§1 Robust Extenders

Let $N = \langle J_\nu^E, F \rangle$ be an active λ-premouse. We call F ω-*complete* iff whenever $U \subset \lambda = lh(F)$ and $W \subset \mathbb{P}(\kappa) \cap N$ are countable sets, then there is a map $g : U \to \kappa$ s.t.

(a) $\langle g(\vec{\alpha}) \rangle \in X \leftrightarrow \langle \vec{\alpha} \rangle \in F(X)$

whenever $\alpha_1, \dots, \alpha_n \in U$ and $X \in W$. (Here $\langle\ \rangle$ is Gödel's tupel function on ordinals.)

[Note We recall that in an active λ-premouse $N = \langle J_\nu^E, F \rangle$, $F = E_\nu^N$ is an extender giving rise to the ultrapower embedding $\pi : J_\tau^E \to_F J_\nu^E$, where $\tau = \tau_\nu = \kappa^{+N}$ and $\kappa = \kappa_\nu = crit(\pi)$. We identify F with $\pi \upharpoonright \mathbb{P}(\kappa)$ and set $\lambda = \lambda_\nu = lh(F) = \pi(\kappa)$. The theorems in this paper can be adapted to other indexings as well.]

ω-completeness is the criterion traditionally used to establish linear iterability. Ralf Schindler has shown that it can also be used to establish the non linear iterability of sufficiently simple premice, (This was also implicit in work of Tony Dodd.) Eventually, however, this criterion appears to fail. In order to handle the "next step", Qi Feng and I introduced the notion "supercomplete" in [FJ]:

F is <u>supercomplete</u> iff for all U, W as above there is $g : U \to \kappa$ s.t. (a) holds and:

(b) Let $v \subset U$. Let t_ξ = the ξ-th element in the canonical well ordering of $L_\lambda[E]$. If $\bigcup_{\xi \in v} t_\xi$ is a well founded relation, then so is $\bigcup_{\xi \in v} t_{g(\xi)}$.

(<u>Note</u> It would make no difference in the application + would perhaps be more natural, if we strengthened the "if" in (b) to an "if and only if".)

For a certain class of premice (a small subclass of the 1-small premice), we were able to carry out Steel's K^c construction in ZFC, using only the condition "supercompleteness" in place of Steel's "background certification". We then showed that the model is iterable and universal wrt. the chosen class of mice.

Both the notions "ω-complete" and "supercomplete" are "self contained" in the sense that they make no reference to objects extraneous to $N = \langle J_\gamma^E, F \rangle$, other than countable sets of ordinals. More precisely: The notion is absolute in any inner model containing N and all countable sets of ordinals. In this paper we introduce the notion of <u>robust</u> extender, which is similarly self contained. This notion appears to have the same efficacy as Steel's "background certification". In particular, we can construct in ZFC the K^c-model of 1-small mice, showing (under the assumption that there is no inner model with a Woodin cardinal) that this model is ∞-iterable and universal. (Mitchell obtained similar results (cf [MSch]), but needed GCH to prove universality. He also replaced "background certification" by a weaker condition.)

In formulating the notion of robustness we shall make use of Chang's model C_∞. This is the smallest inner model containing all countable sets of ordinals. More generally, $C_\infty(u)$ is the smallest inner model containing u and all countable sets of ordinals. If $TC(u)$ is well orderable in $C_\infty(u)$, then $C_\infty(u)$ is, in fact, closed under finite subsets (although it need not satisfy the axiom of choice). The precise definitions are:

<u>Def</u> $C_\xi(u)$ $(\xi \le \infty)$ is defined by:

$$C_0(u) = TC(\{u\})$$

$$C_{\nu+1}(u) = \mathrm{Def}(C_\nu(u)) \cup [\nu]^\omega$$

$$C_\lambda(u) = \bigcup_{\nu<\lambda} C_\nu(u) \text{ for limit } \lambda \le \infty.$$

(Here $\mathrm{Def}(X)$ = the set of $w \subset X$ which are $\langle X, \in \rangle$-definable in parameters from X.)

