§2 Forcing to obtain a $\diamondsuit^+$-sequence from a $\diamondsuit$-sequence.

Let N be a trans. model of
$$ZFC + 2^{\omega} = \omega_1 + 2^{\omega_1} = \omega_2 + \diamondsuit. \quad ^{*)}$$
Let $S = \langle S_\alpha \mid \alpha < \omega_1 \rangle$ be a fixed $\diamondsuit$-sequence,
For $\alpha < \omega_1$ let $d_\alpha : \omega \xrightarrow{\text{onto}} \alpha + 1$, where $d = \langle d_\alpha \mid \alpha < \omega_1 \rangle \in N$, For $\alpha < \omega_1$ set:
$$M_\alpha = L_{\beta_\alpha}[S \restriction \alpha + 1, d \restriction \alpha + 1],$$
where β_α = the least $\beta > \alpha$ s.t.
$\beta > \sup\limits_{\gamma < \alpha} \beta_\gamma$ and $L_\beta[S \restriction \alpha + 1, d \restriction \alpha + 1] \models ZFC^-$.
Define $S^* = \langle S^*_\alpha \mid \alpha < \omega_1 \rangle$ by: $S^*_\alpha = \mathbb{P}(\alpha) \cap M_\alpha$.
Set $M = \langle M_\alpha \mid \alpha < \omega_1 \rangle$.
We shall generically extend N to an $N[G]$ s.t. S^* is a $\diamondsuit^+$-sequence in $N[G]$.

Lemma 1 Let $\mathfrak{A} = \langle A, \in, \ldots \rangle \in H^N_{\omega_2}$ be transitive. Then in N there is $X \prec \mathfrak{A}$ s.t. $\overline{\overline{X}} \leq \omega$, $\alpha = \omega_1 \cap X$ is transitive; and $\overline{\mathfrak{A}} \in M_\alpha$, where $\overline{\mathfrak{A}}$ is the transitivization of X.

*) N plays the role of $N' = N[G]$ in §1

prf. of Lemma 1.

Assume w.l.o.g. that $\omega_1 \subset \Omega$ and let $f : \omega_1 \longleftrightarrow \Omega$. Let $T = T_{\Omega, f} =$ $=$ the complete theory of $\langle \Omega, f, \nu (\nu < \omega_1) \rangle$ (with a constant $\underline{\nu}$ for $\nu < \omega_1$). We suppose this theory to be coded in such a way that $T \subset \omega_1$. Let $C = \{ \alpha \mid \langle \Omega | f''\alpha, f \restriction \alpha \rangle \prec \langle \Omega, f \rangle \}$. Then C is cub in ω_1. Let $\alpha \in C$ s.t. $T \cap \alpha = S_\alpha$. Then, letting $\sigma : \langle \bar{\Omega}, \bar{f} \rangle \overset{\sim}{\longleftrightarrow} \langle \Omega | f''\alpha, f \restriction \alpha \rangle$, we have $\sigma \circ \bar{f} = f \restriction \alpha$, $\sigma \restriction \alpha = \mathrm{id}$, $\sigma(\alpha) = \omega_1$, and $T \cap \alpha =$ the complete theory of $\langle \bar{\Omega}, \bar{f}, \nu (\nu < \alpha) \rangle$. Since $T \cap \alpha \in M_\alpha$, we conclude: $\langle \bar{\Omega}, \bar{f} \rangle \in M_\alpha$, since $\langle \bar{\Omega}, \bar{f} \rangle$ is uniquely recoverable from $T \cap \alpha = \text{theory}(\langle \bar{\Omega}, \bar{f}, \nu (\nu < \alpha) \rangle)$ in any ZFC^- model containing $T \cap \alpha$. Hence the lemma holds with $X = f''\alpha$. QED (Lemma 1)

Def Let $A \subset \omega_2$ in N s.t. $L_{\omega_2}[A] = H_{\omega_2}$.

