\mathcal{I}}$ be a normal iteration of N of limit length. Form $\bar{\mathcal{I}}'$ as above. If $lh(\bar{\mathcal{I}}') < lh(\bar{\mathcal{I}})$, then $\bar{S}(\bar{\mathcal{I}})$ is undefined. If not, set $\bar{S}(\bar{\mathcal{I}}) \simeq S(\bar{\mathcal{I}}')$. Clearly, if $\mathcal{I}$ obeys $\bar{S}$, then $\mathcal{I}'$ obeys S + hence M_i is defined for $i<\theta$. If $\mathrm{Lim}(\theta)$, $b = S(\mathcal{I}')$, then $b = \bar{S}(\mathcal{I})$. N_b is then well founded, since there is $\sigma : N_b \to M_b$ defined by: $\sigma\pi_i = \pi'_i\sigma_i$, where:

$$M_b, \langle \pi'_i \rangle = \lim_{i \le j \text{ in } b} (M_i, \pi'_{ij})$$

$$N_b, \langle \pi_i \rangle = \lim_{i \le j \text{ in } b} (N_i, \pi_{ij}),$$

(Clearly $\bar{S} = S$ if $\sigma = id$, $\mu = ht(M)$, by (A).)

Thus $\bar{S}$ is a normal iteration

Now let $\mathcal{J} = \langle\langle N_i\rangle, \langle \nu_i\rangle, \langle \eta_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a good iteration of M of length Θ. Just as above we define a smooth iteration $\mathcal{J}' = \langle\langle M_i\rangle, \langle \nu_i'\rangle, \langle \eta_i'\rangle, \langle \pi_{ij}'\rangle, T \cap \bar\Theta^2\rangle$ of length $\bar\Theta \leq \Theta$ with maps σ_i + sequences $\vec\rho^{\,i}$ satisfying (A), (B) and:

(C)

(i) If there is $h \leq i$ s.t. $h \notin D$ and $\eta_h < ht(N_h)$ then, letting h be the maximal such, set: $\mu_i = \pi_{hi}' \sigma_h(\eta_h)$ if $\sigma_h(\eta_h) \in \mathrm{dom}(\pi_{hi}')$. In all other cases $\mu_i = ht(M_i)$

[This replaces the def. of μ_i in the earlier version of (B)]

(ii) If $i \notin D$, $\eta_i < ht(N_i)$, then $\eta_i' = ht(M_i)$ (hence $M_{i+1} = M_i$), $\pi_{i,i+1}' = id$

$\mu_{i+1} = \sigma_i(\eta_i)$, $\sigma_{i+1} = \sigma_i \restriction (N_i \| \eta_i)$,

$\vec\rho^{\,i+1} = \min(M_{i+1} \| \mu_i, \sigma_{i+1}, \langle \rho^n_{M_{i+1}\|\mu_i} \mid n<\omega\rangle$

(iii) If $i \notin D$, $\eta_i = ht(N_i)$, then $\eta_i' = ht(M_i$
$\pi'_{i,i+1} = id$, $\mu_{i+1} = \mu_i$, $\sigma_{i+1} = \sigma_i$, $\vec{\rho}^{\,i+1} = \vec{\rho}^{\,i}$.

Exactly as in Lemma 5.1 it then follows that (**) holds whenever M_i is defined. Hence the failure of an M_i to be defined can only be due to a failure of well foundedness. Let S be a smooth iteration strategy for $\mathbf{M}$. Define $\bar{S}$ as before. It follows as before that $\bar{S}$ is a good iteration strategy for M. (Note If $\mathcal{I}$ is itself smooth, then $\mathcal{I}' = \mathcal{I}$ and $\bar{S}(\mathcal{I}) = S(\mathcal{I})$. Thus $\bar{S} \supset S$.)

QED (Lemma 5)

Finally, we note:

If we apply the methods of § 5 using $\sigma : M \xrightarrow[\Sigma^*]{} N \min(\vec{p})$ in place of $\sigma : M \xrightarrow[\Sigma^*]{} N$, we get:

<u>Lemma 6</u> Let N be iterable and let $\sigma : M \xrightarrow[\Sigma^*]{} N \min(\vec{p})$. Then M is iterable.

The proof is straightforward, given § 5.