Letting $L_3^E = L_3[E]$ be the relativized constructible hierarchy, we then set:

$$\bar{C}^E_{\tau,\eta} = C^E_\eta(\langle L^E_\tau, E \cap L^E_\tau \rangle)$$

$$C^E_{\tau,\eta} = \langle \bar{C}^E_{\tau,\eta}, \in, E \cap L^E_\tau, \langle \bar{C}^E_{\tau,\nu} \mid \nu < \eta \rangle \rangle.$$

We are now ready to define:

Def Let $N = \langle J^E_\nu, F \rangle$ be an active premouse. F is <u>robust</u> wrt. N iff whenever $u \subset \lambda = \lambda_\nu$ and $W \subset \mathbb{P}(\kappa) \cap N$ ($\kappa = \kappa_\nu$) are countable, then there is $g : u \to \kappa$ s.t. (a) holds and:

(b) Let $v \subset u$, $\tau = \operatorname{lub} u$, $\bar{\tau} = \operatorname{lub} g''u$.
Let φ be a Σ_1 fmla. Then:
$C^E_{\bar{\tau},\kappa} \models \varphi(g''v, g''u) \leftrightarrow C^E_{\tau,\infty} \models \varphi(v, u)$,

(<u>Note</u> If $\omega \subset u$, it follows from (b) that if $v_1, \ldots, v_n \subset u$ and $\alpha_1, \ldots, \alpha_m \in u$, then:
$C^E_{\bar{\tau}\kappa} \models \varphi(g(\vec{\alpha}), g''\vec{v}) \leftrightarrow C^E_{\tau,\infty} \models \varphi(\vec{\alpha}, \vec{v})$
for Σ_1 fmlae φ. We leave this to the reader.)

(<u>Note</u> If F is robust, then
$$C^E_{\gamma,\kappa} \prec_{\Sigma_1} C^E_{\gamma,\infty} \text{ for all } \gamma < \kappa.$$
This, too, we leave to the reader.)

(<u>Note</u> The hierarchy $C^E_{\gamma,\eta}$ satisfies a condensation principle:

Let $C \prec_{\Sigma_1} C^E_{\gamma,\eta}$ s.t. $C^\omega \subset C$ and $\gamma \subset C$. Then $C \simeq C^E_{\gamma,\bar\eta}$ for an $\bar\eta \leq \eta$. Using this, it follows easily that $C^E_{\gamma,\kappa} \prec C^E_{\gamma\infty}$ for all $\gamma < \kappa$, whenever κ is regular and $2^\alpha < \kappa$ for $\alpha < \kappa$. (This holds, in particular, for $\kappa = (2^\beta)^+$.))

We recall some definitions:

Def A premouse N is weakly iterable (or a weak mouse) iff whenever $\sigma : \bar N \prec N$ and $\bar N$ is countable, then $\bar N$ is ω_1+1-iterable.

Weakly iterable premice are solid and satisfy the condensation lemmas for mice.

Def By an array we mean a sequence $\langle N_i \mid i \leq \theta \rangle$ $(\theta \leq \infty)$ of premice s.t.

(a) N_i is a weak mouse for $i < \theta$

(b) $N_0 = \langle \emptyset, \emptyset \rangle$

(c) Let $i < \theta$ where $M_i = \langle J^E_\gamma, E_{\omega\gamma} \rangle = \mathrm{core}(N_i)$. Then either $N_{i+1} = \langle J^E_{\gamma+1}, \emptyset \rangle$ or else $E_{\omega\gamma} = \emptyset$ and $N_{i+1} = \langle J^E_\gamma, F \rangle$ is active.

(d) Let $\lambda \leq \theta$ be a limit ordinal. For $\xi < \lambda$ set:

$\kappa_\xi = \kappa_{\xi,\lambda} = \inf\{\omega\rho^\omega_{N_i} \mid \xi \le i < \lambda\}$

$\mu_\xi = \mu_{\xi,\lambda} = \kappa_\xi^{+N_\xi}$ (with $\mu_\xi = \kappa_\xi$ if $\kappa_\xi = ht(N_\xi)$).

If $J^{E^{N_\xi}}_{\mu_\xi} = J^{E^{N_i}}_{\mu_\xi}$ for $\xi \le i < \lambda$, then

$N_\lambda = \langle \bigcup_{\xi<\lambda} J^{E^{N_\xi}}_{\mu_\xi}, \emptyset \rangle$.

(Otherwise N_λ is undefined. It can be shown, however, that the condition for defining N_λ is always satisfied.

Following Steel, we call $\mathcal{J} = \langle\langle P_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T\rangle$ a <u>putative</u> <u>normal iteration</u> if it is like a normal iteration except that possibly $lh(\mathcal{J}) = i+2$ and the final model P_{i+1} is ill founded,

<u>Def</u> By a <u>robust array</u> we mean an array $\langle N_i \mid i \le \theta \rangle$ s.t. F is robust whenever $N_{i+1} = \langle J^E_\gamma, F \rangle$ is active.

Our main theorem then reads:

Thm 1 (ZFC) Let $\langle N_i \mid i \leq \theta \rangle$ be a robust array, where $\theta < \infty$. Let $\sigma : P \prec N_\theta$, where P is countable. Let $\mathcal{I} = \langle\langle P_i \rangle, \ldots, T\rangle$ be a countable putative normal iteration of P. Then one of the following holds:

(a) $lh(\mathcal{I}) = i+1$, there is no truncation at an $h \leq_T i$, and there is a "sufficiently elementary" embedding $\sigma' : P_i \to N_\theta$ s.t. $\sigma' \pi_{0i} = \sigma$.