Define $\langle \rho_\nu \mid \nu < \omega_2 \rangle$ by:

ρ_ν = the least $\rho > \omega_1$ s.t. $\rho > \sup_{\xi < \nu} \rho_\xi$,

$M \in L_\rho$, $\mathrm{cf}(\rho) = \omega_1$ and

$L_\rho[A] \models (ZFC^- + \bigwedge x\ \overline{\overline{x}} \leq \omega_1)$

Set: $\tilde{\rho}_\nu = \omega_1 \cup \sup_{\xi < \nu} \rho_\xi$ (Hence $\tilde{\rho}_{\nu+1} = \rho_\nu$.)

$$\mathfrak{A}_\nu = \langle L_{\rho_\nu}[A], \in, A \cap \rho_\nu, M \rangle$$

For $\nu > 0$ set:

$$\tilde{\mathfrak{A}}_\nu = \bigcup_{\nu < \xi} \mathfrak{A}_\nu = \langle L_{\tilde{\rho}_\nu}[A], \in, A \cap \tilde{\rho}_\nu, M \rangle.$$

Then:

<u>Lemma 2</u>

(a) $\langle \rho_\xi \mid \xi < \nu \rangle$ is uniformly $\mathfrak{A}_\nu$-definable

(b) $[x]^\omega \in \mathfrak{A}_\nu$ for $x \in \mathfrak{A}_\nu$

(c) $[\mathfrak{A}_\nu]^\omega \subset \mathfrak{A}_\nu$

prf.

(a) is trivial. (b) follows by $\mathfrak{A}_\nu \models \bigwedge x\ \overline{\overline{x}} \leq \omega_1$ and $[\omega_1]^\omega \in \mathfrak{A}_\nu$ (since $[\omega_1]^\omega \subset \subset \bigcup_{\xi < \omega_1} M_\xi \in \mathfrak{A}_\nu$).

(c) follows by (b) and $\mathrm{cf}(\rho_\nu) = \omega_1$.

Set: f_ν = the $\mathfrak{M}_\nu$-least $f : \omega_1 \longleftrightarrow \tilde{\rho}_\nu$.

a_ξ = the ξ-th $a \subset \omega_1$ in $L_{\omega_2}[a]$.

$\tilde{a}_\nu = \{\langle \xi, \mu \rangle \mid \xi \in a_{f_\nu(\mu)}\}$.

Then:

<u>Lemma 2</u> (d) $\langle f_\xi \mid \xi < \nu \rangle$ is uniformly $\mathfrak{M}_\nu$-definable (and $\tilde{\mathfrak{M}}_\nu$-definable).

(e) $f_\nu \in \mathfrak{M}_\nu$ is uniformly $\mathfrak{M}_\nu$-definable

(f) $\langle a_\xi \mid \xi < \rho_\nu \rangle$ is uniformly $\mathfrak{M}_\nu$-definable

(g) $\tilde{a}_\nu \in \mathfrak{M}_\nu$ is uniformly $\mathfrak{M}_\nu$-definable.

We now define forcing conditions $\mathbb{P} = \mathbb{P}^A$.

<u>Def</u> $\mathbb{P}_\nu = \mathbb{P}_\nu^A =$ The set of countable p s.t. p is closed, bounded in ω_1 and $\alpha \in p \to \tilde{a}_\nu \cap \alpha \in M_\alpha$.

Set: $m_p = \max(p)$ for $p \in \mathbb{P}_\nu$.

$p \le q \leftrightarrow q = p \cap (m_q + 1)$ for $p, q \in \mathbb{P}_\nu$.

<u>Def</u> $\mathbb{P} = \mathbb{P}^A =_{Df}$ the set of maps p s.t. $\mathrm{dom}(p) \subset \omega_2$ is countable, $p(\nu) \in \mathbb{P}_\nu$ for $\nu \in \mathrm{dom}(p)$, and whenever $\nu \in \mathrm{dom}(p)$, then:

(a) $f_\nu{}'' m_{p(\nu)} \subset \mathrm{dom}(p)$

(b) $m_{p(\xi)} \geq m_{p(\nu)}$ for $\xi \in f_\nu{}'' m_{p(\nu)}$

(c) $\alpha \in p(\nu) \to \tilde{c}_{p,\nu} \cap \alpha \in M_\alpha$,

where $\tilde{c}_{p,\nu} = \{\langle \mu, \xi \rangle < m_{p(\nu)} \mid \mu \in p(f_\nu(\xi))\}$.

