§7 The final conclusion

Def By a smooth iteration of length μ we mean $\tilde{I} = \langle I_i \mid i < \mu \rangle$ s.t.

(a) $I_i = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{i_1} \rangle, T_i \rangle$ is a normal iteration.

(b) If $i+1 < \mu$, then I_i has length $\eta_i + 1$ and $I_i^{\eta_i} = I_{i+1}^0$.

(c) If $\gamma < \mu$ is a limit ordinal, then there are at most finitely many $i < \gamma$ s.t. I_i has a truncation on the main branch.

(Def We call $i+1$ a truncation point if I_i has a truncation on the main branch.)

(d) There are partial maps $\tilde{\pi}_{i,j}$ $(i \leq j < \mu)$ s.t.

(i) $\tilde{\pi}_{i,j}$ is a partial map of M_i^0 to M_j^0

(ii) If $h = i+1$, then $\tilde{\pi}_{h,i} = \pi_{0,\eta_h}^{I_h}$

(iii) $\tilde{\pi}_{ij} \circ \tilde{\pi}_{hi} = \tilde{\pi}_{hj}$

(iv) Let $\gamma < \mu$ be a limit ordinal. Let $i_0 < \gamma$ s.t. there is no $h \in (i_0, \gamma]$ which is a truncation point. Then

$$\tilde{\pi}_{ij} : M_i^0 \xrightarrow[\Sigma^*]{} M_j^0 .$$

Moreover, $M_\gamma^0, \langle \tilde{\pi}_{j,\gamma} \mid i_0 \leq j < \gamma \rangle$ is the direct limit of $\langle M_j^0 \mid i_0 \leq j < \gamma \rangle, \langle \tilde{\pi}_{hj} \mid i_0 \leq h \leq j < \gamma \rangle$.

It follows easily that:

<u>Lemma 1</u> Let $\tilde{I}$ be a smooth iteration of limit length μ. If $\xi_0 < \mu$ s.t. (ξ_0, μ) has no truncation point, then $\pi_{ij} : M_i \to_{\Sigma^*} M_j$ for $\xi_0 \le i \le j < \mu$.

<u>Def</u> Σ is a <u>smooth iteration strategy</u> iff Σ is a partial function whose domain consists of smooth iterations $\tilde{I} = \langle I_i \mid i \le \gamma \rangle$ of successor length γ, where I_γ is of limit length. We say that a smooth iteration $\langle I_i \mid i < \mu \rangle$ is Σ-<u>conforming</u> if whenever $i < \mu$ and $\gamma < lh(I_i)$ is a limit, then $T_i{}''\{\gamma\} = \Sigma((\tilde{I} \restriction i)^\frown \langle I_i \mid \gamma \rangle)$.

Σ is <u>successful</u> <u>for</u> M iff every Σ-conforming smooth iteration strategy for M can be extended to a longer Σ-conforming smooth iteration — in other words:

(1) If $\tilde{I}$ is of length $\mu+1$ and I_μ has length $\eta_\mu+1$ and $E_\nu^{M'} \neq \emptyset$, where M' is the final model of M_μ, and $\nu > \nu_i^{I_\mu}$ for $i < \eta_\mu$; then I_μ extends to an iteration of length $\eta_\mu+2$ with $\nu = \nu_{\eta_\mu}$.

(2) If $\tilde{I}$ is of length $\mu+1$ and I_μ is of limit length λ, then $\Sigma(\tilde{I})$ exists and I_μ extends to I' of length $\lambda+1$ with $T'^{u}\{\lambda\} = \Sigma(\tilde{I})$

(3) If $\tilde{I}$ is of limit length μ, then:

(a) There at most finitely many truncation points below μ

(b) If (ξ, μ) is truncation free, then $\langle M_i^{\circ} \mid \xi \le i < \mu \rangle, \langle \pi_{ij} \mid \xi \le i \le j < \mu \rangle$ has a transitivized direct limit:
$M_\mu^{\circ}, \langle \pi_{i,\mu} \mid \xi \le i < \mu \rangle$.