(b) $lh(\mathcal{I}) = i+1$, there is a truncation at an $h \leq_T i$, and there is a "sufficiently elementary $\sigma : P_i \to N_\xi$, where $\xi < \theta$.

(c) $\mathcal{I}$ has a maximal branch b of limit length s.t. b has no truncation pts. and there is "suff. elementary" $\sigma' : P_b \to N_\theta$ s.t. $\sigma' \pi_b = \sigma$ (where P_b is the limit model and π_b the canonical embedding);

(d) $\mathcal{I}$ has a maximal branch b of limit length with a truncation point and there is a "suff. elementary" $\sigma' : P_b \to N_\xi$, where $\xi < \theta$.

(Note We shall not comment on the notion "sufficiently elementary", other than to say that it entails Σ_0-elementarity.)

Note $\mathcal{I}$ is called realizable wrt. $\sigma : P \prec N_\theta$ iff either (a) or (b) holds, or else (c) or (d) holds for a branch which lies cofinally in $\mathcal{I}$. Thus fr. ins. if $\mathcal{I}|\lambda$ has at most one realizable branch for $\lambda < lh(\mathcal{I})$, it follows that $\mathcal{I}$ is realizable. It is obvious that a putative iteration is an iteration if it is realizable.]

This is identical with Steel's main Theorem, except that he requires that each extender F used in the construction be "background certified" rather than merely robust. He also does not prove the Theorem in ZFC, but works in V_Ω, where Ω is an inaccessible.

We now list some well known consequences for 1-small premice: If $\sigma : P \prec N_\theta$, P being countable, N_θ is 1-small and $\sigma : P \prec N_\theta$, P being countable, we can devise a strategy for countable putative iterations $\mathcal{I}$ of P s.t. each $\mathcal{I}$ which follows the strategy is realizable. Hence P is ω_1-iterable. A forcing argument shows, in fact, that P is ω_1+1 iterable. Hence:

<u>Corollary 2</u> If N_θ is 1-small, then N_θ is a weak mouse.

By known methods this yields:

<u>Corollary 3</u> Let $\langle N_i \mid i \leq \theta \rangle$ $(\theta \leq \infty)$ be a robust array in which each N_i is 1-small. Assume either that there is no inner model with a Woodin cardinal or that V is closed under #. Then N_θ is ∞-iterable.

<u>Def</u> The 1-small <u>robust</u> K^c-model is $K = N_\infty$, where N_i $(i \leq \infty)$ is the array formed by taking $N_{i+1} = \langle J_\nu^E, F \rangle$ whenever $M_i = \mathrm{core}(N_i) = \langle J_\nu^E, \emptyset \rangle$, is 1-small, F is robust, and $\langle J_\nu^E, F \rangle$ is 1-small. (A bicephalus argument shows that the choice of F is unique.) The construction cannot break down by Cor 2.

<u>Def</u> Let K be the 1-small robust K^c-model. K is <u>universal</u> iff

(a) K is ∞-iterable

(b) The coiteration of K with any 1-small ∞-iterable premouse terminates below ∞.

(By Cor 3, (a) is satisfied if there is no inner model with a Woodin cardinal or V is closed under #.)

We now sketch a proof of:

<u>Thm 4</u> Let K be the 1-small robust K^c model. If K is ∞-iterable, then K is universal.

prf.

Let N be an iterable premouse.

Let $\bar{\bar{N}} < \theta$ where $2^{2^\omega} < \theta$, $\alpha^\omega < \theta$ for $\alpha < \theta$. [E.g. $\theta = (2^\beta)^+$, where $\beta \geq 2^\omega, \bar{\bar{N}}$.]

<u>Claim</u> The coiteration of N, K terminates below θ.

prf. Suppose not.

Let $\mathcal{J}_0, \mathcal{J}_1$ be the coiteration of $N^0 = N$, $N^1 = K \| \theta^+$ up to $\theta + 1$, (Here $K \| \xi = \langle J_\xi^E, E_{\omega\xi} \rangle$, where $K = J_\infty^E$..)

Let π^0_{ij}, π^1_{ij} be the iteration maps. ~~...~~ Then

Let $\Omega > \bar{\bar{H}}_\theta$ be regular.

(1) $\mathcal{J}_0, \mathcal{J}_1 \in H_\Omega$.