$p \le q \leftrightarrow_{Df} (\mathrm{dom}(q) \subset \mathrm{dom}(p) \wedge \bigwedge \nu \in \mathrm{dom}(q)\ p_\nu \le q_\nu$ in $\mathbb{P}_\nu)$

<u>Def</u> For $p \in \mathbb{P}$ set:

$m_p = \min\{m_{p(\nu)} \mid \nu \in \mathrm{dom}(p)\}$

$l_p = \mathrm{lub}(\mathrm{dom}(p))$

It is immediate from the definition that $\mathbb{P} \subset L_{\omega_1}[A]$ is $L_{\omega_1}[A]$ definable. It is also clear that, if W is an inner model of N with $[W]^{\omega} \subset W$ in N, then $\mathbb{P} \subset W$. Hence (A), (I) of §1 are proven. (B) is also trivial from the definition of $\mathbb{P}$. For $\nu < \omega_1$ set:

$\tilde{\mathbb{P}}_{\nu} =_{\mathrm{Df}} \{p \in \mathbb{P} \mid \mathrm{dom}(p) \subset \nu\}$. It is apparent that

$\tilde{\mathbb{P}}_{\nu} = \mathbb{P}^{\nu} =_{\mathrm{Df}} \{p \upharpoonright \nu \mid p \in \mathbb{P}\}$. (Hence $\mathbb{P}^{\nu} \subset \mathbb{P}$)

(In this section we generally write $\tilde{\mathbb{P}}_{\nu}$ instead of $\mathbb{P}^{\nu}$. The reason is that §2 was largely written before §1.) It is apparent that if $p \in \mathbb{P}$ and $q \leq p \upharpoonright \nu$ in $\tilde{\mathbb{P}}_{\nu}$, then $p \cup q \in \mathbb{P}$. Hence (D), (E) are proven. Before making the further verification we prove:

<u>Lemma 3</u> Let $p \in \mathbb{P}_A$, $\gamma < \omega_1$, $\delta < \omega_2$. There is $p' \le p$ s.t. $\gamma \le m_{p'}$, $\delta \le l(p')$.

proof.

Assume w.l.o.g. $l(p) < \delta$, $m_p < \gamma$.

Let $\mathfrak{A} = \langle L_\beta[A], A, M, p, \delta\rangle$, where β is least s.t.

$\langle L_\beta[A], A, M, p, \delta\rangle \prec \langle L_{\omega_2}[A], A, M, p, \delta\rangle$.

Then $\mathfrak{A} = \tilde{\mathfrak{A}}_\beta$, $\beta = \tilde{\beta}_\beta$.

Let $X \prec \mathfrak{A}$ be countable s.t. $\delta, \gamma \in X$ and $\alpha = X \cap \omega_1$ is transitive and $\bar{\mathfrak{A}} \in M_\alpha$, where $\bar{\mathfrak{A}}$ is the transitivization of $\mathfrak{A}|X$.

Let $\sigma : \bar{\mathfrak{A}} \stackrel{\sim}{\longleftrightarrow} \mathfrak{A}|X$. Let $\bar{\mathfrak{A}} = \langle L_{\bar\beta}[\bar A], \bar A, \bar M, \bar p\rangle$. Define p' by: $\mathrm{dom}(p') = \omega_2 \cap X$,

$$p'(\nu) = \begin{cases} p(\nu) \cup \{\alpha\} & \text{if } \nu \in \mathrm{dom}(p) \\ \{\alpha\} & \text{otherwise} \end{cases}.$$

<u>Claim</u> $p' \in \mathbb{P}$ (hence $p' \le p$).

(a) $p'(\nu) \in \mathbb{P}_\nu$ (trivial)

(b) Let $\nu \in \mathrm{dom}(p')$. Then $f_\nu \in X$. Let $\sigma(\bar f) = f_\nu$. Then $\bar f : \alpha \leftrightarrow \bar\rho$, where $\sigma(\bar\rho) = \rho_\nu$.

But $\tilde{a}_\nu \in X$. Let $\sigma(\bar{a}) = a_\nu$. Then

$\bar{a} = \{\langle\mu,\xi\rangle \in \alpha \mid \mu \in \bar{f}(\xi)\} =$

$= \{\langle\mu,\xi\rangle \in \alpha \mid \mu \in f_\nu(\xi)\} =$

$= \tilde{a}_\nu \cap \alpha$. Hence

(c) $\tilde{a}_\nu \cap \alpha \in M_\alpha$

(d) $\tilde{C}_{p',\nu} \in M_\alpha$, since

$\tilde{C}_{p',\nu} = \{\langle\mu,\xi\rangle \in \alpha \mid \nu \in p'(f_\nu(\xi))\} =$

$= \{\langle\mu,\xi\rangle \in \alpha \mid f_\nu(\xi) \in \mathrm{dom}(p) \wedge \nu \in p(f_\nu(\xi))\}$

$= \{\quad " \quad \mid \dot{f}(\xi) \in \mathrm{dom}(\bar{p}) \wedge \nu \in \bar{p}(\bar{f}(\xi))\}$

$\in \bar{\Omega} \subset M_\alpha$.

