§2 Iterating Subproper and Subcomplete Forcing

We essentially repeat our earlier definition.

<u>Def</u> A complete BA $\mathbb{B}$ is <u>subproper</u> iff for sufficiently large cardinals θ:
Let $\mathbb{B} \in H_\theta$ and $N = \langle L_\tau[A], A, \in \rangle$ where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ be countable and full. Let $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(1 \le i \le m < \omega)$ where $\lambda_i \in (\omega_1, \theta)$ is regular and $\overline{\overline{\mathbb{B}}} < \lambda_i$ for $i = 1, \dots, m$.
Set: $\bar{\lambda}_0 = \mathrm{On} \cap \bar{N}$. For any $a \in \bar{\mathbb{B}} \setminus \{0\}$ there is $b \subset \sigma(a)$ which forces that if $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \dots, m)$

(c) $\sup \sigma_0{}''\bar{\lambda}_i = \sup \sigma''\bar{\lambda}_i$ $(i = 0, \dots, m)$

(d) $\bar{G} = \sigma_0^{-1}{}''G$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$.

It is trivial to see that a subproper forcing cannot collapse ω_1. Clearly every proper forcing is subproper, since properness is just the case $\sigma_0 = \sigma$.

<u>Note</u> σ_0 extends uniquely in $V[G]$ to a map $\sigma_0^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma_0^*(\bar{G}) = G$. But $\bar{N}[\bar{G}]$ is then full in $V[G]$. Thus the stage is set to handle a subproper $\tilde{\mathbb{B}} \in V[G]$.

Def We say that μ verifies the subproperness of $\mathbb{B}$ if the above holds for all cardinals $\theta \geq \mu$.

Def Let $N = L_\tau[A]$, $\sigma : \bar{N} \prec N$ be as above. We then say that σ witnesses the subproperness of $\mathbb{B}$ w.r.t. N. We also say that $X = \operatorname{rng}(\sigma)$ witnesses the subproperness of $\mathbb{B}$.

An apparent weakening of the notion of subproperness is:

<u>Def</u> $\mathbb{B}$ is <u>weakly subproper</u> iff there is z s.t. for sufficiently large cardinals θ, if $N = L_\tau[A]$ is as above and $X \prec N$ is countable and full s.t. $z, \theta, \mathbb{B} \in N$, then X witnesses the subproperness of $\mathbb{B}$.

However:

<u>Lemma 1</u> If $\mathbb{B}$ is weakly subproper, then it is subproper.

prf.

Let θ_0 be least s.t. for some $z \in H_{\theta_0}$ if $\theta \geq \theta_0$ is a cardinal and N is as above and $X \prec N$ is countable and full s.t $\theta, \mathbb{B}, z \in X$, then X witnesses subproperness.

If $\mu > \theta_0$ s.t. $\mu = \overline{\overline{V_\mu}}$, let θ_0^μ be defined as above in V_μ rather than V. Clearly $\theta_0^\mu \leq \theta_0^{\mu'} \leq \theta_0$ for $\mu \leq \mu'$. But then there is μ_0 s.t $\theta_0^{\mu_0} = \theta_0$. If $\mu \geq \mu_0$, let A_μ = the set of $z \in H_{\theta_0}$ s.t. if $\theta_0 \leq \theta < \mu$, θ is a cardinal, N is as above, $X \prec N$ is countable and full s.t. $\theta, \mathbb{B}, z \in X$, then X witnesses subproperness. Then $\mu_0 \leq \mu \leq \mu' \rightarrow$ $\rightarrow A_\mu \supset A_{\mu'} \subset H_{\theta_0}$. Hence there is μ_1 s.t. $A_\mu = A_{\mu_1}$ for $\mu_1 \leq \mu$. But then $A_{\mu_1} = A_\infty$, where A_∞ is defined as above with ∞ in place of μ.

Now let $\theta > \mu_1$ be a cardinal + let $N = L_\tau[A]$ be as above. Then $\langle \theta_0^\mu \mid \mu = \bar{\bar{V}}_\mu < \theta \rangle$ is N-definable in $\theta, \mathbb{B}$. Hence μ_0 is N-definable in $\theta, \mathbb{B}$. Hence $\langle A_\mu \mid \mu_0 \leq \mu \wedge V_\mu \in H_\theta \wedge \bar{\bar{V}}_\mu = \mu \rangle$ is N-definable in $\theta, \mathbb{B}$. Hence μ_1, $A_{\mu_1} = A_\infty$ are N-definable in $\theta, \mathbb{B}$. Now let $X \prec N$ be full and countable s.t. $\theta, \mathbb{B} \in X$. Then $A_\infty \cap X \neq \emptyset$. Hence X witnesses subproperness. QED

<u>Lemma 2</u> Let $\mathbb{B}_0 \subseteq \mathbb{B}_1$ where $\mathbb{B}_0$ is subproper and $\Vdash_0 \check{\mathbb{B}}_1 / \dot{G}_0$ is subproper. Then $\mathbb{B}_1$ is subproper.

pf.