Let $X \prec H_\Omega$ s.t. $\mathcal{J}_0, \mathcal{J}_1 \in X$, $\bar{\bar{X}} < \theta$, $[X]^\omega \subset X$, $\mathcal{P}(2^\omega) \subset X$, $X \cap \theta$ is transitive, (Such X exists by our assumptions on θ.)

Set: $\sigma : \bar{H} \xrightarrow{\sim} X$ with $\bar{H}$ transitive.

Then $\sigma : \bar{H} \prec H_\Omega$, $\sigma \upharpoonright \bar{\theta} = \mathrm{id}$, $\sigma(\bar{\theta}) = \theta$.

Let $\sigma(\bar{\mathfrak{I}}_0, \bar{\mathfrak{I}}_1) = \mathfrak{I}_0, \mathfrak{I}_1$. Set:

$$\mathfrak{I}_0 = \langle \langle N_i \rangle, \langle \nu_i^0 \rangle, \langle \pi_{ij}^0 \rangle, T^0 \rangle$$

$$\bar{\mathfrak{I}}_0 = \langle \langle \bar{N}_i \rangle, \langle \bar{\nu}_i^0 \rangle, \langle \bar{\pi}_{ij}^0 \rangle, \bar{T}^0 \rangle$$

Since $\sigma \upharpoonright H_{\bar{\theta}}^{\bar{H}} = \mathrm{id}$, we have:

$\bar{\mathfrak{I}}_0 | \bar{\theta} = \mathfrak{I}_0 | \theta$ (i.e. $\bar{N}_i = N_i$, $\bar{\pi}_{ij} = \pi_{ij}$, $i \leq_{\bar{T}^0} j \leftrightarrow i \leq_{T^0} j$ for $i, j < \bar{\theta}$.)

Clearly $i \leq_{\bar{T}^0} \bar{\theta} \leftrightarrow i = \sigma(i) \leq_{T^0} \theta$ for $i < \theta$.

Hence $\bar{\theta} = \sup\{i \mid i \leq_{T^0} \theta \wedge i < \theta\} \leq_{T^0} \theta$.

Hence $i \leq_{\bar{T}^0} \bar{\theta} \leftrightarrow i \leq_{T^0} \bar{\theta}$.

(3) $N_{\bar{\theta}} = N_\theta$, $\bar{\pi}_{i\bar{\theta}}^0 = \pi_{i\bar{\theta}}^0$ for $i \leq_{T^0} \bar{\theta}$, since

$\langle \bar{N}_\theta, \langle \bar{\pi}_{i\bar{\theta}}^0 \mid i \leq_{\bar{T}^0} \bar{\theta} \rangle \rangle =$

$= \lim(\langle N_i \mid i \leq_{T^0} \bar{\theta} \rangle, \langle \pi_{ij}^0 \mid i \leq_{T^0} j \leq_{T^0} \bar{\theta} \rangle)$

$= \langle N_\theta, \langle \pi_{i\theta}^0 \mid i \leq_{T^0} \bar{\theta} \rangle \rangle$.

(4) $\sigma \upharpoonright N_{\bar{\theta}} = \pi_{\bar{\theta}\theta}^0$, since for $x \in N_{\bar{\theta}}$ there is $i \leq_{T^0} \bar{\theta}$ with $x = \pi_{i\bar{\theta}}(x')$. Hence $\sigma(x) = \sigma(\pi_{i\bar{\theta}}(x')) = \pi_{i\theta}(x') = \pi_{\bar{\theta}\theta}(x)$.

(5) Set: $\tau_{\bar{\theta}} = \bar{\theta}^{+N_{\bar{\theta}}}$, $\tau_\theta = \theta^{+N_\theta}$.

Then $\tau_\theta = \sup \pi_{\bar{\theta}\theta} `` \tau_{\bar{\theta}}$.

prf. (sketch)

Set $\theta_i = \pi_{\bar{\theta}i}(\bar{\theta})$ for $i \in [\bar{\theta}, \theta]_{T^0}$.

By ind. on i: $\theta_i = \mathrm{crit}(\pi_{i\theta})$ and $\pi_{ji} \upharpoonright J_{\tau_j}^{E^{N_j}} : J_{\tau_j}^{E^{N_i}} \xrightarrow[\Sigma_1]{} J_{\tau_i}^{E^{N_i}}$ cofinally for $j \leq_{T^0} i$ (where $\tau_i = \theta_i^{+N_i}$).

We now attempt the corresponding analysis on the K-side of the coiteration.