(e) At $\eta \in p'(\nu) \cap \alpha$, then:

$p'(\nu) \cap \alpha = p(\nu)$ + hence

$\tilde{C}_{p',\nu} \cap \eta = \tilde{C}_{p,\nu} \cap \eta \in M_\eta$.

QED (Lemma 3)

We now verify (c) of §1:

Lemma 4 $\mathbb{P}$ is ω_1 - distributive.

prf.
Let $p \in \mathbb{P}$. Let Δ_i be strongly dense in $\mathbb{P}$ for $i < \omega$. Claim $\bigvee p' \leq p$ $p' \in \bigcap_i \Delta_i$.

Fix $\langle\langle \gamma_i, m_i \rangle \mid i < \omega_1 \rangle$ s.t.
(a) $m_i < \omega$; (b) $\gamma_i < i$ for $0 < i < \omega_1$;
(c) $\forall \langle \gamma, n \rangle \in \omega_1 \times \omega$, then $\{ i \mid \langle \gamma_i, m_i \rangle = \langle \gamma, n \rangle \}$ is unbounded in ω_1.
Pick $\langle p_i \mid i < \omega_1 \rangle$ s.t. $p_0 = p$ and $p_i \leq p_{\gamma_i}$, $p_i \in \Delta_{n_i}$, $m_{p_i} \geq i$ for $0 < i < \omega_1$. Let
$\mathfrak{A} = \langle L_\beta [A], A, \langle p_i \mid i < \omega_1 \rangle, M \rangle \prec$
$\prec \langle L_{\omega_2} [A], A,$ " , " $\rangle$
s.t. $\beta < \omega_2$, $cf(\beta) = \omega_1$.
Let $X \prec \mathfrak{A}$ be countable s.t. $\alpha = \omega_1 \cap X$ is transitive and $\bar{\mathfrak{A}} \in M_\alpha$, where $\bar{\mathfrak{A}}$ is the transitivation of X. Then
$\bar{\mathfrak{A}} = \langle L_{\bar{\beta}} [\bar{A}], \bar{A}, \langle \bar{p}_i \mid i < \alpha \rangle, M \upharpoonright \alpha \rangle$.
Let $\alpha = \sup_{i<\omega} \alpha_i$, where $\langle \alpha_i \mid i < \omega \rangle \in$ $\in M_\alpha$ is monotone. In M_α select $\langle \eta_i \mid i < \omega \rangle$

s.t. $\alpha_i < \eta_i$, $\gamma_{\eta_{i+1}} = \eta_i$, $n_{\eta_{i+1}} = i$.

Define p' by: $\mathrm{dom}(p') = \bigcup_{i<\omega} \mathrm{dom}(p_{\eta_i})$

$$p'(\nu) = \{\alpha\} \cup \bigcup_{\substack{i<\omega \\ \nu \in \mathrm{dom}(p_{\eta_i})}} p_{\eta_i}(\nu) \quad \text{for } \nu \in \mathrm{dom}(p').$$

It suffices to show:

<u>Claim</u> $p' \in \mathbb{P}$ (Hence $p' \leq p_{\eta_{i+1}} \in \Delta_i$ for $i<\omega$.)

prf.
$\mathrm{dom}(p') \subset \omega_1$ is trivially countable.
Moreover:
(a) $p'(\nu)$ is closed, bounded in ω_1 for $\nu \in \mathrm{dom}(p')$,
since $p'(\nu) \cap \alpha$ is cub in $\alpha = \max(p'(\nu))$
(b) $\tilde{a}_\nu \cap \gamma \in M_\gamma$ for $\gamma \in p'(\nu)$

prf.
Case 1 $\gamma < \alpha$. Then $\gamma \in \mathrm{dom}(p_{\eta_i}(\nu))$ and hence $\tilde{a}_\nu \cap \gamma \in M_\gamma$.
<u>Case 2</u> $\gamma = \alpha$.
Let $\nu \in \mathrm{dom}(p_{\eta_i})$. Then $\nu \in X$ since $p_{\eta_i} \in X$. Hence $\tilde{a}_\nu \in X$. Let $\sigma(\bar{a}) = \tilde{a}_\nu$. Then $\bar{a} = \tilde{a}_\nu \cap \alpha \in M_\alpha$.
QED(b)

(c) $f_\nu{}''\, m_{p'(\nu)} = f''\alpha \subset \mathrm{dom}(p')$ for $\nu \in \mathrm{dom}(p)$.