Def We say that M is <u>smoothly iterable</u> iff it has a successful smooth iteration strategy.

We prove:

<u>Theorem 1</u> Let M be uniquely normally iterable. Let $I^* = \langle\langle M_i^*\rangle, \langle\nu_i^*\rangle, \langle\pi_{ij}^*\rangle, T^*\rangle$ be a normal iteration of M of length η^*+1. Let $\sigma^*: N \xrightarrow[\Sigma^*]{} M_{\eta^*}^* \; \mathrm{min}(\rho^*)$. Then N is smoothly iterable.

proof.

We build upon §4. We there defined what it means for a pair $\langle S, I'\rangle$ to be a <u>justification</u> of a normal iteration I of N wrt. I^*, σ^*, ρ^*. We note that $\langle S, I'\rangle$, if it exists, is uniquely determined by $\langle I^*, \sigma^*, \rho^*\rangle$ and I.

We call I <u>justifiable</u> wrt. I^*, σ^*, ρ^* iff it has a justification.

We then noted:

(A) If I is justifiable and of length $\mu+1$ and $E_\nu^{N_\mu} \neq \emptyset$ where $\nu > \nu_i$ for $i < \mu$, then I extends ~~extends~~ uniquely to an iteration I'' of length $\mu+2$ with $\nu = \nu_\mu''$. Moreover, I'' is justifiable. (Hence the justification of I' extends the justification of I in the obvious way.)

(B) Let I be justifiable and of limit length μ. Let $\langle \mathcal{S}, I' \rangle$ be the justification of I. Let b be the unique well founded branch in $\mathcal{S}$. Then b is a cofinal well founded branch in I and I can be extended to a justifiable iteration I'' of length $\mu+1$ by setting: T'' "$\{\mu\} = b$.

This gave us an obvious successful normal iteration strategy for N. We then considered smooth iterations of N of finite length. Let $\tilde{I} = \langle I_0, \ldots, I_m \rangle$ be such a smooth iteration. By a <u>justification</u> of $\tilde{I}$ wrt. I, I^*, σ^*, ρ^* we mean

$\langle \langle \mathcal{S}_0, I_0' \rangle, \ldots, \langle \mathcal{S}_m, I_m' \rangle \rangle$ s.t.

(a) $\langle \mathcal{S}_0, I_0' \rangle$ is a justification of I_0 wrt. $\langle I^*, \sigma^*, \rho^* \rangle$

(b) $\langle \mathcal{S}_{i+1}, I' \rangle$ is a justification of I_{i+1} wrt $\langle I^*_{i+1}, \sigma^*_{i+1}, \rho^*_{i+1} \rangle$ where:

- I^*_{i+1} = the final iteration in $\mathcal{S}_i$
- $\sigma^*_{i+1} = \sigma^i_{\eta_i}$ where $I_i' = \langle \langle N_h'^i \rangle, \langle \pi_h'^i \rangle, \langle \sigma_h^i \rangle, \rho^{i,h} \rangle$ and $\eta_i + 1 = \mathrm{lh}\, I_i'$
- $\rho^*_{i+1} = \rho^{i,\eta_i}$.

This, again, gives us an obvious strategy for finite length smooth iteration of N. (We note that $\langle\langle \mathcal{S}_0, I_0' \rangle, \ldots, \langle \mathcal{S}_n, I_n' \rangle\rangle$, if it exists, is uniquely determined by $\langle I_0, \ldots, I_n \rangle$ and $\langle I^*, \sigma^*, \rho^* \rangle$.)