Let: $\mathcal{J}^1 = \langle\langle K_i\rangle, \langle \nu_i^1\rangle, \langle \pi^1_{ij}\rangle, T^1\rangle$

$\bar{\mathcal{J}}^1 = \langle\langle \bar{K}_i\rangle, \langle \bar{\nu}_i^1\rangle, \langle \bar{\pi}_{ij}\rangle, \bar{T}^1\rangle.$

As before: $i \leq_{\bar{T}^1} j \leftrightarrow i \leq_{T^1} j$ for $i, j < \bar{\theta}$.

Moreover $\bar{\theta} \leq_{T^1} \theta$ and $i \leq_{\bar{T}^1} \bar{\theta} \leftrightarrow i \leq_{T^1} \bar{\theta}$ as before. For $\xi < \bar{\theta}$, $i, j < \bar{\theta}$, $i \leq_{\bar{T}^1} j$, we have: $\bar{K}_i \| \xi = \sigma(\bar{K}_i \| \xi) = K_i \| \xi$,

$\bar{\pi}^1_{ij} \upharpoonright (\bar{K}_i \| \xi) = \sigma(\pi^1_{ij} \upharpoonright (\bar{K}_i \| \xi)) = \pi^1_{ij} \upharpoonright (K_i \| \xi)$.

Hence:

(6) $J_{\bar{\theta}}^{E^{\bar{K}_i}} = J_{\bar{\theta}}^{E^{K_i}}$, $\bar{\pi}^1_{ij} \upharpoonright J_{\bar{\theta}}^{E^{\bar{K}_i}} = \pi^1_{ij} \upharpoonright J_{\bar{\theta}}^{E^{K_i}}$

for $i, j < \bar{\theta}$.

(7) Set $\tilde{\theta} = \bar{\pi}^1_{0\bar{\theta}}(\bar{\theta})$. Then $J_{\tilde{\theta}}^{E^{\bar{K}_{\bar{\theta}}}} = J_{\tilde{\theta}}^{E^{K_{\bar{\theta}}}}$

and $\bar{\pi}^1_{i\bar{\theta}} \upharpoonright J_{\bar{\theta}}^{E^{\bar{K}_i}} = \pi^1_{i\theta} \upharpoonright J_{\theta}^{E^{K_i}}$ for $i \leq_{T^1} \bar{\theta}$,

and $\tilde{\theta} = \pi^1_{0\theta}(\bar{\theta})$, since:

$\langle J_{\tilde{\theta}}^{E^{\bar{K}_{\bar{\theta}}}}, \langle \bar{\pi}^1_{i\bar{\theta}} \upharpoonright J_{\bar{\theta}}^{E^{\bar{K}_i}} \mid i \leq_{\bar{T}^1} \bar{\theta}\rangle\rangle =$

$\lim(\langle J_{\bar{\theta}}^{E^{K_i}} \mid i <_{T^1} \bar{\theta}\rangle, \langle \pi^1_{ij} \upharpoonright J_{\bar{\theta}}^{E^{K_i}} \mid i \leq_{T^1} j <_{T^1} \bar{\theta}\rangle)$

$= \langle J_{\tilde{\theta}'}^{E^{K_{\bar{\theta}}}}, \langle \pi^1_{i\theta} \upharpoonright J_{\bar{\theta}}^{E^{K_i}} \mid i <_{T^1} \bar{\theta}\rangle\rangle$,

where $\tilde{\theta}' = \pi^1_{0\bar{\theta}}(\bar{\theta})$.

Hence:

(8) $\sigma \upharpoonright J_{\tilde{\theta}}^{E^{K_{\tilde{\theta}}}} = \pi^1_{\bar{\theta}\theta} \upharpoonright J_{\tilde{\theta}}^{E^{K_{\bar{\theta}}}}$, since for $x \in J_{\tilde{\theta}}^{E^{K_{\tilde{\theta}}}}$, $x = \bar{\pi}^1_{i\bar{\theta}}(x')$, $i \leq_{T^1} \bar{\theta}$, we have: $\sigma(x) = \sigma(\pi^1_{i\bar{\theta}}(x')) = \pi^1_{i\theta}(x') = \pi_{\bar{\theta}\theta}(x)$.

Hence:

(9) $\mathrm{crit}(\pi^1_{\bar\theta\theta}) \geq \bar\theta$. Moreover:

$\mathrm{crit}(\pi^1_{\bar\theta\theta}) > \bar\theta \longrightarrow \tilde\theta = \bar\theta$.