Let $\xi = f_\nu(\mu)$, $\mu < \alpha$. Let $\nu \in \mathrm{dom}(p_{\eta_i})$, where $\eta_i > \mu$ (hence $m_{p_{\eta_i}} \geq \eta_i > \mu$).

Then $m_{p_{\eta_i}(\nu)} > \mu$ and $f_\nu(\mu) \in$

$\in \mathrm{dom}(p_{\eta_i}) \subset \mathrm{dom}(p')$

(d) $m_{p'(\xi)} = m_{p'(\nu)} = \alpha$ for $\xi \in f_\nu{}''\alpha$

(e) $\gamma \in p'(\nu) \to \tilde{c}_{p',\nu} \cap \gamma \in M_\gamma$

<u>Case 1</u> $\gamma < \alpha$. Trivial since $\gamma \in p_{\eta_i}(\nu)$ for an $i < \omega$ and

$\tilde{c}_{p',\nu} \cap \gamma = \tilde{c}_{p_{\eta_i},\nu} \cap \gamma$.

<u>Case 2</u> $\gamma = \alpha$.

$$\tilde{c}_{p',\nu} = \bigcup_{\substack{i<\omega \\ \nu \in \mathrm{dom}(p_{\eta_i})}} \tilde{c}_{p_i,\nu}\,.$$ But

The function $\langle \tilde{c}_{p_\xi,\nu} \mid \xi < \omega_1 \wedge \nu \in \mathrm{dom}(p_\xi)\rangle$ is $\mathfrak{A}$-definable. Let $\langle \bar{c}_{\bar{p}_\xi,\nu} \mid \xi < \alpha \wedge \nu \in \mathrm{dom}(p_\nu)\rangle$ have the same def. in $\bar{\mathfrak{A}}$.

Then $\sigma(\bar{c}_{\bar{p}_\xi, \nu}) = \tilde{c}_{p_\xi, \sigma(\nu)}$; hence

$\bar{c}_{\bar{p}_\xi, \nu} = \tilde{c}_{p_\xi, \sigma(\nu)} \cap \alpha$. Since $\langle \bar{p}_{\eta_i} \mid i < \omega \rangle \in M_\alpha$,

we have: $\tilde{c}_{p', \nu} = \bigcup_{\substack{i<\omega \\ \nu \in \mathrm{dom}(\bar{p}_{\eta_i})}} \bar{c}_{\bar{p}_{\eta_i}}$, where $\sigma(\bar{\nu}) = \nu$.

QED (Lemma 4)

<u>Note</u> a modification of this proof shows:

<u>Cor. 4.1</u> Let $\nu < \omega_1$. Set $\mathbb{Q}$ = the set of $\langle p, q \rangle \in \mathbb{P} \times \mathbb{P}$ s.t. $p \restriction \nu = q \restriction \nu$, with the ordering: $\langle p, q \rangle \leq \langle p', q' \rangle \leftrightarrow (p \leq p' \wedge q \leq q')$.

Then $\mathbb{Q}$ is ω_1-distributive.

(<u>Note</u> $\mathbb{Q} = \mathbb{P} \times \mathbb{P}$ for $\nu = 0$)

As in §1, a standard proof using (D), (E) gives:

<u>Lemma 5</u> $\mathbb{P}$ satisfies the ω_2-cc (i.e. every antichain has cardinality $\leq \omega_1$).

Hence $\mathbb{P}$ preserves cardinals.

Lemma 6 Let G be $\mathbb{P}_\nu$ - generic over N. Set $C = \bigcup G$. Then

(a) C is cub in ω_1 and $C \cap \alpha \in M_\alpha$ for $\alpha \in C$.

(b) Let $a \subset \omega_1$, $a \in L_{\rho_\nu}[A]$. Then there is $\alpha_0 \in C$ s.t. $a \cap \alpha \in M_\alpha$ for all $\alpha \in C \setminus \alpha_0$.

prf. Trivial.

Def Let G be $\mathbb{P}$ - generic over N.