In §6 we developed the general notion of smooth iteration. $\langle \mathcal{S}_0, \ldots, \mathcal{S}_n \rangle$ is, then a smooth iteration, and we can. the insertions e_{ij} $(0 \leq i \leq n)$ of I_i^0 into I_j^0 as we did there. We can also define a partial map $\hat{\pi}_{ij}$ of M_i^0 into M_j^0, (M_i^0 being the final model of I_i^0), by:

$$\hat{\pi}_{i,i+1} = \pi_{0,\eta_i}^{I_i'}, \quad \hat{\pi}_{h,i+1} = \hat{\pi}_{i,i+1} \cdot \hat{\pi}_{h,i}.$$

It is then easily seen that if $i+1$ is not a truncation point in $\langle \mathcal{S}_0, \ldots, \mathcal{S}_n \rangle$ (i.e. there is no truncation on the main branch of $\mathcal{S}_i$), then

$$\hat{\pi}_{i,i+1} : M_i^0 \xrightarrow[\Sigma^*]{} M_{i+1}^0 .$$

Hence, if there is no truncation in $(h, j]$, then $\tilde{\pi}_{ij} : M_i^0 \longrightarrow_{\Sigma^*} M_j^0$. If we set:
$\hat{\rho}^i = (\rho^0)^{S_i} = \langle \rho_n^i \mid n < \omega \rangle$, $\hat{\sigma}_i = \sigma_0^{S_i}$,
then $\hat{\sigma}_i : N_i^0 \longrightarrow_{\Sigma^*} M_i^0$ min $\hat{\rho}^i$. It is easily seen that, if $(i, j]$ has no truncation, then: $\hat{\pi}_{ij}{}''\hat{\rho}_n^i \leq \rho_n^j \leq \hat{\pi}_{ij}(\rho_n^i)$ for $n < \omega$.

With this clue, we can define the general notion of <u>justification</u>:

<u>Def</u> Let $\tilde{I}$ be a smooth iteration of N, where I^* is a normal iteration of M, (M being uniquely normally iterable), and $\sigma^* : N \longrightarrow_{\Sigma^*} M_{\gamma^*}^*$ min (ρ^*). ($M_{\gamma^*}^*$ being the final model of I^*). Let $\tilde{I} = \langle I_i \mid i < \mu \rangle$. By a <u>justification</u> of $\tilde{I}$ wrt I^*, σ^*, ρ^* we mean a sequence $\langle\langle S_i, I_i' \rangle \mid i < \mu \rangle$ s.t. (a), (b) hold and:

Let e_{ij} be the insertion of I_i^0 into I_j^0 defined in §6, where $S_i = \langle I_i^h \mid h < \mu_i \rangle$
Set: $\hat{\rho}^i = (\rho^0)^{S_i}$, $\hat{\sigma}_i = \sigma_0^{S_i}$,

We define a strategy Σ for smooth iterations $\hat{I}$ of N by:

Let $\tilde{I}$ be of length $\mu+1$ where I_μ is of limit length. If $\hat{I}$ has a justification $\langle\langle s_i, I_i'\rangle \mid i \le \mu\rangle$ w.r.t. $\langle I^*; \rho^*; \sigma^*\rangle$, set:

$\Sigma(\tilde{I}) =: b$, where b is the unique cofinal well founded branch in I_μ. If no such justification exists, then $\Sigma(\tilde{I})$ is undefined.

We leave it to the reader to show that, if $\hat{I}$ is Σ-conforming (i.e. all infinite branches are chosen by Σ), then $\tilde{I}$ has a justification wrt $\langle I^*, \rho^*, \sigma^*\rangle$. Hence Σ is a successful strategy for N. Thus we have shown:

<u>Lemma 2</u> If I^*, ρ^*, σ^*, N are as above, then N is smoothly iterable.

Taking $N = M$, $I^* = \langle\langle M\rangle, \emptyset, \langle id\rangle, \emptyset\rangle$ as the 1-step iteration of M, $\rho^* = \langle \rho_M^n \mid n < \omega\rangle$ and $\sigma^* = id \restriction M$, we get:

<u>Corollary</u> 3 If M is uniquely normally iterable, then it is smoothly iterable.

(<u>Note</u> It would have been notationally a bit easier to prove Corollary 3 directly, but we chose to stick with the notation of §4.)

We leave it to the reader to conclude:

<u>Lemma 4</u> If M has a successful insertion invariant normal iteration strategy, then it is smoothly iterable.