By (5), $\pi^0_{\bar\theta\theta}$ takes $J^{EN_{\bar\theta}}_{\tau_{\bar\theta}}$ cofinally to $J^{EN_\theta}_{\tau_\theta}$. By the condensation lemma we conclude: $J^{EN_{\bar\theta}}_{\tau_{\bar\theta}} = J^{EN_\theta}_{\tau_{\bar\theta}}$. But $J^{EN_\theta}_\theta = J^{EK_\theta}_\theta$. Hence:

(10) $J^{EN_{\bar\theta}}_{\tau_{\bar\theta}} = J^{EK_\theta}_{\tau_{\bar\theta}}$. But then

(11) $J^{EN_{\bar\theta}}_{\tau_{\bar\theta}} = J^{EK_{\bar\theta}}_{\tau_{\bar\theta}}$, since if not, we would have $\lambda_{\bar\theta} < \tau_{\bar\theta}$ and hence $\tau_{\bar\theta} \leq \bar\theta^{+K_\theta} = \theta^{+J^{EK_{\bar\theta}}_{\lambda_\theta}} < \tau_{\bar\theta}$. Contr!

(12) $\tau_\theta = \bar\theta^{+K_{\bar\theta}}$

prf. If not, $\mathrm{crit}(\pi^1_{\bar\theta\theta}) = \bar\theta$ and $\bar\theta$ is a truncation pt. on the branch $\{i \mid i <_{T^1} \theta\}$. But there are only finitely many such points and they lie below $\bar\theta$, since $\sigma : \bar H \prec H_\Omega$, $\sigma(\bar\theta) = \theta$. QED (12)

(13) $\mathrm{crit}(\pi^1_{\bar\theta\theta}) > \bar\theta$ (hence $\bar\theta = \pi_{0\bar\theta}(\bar\theta)$)

prf. Suppose not. Then $\mathrm{crit}(\pi^1_{\bar\theta\theta}) = \bar\theta$. By (12), (11) we have:

$\mathbb{P}(\bar\theta) \cap N_{\bar\theta} = \mathbb{P}(\bar\theta) \cap K_{\bar\theta}$. Let $i_n + 1 \leq_{T^n} \theta$ s.t. $T^n(i_n+1) = \bar\theta$ for $n = 0, 1$. Then $E^{W^n_{i_n}}_{\nu_{i_n}}(X) = \sigma(X) \cap \lambda_{i_n}$, But $i_0 = i_1$ by the usual argument, using the initial segment condition. Hence for $i = i_0 = i_1$ we have $E^{N_i^0}_{\nu_i} = E^{N_i^1}_{\nu_i}$, which is impossible in a coiteration. QED(13)

(14) $\mathbb{P}(\bar\theta) \cap K \subset \bar H$

Proof.
Let $X \in \mathbb{P}(\bar\theta) \cap K$. Set $Y = \pi^1_{0\bar\theta}(X) =$
$= \bigcup_{\xi < \bar\theta} \pi^1_{0\bar\theta}(X \cap \xi)$, where $\pi^1_{0\bar\theta} \upharpoonright J^{E^K}_{\bar\theta} =$
$= \bar\pi^1_{0\theta} \upharpoonright J^{E^K}_{\bar\theta} = \sigma^{-1}(\pi^1_{0\theta} \upharpoonright J^{E^K}_{\theta}) \in \bar H$,
Then $Y \in \operatorname{rng}(\pi^1_{0\bar\theta}) \subset K_{\bar\theta}$. Hence
$Y \in \mathbb{P}(\bar\theta) \cap K_{\bar\theta} \subset J^{N_{\bar\theta}}_{\tau_{\bar\theta}} \subset \bar H$, since $N_{\bar\theta} \in \bar H$.
Hence $X = \pi^{1\,-1}_{0\theta}{}''Y \in \bar H$. QED(14)

(14) is the key observation, due to Mitchell.

Now select $X' \prec H_\Omega$ with the same properties as X except that $\bar\theta \in X'$. Hence $\theta' = \theta \cap X' > \bar\theta$. Again let $\sigma' : H' \overset{\sim}{\longleftrightarrow} X' \prec H_\Omega$, with H' transitive. Set $\tilde\sigma = \sigma'^{-1}\sigma$. It follows easily that:

(15) (a) $\tilde\sigma \upharpoonright N_{\bar\theta} = \pi^0_{\bar\theta\theta'}$, (b) $\tilde\sigma(\pi^1_{0\bar\theta} \upharpoonright J^E_{\bar\theta}) = \pi^1_{0\theta'} \upharpoonright J^E_{\theta'}$

where $E = E^K$;

Set: $\tau = \bar\theta^{+K}$, $\nu = \theta'^{+K}$.