$G_\nu =_{Df} \{p(\nu) \mid p \in G \wedge \nu \in dom(p)\}$

$C_\nu = C_\nu^G = \bigcup G_\nu$

$\tilde{C}_\nu = \tilde{C}_\nu^G = \{\langle \mu, \tau \rangle < \omega_1 \mid \mu \in C_{f_\nu(\tau)}\}$.

Def $\tilde{\mathbb{P}}_\nu = \tilde{\mathbb{P}}_\nu^A = \{p \in \mathbb{P} \mid dom(p) \subset \nu\}$.

(Hence $\tilde{\mathbb{P}}_\nu \in M_\nu$ by Lemma 2)

Lemma 7 Let G be $\mathbb{P}$ - generic over N.

(a) G_ν is $\mathbb{P}_\nu$ - generic over N

(b) $\alpha \in C_\nu \to \alpha \cap \tilde{C}_\nu,\ \alpha \cap C_\nu \in M_\alpha$

(c) $N[G] = N[\langle C_\nu \mid \nu < \omega_1 \rangle]$

(d) $N[G \cap \tilde{\mathbb{P}}_\nu] = N[\tilde{C}_\nu]$

proof

(a) Let $p \in \mathbb{P}$, $q \leq p(\nu)$ in $\mathbb{P}_\nu$.

Claim There is $p' \leq p$ in $\mathbb{P}$ s.t. $p'(\nu) \leq q$ in $\mathbb{P}_\nu$.

Imitating the proof of Lemma 3 (applied to $p \upharpoonright \nu$, we find an $\alpha > m_q$ and a $p'' \in \tilde{\mathbb{P}}_\nu$ s.t. $p'' \le p \upharpoonright \nu$ in $\mathbb{P}$ and $m_{p''(\xi)} = \alpha$ for all $\xi \in \mathrm{dom}(p'')$ and $p''(\xi) \cap \alpha = \begin{cases} p(\xi) & \text{if } \xi \in \mathrm{dom}(p) \\ \emptyset & \text{if not} \end{cases}$ for $\xi \in \mathrm{dom}(p'')$. Set $p' = p'' \cup \{\langle q, \nu\rangle\}$. It is easily verified that $p' \in \mathbb{P}$. Hence $p' \le p$ and $p'_\nu = q$. QED (a)

(b) trivial

(c) ($\supset$) trivial. ($\subset$) follows by:

For $p \in \mathbb{P}$ we have:

$$p \in G \leftrightarrow \bigwedge \nu \in \mathrm{dom}(p)\, (p(\nu) = C_\nu \cap (m_{p(\nu)}+1))$$

(d) $N[G \cap \tilde{\mathbb{P}}_\nu] = N[\langle C_\xi \mid \xi < \nu\rangle]$ follows as in (c). But $N[\langle C_\xi \mid \xi < \nu\rangle] = N[\tilde{C}_\nu]$. QED (Lemma 7)

Using this, we verify (G) of §1:

<u>Lemma 8</u> Assume $L_{\omega_2}[A] = H_{\omega_2}$ in N. Then $S^* = \langle S^*_\xi \mid \xi < \omega_1 \rangle$ is a $\diamondsuit^+$-sequence in $N[G]$.

prf.

Let $B \subset \omega_1$ in $N[G]$. <u>Claim</u> There is a cub $C \subset \omega_1$ in $N[G]$ s.t.

$$\bigwedge \alpha \in C \quad B \cap \alpha, C \cap \alpha \in M_\alpha.$$

Let $B = \dot{B}^G$. For $\nu < \omega_1$ choose in N a maximal antichain X_ν in $\{p \mid p \Vdash \check{\nu} \in \dot{B}\}$. Then $\overline{\overline{X}}_\nu \leq \omega_1$, where $X_\nu \subset H_{\omega_2}$. Hence $\langle X_\nu \mid \nu < \omega_1 \rangle \in L_{\omega_2}[A]$. We know:

(1) $\nu \in B \longleftrightarrow G \cap X_\nu \neq \emptyset$

Pick $\beta < \omega_2$ s.t.

$\mathfrak{M} = \langle L_\beta[A], A \cap \beta, M, X \rangle \prec \langle L_{\omega_1}[A], A, M, X \rangle$

where $X = \langle X_\nu \mid \nu < \omega_1 \rangle$, and $cf(\beta) = \omega_1$.

Clearly $\beta = \sup_{\nu < \beta} \rho_\nu$, $\mathfrak{M} = \langle \tilde{\mathfrak{M}}_\beta, X \rangle$.

Since $cf(\beta) = \omega_1$, we have

(2) $[\mathfrak{M}]^\omega \subset \mathfrak{M}$; $x \in \mathfrak{M} \to [x]^\omega \in \mathfrak{M}$

Hence:

(3) $\tilde{\mathbb{P}}_\beta \subset \mathfrak{M}$ is $\mathfrak{M}$-definable.

Clearly (1) can be improved to

(4) $\nu \in B \leftrightarrow (G \cap \tilde{\mathbb{P}}_\beta) \cap X_\nu \neq \emptyset$.

Note that $\rho_\beta > \beta$ and $f_\beta \in L_{\rho_\beta}[A]$, where $f_\beta : \omega_1 \leftrightarrow \beta$.

Let $D \subset \omega_1$ code the complete theory of $\langle \mathfrak{A}, f_\beta, \nu \mid \nu < \omega_1 \rangle$. Then $\mathfrak{A}$ is uniquely recoverable from D in any transitive ZFC$^-$ model containing D as a set.

Then $D \in L_{\rho_\beta}[A]$. Hence there is $\alpha_0 \in C_\beta$ s.t. $D \cap \alpha \in M_\alpha$ for all $\alpha \in C_\beta \setminus \alpha_0$.

Set: $\mathfrak{A}^* = \langle \mathfrak{A}, D, G \cap \tilde{\mathbb{P}}_\beta \rangle$. Set:

<u>Def</u> Y_α = the smallest $Y \prec \mathfrak{A}^*$ s.t. $\alpha \subset Y$ $(\alpha \leq \omega_1)$.

Then $\mathfrak{A}^* = Y_{\omega_1}$ and Y_α is countable for $\alpha < \omega_1$. Set:

$C = \{\alpha \in C_\beta \mid \alpha_0 \leq \alpha = \omega_1 \cap Y_\alpha\}$.

Then C is cub in ω_1

<u>Claim</u> Let $\alpha \in C$. Then $B \cap \alpha, C \cap \alpha \in M_\alpha$.

prf.
Let $\sigma : \bar{\mathfrak{A}}^* \overset{\sim}{\longleftrightarrow} Y_\alpha$, where $\bar{\mathfrak{A}}^* = \langle \bar{\mathfrak{A}}, \bar{G}, \bar{D} \rangle$ and $\bar{\mathfrak{A}} = \langle L_{\bar{\beta}}[\bar{A}], \bar{A}, \bar{M}, \bar{X} \rangle$. Then

$\alpha = \mathrm{crit}(\sigma)$, $\sigma(\alpha) = \omega_1$, $\sigma(\bar{A}) = A \cap \beta$,

$\sigma(\bar{M}) = M$ (hence $\bar{M} = M \upharpoonright \alpha$),

$\sigma(\bar{X}) = X$ (hence $\bar{X} = \langle \bar{X}_\nu \mid \nu < \alpha \rangle$, where $\bar{X}_\nu$ is a maximal antichain in $\bar{\mathbb{P}}$, and $\bar{\mathbb{P}}$ is defined in $\bar{\mathfrak{A}}$ as $\tilde{\mathbb{P}}_\beta$ in $\mathfrak{A}$.

Finally, we note that $\bar{D} = D \cap \alpha \in M_\alpha$, since $\sigma(\bar{D}) = D$ and $\alpha \in C_\beta \setminus \alpha_0$. Hence

(5) $\bar{\sigma} \in M_\alpha$,

since $\bar{\sigma}$ is recoverable from $\bar{D}$ as σ was recoverable from D.

(6) $\bar{G} \in M_\alpha$

prf.

By the proof of Lemma 7 (d), $G \cap \tilde{\mathbb{P}}_\beta$ is $\langle \sigma, \tilde{C}_\beta \rangle$-definable, where $\tilde{C}$ is, in turn, σ^*-definable. Let $\bar{C}$ have the same definition in $\bar{\sigma}^*$. Then $\bar{C} = \sigma^{-1}{}''\tilde{C}_\beta = \tilde{C}_\beta \cap \alpha \in M_\alpha$, since $\alpha \in C_\beta$. But then $\bar{G}$ is $\langle \bar{\sigma}, \bar{C} \rangle$-definable as $G \cap \tilde{\mathbb{P}}_\beta$ was defined in $\langle \sigma, \tilde{C}_\beta \rangle$, since $\sigma(\bar{G}) = G \cap \tilde{\mathbb{P}}_\beta$ and, $\sigma : \langle \bar{\sigma}, \bar{C}, \bar{G} \rangle \prec \langle \sigma, \tilde{C}_\beta, G \cap \tilde{\mathbb{P}}_\beta \rangle$.