(16) $\tilde{\sigma} \upharpoonright J_\tau^E : J_\tau^E \xrightarrow[\Sigma_0]{} J_\nu^E$ cofinally.

proof:
$\pi^1_{0\bar{\theta}}$ takes J_τ^E cofinally to $J^{E\bar{\theta}}_{\tau_{\bar{\theta}}}$;
$\pi^1_{0\theta'}$ " J_ν^E " " $J^{E\theta'}_{\tau_{\theta'}}$;
$\tilde{\sigma}$ takes $J^{E\bar{\theta}}_{\tau_{\bar{\theta}}}$ to $J^{E\theta'}_{\tau_{\theta'}}$ cofinally, since $\tilde{\sigma} \upharpoonright M_{\bar{\theta}} = \pi_{\bar{\theta}\theta}$, $J^{E\bar{\theta}}_{\tau_{\bar{\theta}}} = J^{E^{M_\theta}}_{\tau_\theta}$, $J^{E\theta'}_{\tau_{\theta'}} = J^{E^{M_{\theta'}}}_{\tau_{\theta'}}$. Hence $(\pi^1_{0\theta})^{-1}\tilde{\sigma}\pi^1_{0\theta}$ takes J_τ^E cofinally to J_ν^E. At remains only to prove:

<u>Claim</u> $(\pi^1_{0\theta})^{-1}\tilde{\sigma}\pi^1_{0\theta} \upharpoonright \tau = \tilde{\sigma} \upharpoonright \tau$.

Let $\xi < \tau$. Then ξ is coded by an $X \in \mathbb{P}(\bar{\theta}) \cap K$. Hence $\pi_{0\bar{\theta}}(\xi)$ is coded by $Y = \pi_{0\bar{\theta}}(X)$. Hence $\tilde{\sigma}\pi_{0\bar{\theta}}(\xi)$ is coded by $\tilde{\sigma}(Y)$. Hence $(\pi_{0\theta'})^{-1}\tilde{\sigma}\pi_{0\bar{\theta}}(\xi)$ is coded by $(\pi^1_{0\theta'})^{-1}{}''\tilde{\sigma}(Y) = \tilde{\sigma}((\pi^1_{0\bar{\theta}})^{-1}{}''Y) = \tilde{\sigma}(X)$. But $\tilde{\sigma}(X)$ codes $\sigma(\xi)$. QED (16)

Set: $F = \tilde{\sigma} \upharpoonright (\mathbb{P}(\bar{\theta}) \cap K)$. Then $M = \langle J_\nu^E, F \rangle$ is a "prepremouse" – i.e. M satisfies all premouse conditions except, possibly, the initial segment condition for the top extender. Clearly we have: $\tilde{\sigma} \restriction J_{\bar\tau}^{E} : J_{\bar\tau}^{E} \longrightarrow_{F} J_{\gamma}^{E}$. The notion "robust" obviously makes sense for such structures and we prove:

<u>Lemma 4.1</u> F is robust for M.

prf.

By our choice of θ:

(17) $C^{a}_{\tau,\theta} \prec_{\Sigma_1} C^{a}_{\tau,\infty}$ for $a \in H_\theta$, $\tau < \theta$

Since $\bar H$ is countably closed we have:

(18) $C^{a}_{\tau,\alpha} = (C^{a}_{\tau,\alpha})_{\bar H}$ for $\tau < \alpha \in \bar H$, $a \in \bar H$.

Since $\sigma : \bar H \prec H_\Omega$ and $\sigma(\bar\theta) = \theta$ we have:

(19) $C^{a}_{\tau,\bar\theta} \prec_{\Sigma_1} C^{a}_{\tau,\infty}$ for $a \in \bar H$, $\tau < \bar\theta$.

by (17), (18), (19) hold in particular for $a = E|\bar\theta$, where $\bar K = J_\infty^E$. All of this holds mutatis mutandis for H' in place of $\bar H$.

Now let $u \subset \theta'$ be countable. (Recall: $\theta' = \tilde\sigma(\bar\theta) = \lambda_F$.) Let $g : \omega \xrightarrow{\text{onto}} u$. Then $g \in \bar H$ by countable closure.

Let $\vec{w} = \langle w_\xi \mid \xi < 2^\omega \rangle$ enumerate the subsets of ω. Then w is coded by a subset of 2^ω & hence $w \in \bar H$.

Set: $\mu = \mathrm{lub}\, u$.

Set: $\tilde{w} = \{\langle\varphi, \xi\rangle \mid \varphi$ is a Σ_1-fmla $\wedge\ \xi < 2^{\omega_1}$
$\wedge\ C^{E}_{\mu,\infty} \models \varphi(g''w_\xi, u)\}$.

Then $\tilde{w} \in \bar{H}$, since $\tilde{w} < 2^{\omega}$. Finally, let W be a countable subset of $\mathcal{P}(\bar{\theta}) \cap K$ and let $X = \langle X_i \mid i < \omega\rangle$ be an enumeration of W. Then $X \in \bar{H}$ by countable closure and $\tilde{\sigma}(X) = \langle \tilde{\sigma}(X_i) \mid i < \omega\rangle$.