Hence $G \in M_\alpha$, since $\langle \bar{\sigma}, \bar{C} \rangle \in M_\alpha$

QED (6)

By the proof of Lemma 7 (c):

$$\nu \in B \longleftrightarrow (G \cap \tilde{\mathbb{P}}_\nu) \cap X_\nu \neq \emptyset$$
$$\longleftrightarrow \bar{G} \cap \bar{X}_\nu \neq \emptyset \quad \text{for } \nu < \alpha.$$

Hence:

(7) $B \cap \alpha \in M_\alpha$, since $\bar{G}, \langle \bar{X}_\nu \mid \nu < \alpha \rangle \in M_\alpha$.

Finally we note that:

(8) $C \cap \alpha \in M_\alpha$,

since $\bar{\mathfrak{A}}^* \in M_\alpha$ by (6), $C_\beta \cap \alpha, \alpha_0 \in M_\alpha$, and $C \cap \alpha$ is definable from $\bar{\mathfrak{A}}^*$, $C_\beta \cap \alpha, \alpha_0$ as C was defined from $\mathfrak{A}^*, C_\beta, \alpha_0$. QED (Lemma 8)

It remains only to verify (H), which will follow by Corollary 4.1:

<u>Lemma 9</u> Let $\bar{G}$ be $\bar{\mathbb{P}}_\nu$ - generic over N. Set: $\mathbb{P}_{\bar{G}} = \{p \in \mathbb{P} \mid p \restriction \nu \in \bar{G}\}$. Then $\mathbb{P}_{\bar{G}} \times \mathbb{P}_{\bar{G}}$ is ω_1 - distributive.

proof. Let $G_0 \times G_1$ be $\mathbb{P}_{\bar{G}} \times \mathbb{P}_{\bar{G}}$ - generic over $N[\bar{G}]$. We must show that $N[\bar{G}][G_0 \times G_1] = N[G_0 \times G_1]$ contains no new countable subsets of $N[\bar{G}]$. Let $\mathbb{Q} = \{\langle p, q\rangle \in \mathbb{P} \times \mathbb{P} \mid p \restriction \nu = q \restriction \nu\}$ be as in Lemma 4.1. Set $\mathbb{Q}_{\bar{G}} = \mathbb{Q} \cap (\mathbb{P}_{\bar{G}} \times \mathbb{P}_{\bar{G}})$. It is easily seen that $\mathbb{Q}_{\bar{G}}$ is dense in $\mathbb{P}_{\bar{G}} \times \mathbb{P}_{\bar{G}}$. Hence $N[G_0 \times G_1] = N[\tilde{G}]$, where $\tilde{G} = (G_0 \times G_1) \cap \mathbb{Q}$, and $\tilde{G}$ is $\mathbb{Q}_{\bar{G}}$ - generic over $N[\bar{G}]$.

It is apparent from the def. of $\mathbb{Q}$ that:

(E') Let $\langle p, p' \rangle \in \mathbb{Q}$. Let $q \leq p \restriction \gamma = p' \restriction \gamma$ in $\tilde{\mathbb{P}}_\gamma$. Then $\langle p \cup q, p' \cup q \rangle \in \mathbb{Q}$.

Using this we can repeat the proof of (2) (following (G1)) in §1 to get:

$\mathbb{Q} \cap (G_0 \times G_1)$ is $\mathbb{Q}$-generic over N iff $\bar{G} = G_i \cap \tilde{\mathbb{P}}_\gamma$ is $\tilde{\mathbb{P}}_\gamma$-generic over N and $\mathbb{Q} \cap (G_0 \times G_1)$ is $\mathbb{Q}_{\bar{G}}$-generic over $N[\bar{G}]$. Hence $\tilde{G} = \mathbb{Q} \cap (G_0 \times G_1)$ is $\mathbb{Q}$-generic over N, where $\mathbb{Q}$ is ω_1-distributive in N. Hence $N[G_0 \times G_1] = N[\tilde{G}]$ contains no new countable sets of ordinals.

QED