Set: $\tilde{X}$ = the set of $\langle\langle i_1, \ldots, i_n\rangle, j\rangle$ s.t. $\langle g(\vec{i})\rangle \in \tilde{\sigma}(X_j)$. Then $\tilde{X} \in \bar{H}$.

Clearly: $\tilde{\sigma}(\tilde{w}) = \tilde{w}$, $\tilde{\sigma}(\tilde{X}) = X$. But
$\sigma(w) = w$,

in H' there is $g: \omega \to \theta'$ s.t.

(a) $\langle g(i_1), \ldots, g(i_n)\rangle \in \tilde{\sigma}(X)_j \longleftrightarrow \langle \vec{i}, j\rangle \in \tilde{X}$

(b) $C^{E|\theta'}_{\mu,\theta'} \models \varphi(g''w_\xi, u) \longleftrightarrow \langle\varphi, \xi\rangle \in \tilde{w}$.

But then since $g \in H'$ and $\tilde{\sigma}: \bar{H} \prec H'$, and elementarity gives $\tilde{\sigma}(E|\bar{\theta}) = E|\theta$,

is $\bar{g} \in \bar{H}$ s.t. $\bar{g}: \omega \to \bar{\theta}$ and

(c) $\langle \bar{g}(\vec{i})\rangle \in X_j \longleftrightarrow \langle \vec{i}, j\rangle \in \tilde{X}$

(d) $C^{E|\bar{\theta}}_{\bar{\mu},\bar{\theta}} \models \varphi(\bar{g}''w_\xi, u) \longleftrightarrow \langle\varphi, \xi\rangle \in \tilde{w}$,

where $\bar{\mu} = \sup \bar{g}''\omega$. Let $h: u \to \bar{\theta}$ be defined by: $h(g(i)) = \bar{g}(i)$. (This will be defined & unique if we assume that $\{\langle i, j\rangle < \bar{\theta} \mid i = j\}$ is in X_i.) Then h verifies robustness wrt. u, W.

QED (Lemma 4.1)

Clearly $F \notin K$, since otherwise $\pi \in K$, where $\pi = \tilde{\sigma} \upharpoonright J_\nu^E : J_\tau^E \xrightarrow[F]{} J_\nu^E$ cofinally; hence $cf(\nu) \leq \tau < \nu$ in K, where $\nu = \theta'^{+K}$. Contr!

Thus there must be a least $\lambda \leq \theta'$ s.t. $\bar{F} = F|\lambda \notin K$ and there is $\bar{\pi} : J_\tau^E \xrightarrow[\bar{F}|\lambda]{} J_{\bar{\nu}}^{\bar{E}}$ with $\bar{\pi}(\bar{\theta}) = \lambda$. The structure $\bar{M} = \langle J_{\bar{\nu}}^{\bar{E}}, \bar{F} \rangle$ will then satisfy the initial segment condition. But there is $\delta : M \xrightarrow[\Sigma_1]{} M$ defined by: $\delta(\bar{\pi}(f)(\alpha)) = \pi(f)(\alpha)$. At $\bar{\theta} < \xi < \bar{\tau}$ s.t. $\rho^1_{J_\xi^{\bar{E}}} = \bar{\theta}$, then $\rho^1_{J^E_{\delta(\xi)}} = \theta'$. Hence $\delta \upharpoonright J_\xi^{\bar{E}} \in M$, since $\delta(\bar{h}(i, \langle \xi, \bar{p} \rangle)) = h(i, \langle \xi, p \rangle))$, where $\bar{h} = h_{J_\xi^{\bar{E}}}$, $h = h_{J^E_{\delta(\xi)}}$, $\bar{p} = p_{J_\xi^{\bar{E}}}$, $p = p_{J^E_{\delta(\xi)}}$, and $\xi < \bar{\theta}$. By the internal condensation lemma of K, it follows that $J_\xi^{\bar{E}} = J_\xi^E$. Hence $\bar{M} = \langle J_{\bar{\nu}}^E, \bar{F} \rangle$. Since λ is a limit cardinal in K, it follows by standard methods (cf. [MI] §1, Fact 5 - Fact 9) that there is $\eta < \infty$ with $N_\eta = \langle J_{\bar{\nu}}^E, \emptyset \rangle$ and $\omega\rho^\omega_{N_i} \geq \lambda$ for all $i \geq \eta$. By soundness, $N_{\eta+1} = \bar{M} = M_{\eta+1} = N_i \| \bar{\nu}$ for all $i > \eta$.

Hence $\bar{M} = K \| \bar{\nu}$. Hence $\bar{F} \in K$.
Contradiction! Q.E.D (Lemma 4)