Choose θ large enough that $\mathbb{B}_1 \in H_\theta$, θ verifies the subproperness of $\mathbb{B}_0$ and the subproperness of $\check{\mathbb{B}}_1 / \dot{G}_0$.

$\Vdash_0 \check{\theta}$ verifies the subproperness of

Let $N = L_\tau[A]$ where $\tau > \theta$ is regular and $H_\theta \subset N$. Let $X \prec N$ be full and countable s.t. $\theta, \mathbb{B}_0, \mathbb{B}_1 \in X$. By Lemma 1 it suffices to show that X witnesses the subproperness of $\mathbb{B}_1$. Let $a \in X \cap (\mathbb{B} \setminus \{0\})$. Let $s, \lambda_1, \ldots, \lambda_m \in X$ s.t. $\lambda_i \in (\omega_1, \theta)$ is regular and $< \lambda_2$.

Let $\sigma : \bar{N} \xrightarrow{\sim} X$, $\sigma(\bar{s}, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{s}, \bar{\lambda}_i, \bar{\theta}) = s, \mathbb{B}_0, \mathbb{B}_1, s, \lambda_i, \theta$. Let $\bar{\lambda}_0 = On \cap \bar{N}$.

By the subproperness of $\mathbb{B}_0$ There is $b_0 \subset a_0 =_{\mathrm{Df}} h_0(a)$ which forces the existence of $\sigma_0 \in V[G_0]$ s.t.

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \theta$ $(i = 1, \ldots, m)$

(c) $\sup \sigma_0{}'' \bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, \ldots, m)$

(d) $\bar{G}_0 = \sigma_0^{-1}{}'' G_0$ is $\bar{\mathbb{B}}_0$-generic over $\bar{N}$,

where G_0 is $\mathbb{B}_0$-generic over V,

But then σ_0 extends uniquely to a $\sigma_0{}^* : \bar{N}[\bar{G}] \prec N[G]$ s.t. $\sigma_0^*(\bar{G}) = G$.

Hence $X_0 = \mathrm{rng}(\sigma_0^*) \prec N[G_0]$ is full in $V[G_0]$.

Note that $b_0 \subset h_0(a) = [\![\check{a}/\dot{G}_0 \neq 0]\!]_{\mathbb{B}_0}$. Hence $a' = a/G_0 \neq 0$ in $\mathbb{B}' = \mathbb{B}_1/G_0$ and $a' \in X_0$.

But then there is $b' \subset a'$ in $\mathbb{B}'$ which forces the existence of $\sigma' \in V[G_0][G']$ (where G' is $\mathbb{B}'$-generic ~~over~~ $V[G_0]$) s.t.

(a') $\sigma' : \bar{N}[\bar{G}_0] \prec N[G_0]$

(b') $\sigma'(\bar{\lambda}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\mathbb{B}}', \bar{G}_0, \bar{\theta}) = \lambda, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \mathbb{B}', G_0, \theta$

where $i = 1, \ldots, m$ and $\bar{\mathbb{B}}' = \sigma'^{-1}(\mathbb{B}') = \bar{\mathbb{B}}_1/\bar{G}_0$.

(c') $\sup \sigma'{}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, m)$

(d') $\bar{G}' = \sigma'^{-1}{}'' G'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$.

We may assume $b' = \mathring{b}^{G_0}$, where $\Vdash_0 \mathring{b} \in \check{\mathbb{B}}_1/\dot{G}_0$ and b_0 forces that $\mathring{b}$ forces the above to hold. But then there is $b_1 \in \mathbb{B}_1$ s.t. $\Vdash_0 \check{b}_1/\dot{G}_0 = \mathring{b}$. Since $b_0 \Vdash \dot{b} = \check{b}_1/\dot{G}_0 \neq 0$, we have $b_0 \subset h_0(b_1)$. Hence $b \neq 0$, where $b = b_0 \cap b_1$. Since $b_0 \Vdash \check{b}_1/\dot{G}_0 \subset \check{a}/\dot{G}_0$, we have $b = b_0 \cap b_1 \subset a$. Now let $G_1 \ni b$ be $\mathbb{B}_1$-generic. Set:

$$G_0 = G_1 \cap \mathbb{B}_0 \;;\; G' = \{ b/G_0 \mid b \in G_1 \}$$

Then G_0 is $\mathbb{B}_0$-generic and G' is $\mathbb{B}' = \mathbb{B}_1/G_0$-generic over $V[G_0]$. Since $b_0 \in G_0$, there is $\sigma_0 \in V[G_0]$ satisfying (a)-(d) above. This gives $\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G_0]$ s.t. $\sigma_0^*(\bar{G}_0) = G_0$, and $b' = \mathring{b}^{G_0} = b_1/G_0$. Since $b' \in G'$, there is $\sigma' \in V[G_0][G'] = V[G_1]$

satisfying (a') – (d') above. Set:
$\sigma_1 = \sigma' \upharpoonright \bar{N}$. Then

(a) $\sigma_1 : \bar{N} \prec N$

(b) $\sigma_1(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_1, \theta \quad (i = 1, \ldots, n)$

(c) $\sup \sigma_1 `` \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, m)$

It remains only to show:

<u>Claim</u> $\bar{G}_1 = \sigma_1^{-1} `` G_1$ is $\bar{\mathbb{B}}_1$-generic over $\bar{N}$.

pf.
Clearly $G_1 = G_0 * G' = \{b \mid b/G_0 \in G'\}$.
Set $\bar{G}_1 = \bar{G}_0 * \bar{G}'$. Then $\bar{G}_1$ is $\bar{\mathbb{B}}_1$-generic over $\bar{N}$, since $\bar{G}_0$ is $\bar{\mathbb{B}}_0$-generic over $\bar{N}$ and $\bar{G}'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$. But

$$b \in \bar{G}_1 \longleftrightarrow b/\bar{G}_0 \in \bar{G}' \longleftrightarrow \sigma'(b/\bar{G}_0) = \sigma_1(b)/G_0 \in G'$$

$$\longleftrightarrow \sigma_1(b) \in G_1.$$

QED (Lemma 2)

We recall the definition of <u>subcompleteness</u>:

<u>Def</u> $\mathbb{B}$ is <u>subcomplete</u> iff for suff. large cardinals θ: If $\mathbb{B} \in H_\theta$ and $N, \bar{s}, \sigma, \lambda_i, \bar{N}$, $\bar{\mathbb{B}}, \bar{\theta}, \bar{s}, \bar{\lambda}_i$ are as in the def of "subproper", and $\bar{G}$ is $\bar{\mathbb{B}}$-generic over $\bar{N}$, then there is $b \in \mathbb{B}$ which forces the existence of σ_0 satisfying (a) – (d).
[The difference is that $\bar{G}$ was chosen in advance.]

We leave it to the reader to see that subcomplete forcing adds no reals. It is also straightforward to define the notion "weakly subcomplete" and prove the analogue of Lemma 1.

By a slight modification of the proof just given we obtain:

<u>Lemma 3</u> Lemma 2 holds with "subcomplete" in place of "subproper".

proof (sketch)

$\bar{G}_1$ is given at the outset and we set:

$\bar{G}_0 = \bar{G}_1 \cap \bar{\mathbb{B}}_0$; $\bar{G}' = \{ b/\bar{G}_0 \mid b \in \bar{G}_1 \}$.

Then $\bar{G}_1 = \bar{G}_0 * \bar{G}'$, $\bar{G}_0$ is $\bar{\mathbb{B}}_0$ - generic over $\bar{N}$ and $\bar{G}'$ is $\bar{\mathbb{B}}'$-generic over $\bar{N}[\bar{G}_0]$, where $\bar{\mathbb{B}}' = \bar{\mathbb{B}}_1 / \bar{G}_0$. As before we are assuming $\sigma : \bar{N} \prec N$ to be full and $\sigma(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{B}}_0, \bar{\mathbb{B}}_1, \bar{\theta}) = s, \lambda_i, \mathbb{B}_0, \mathbb{B}_1, \theta$.

Let $b_0 \in \mathbb{B}_0 \setminus \{0\}$ force the existence of $\sigma_0 \in V[G_0]$ satisfying (a)–(d). Then σ_0 extends uniquely to $\sigma_0^* : \bar{N}[\bar{G}_0] \prec N[G_0]$ as before, where $\bar{N}[\bar{G}_0]$ is full in $V[G_0]$. Thus $\sigma_0^*(\bar{\mathbb{B}}') = \mathbb{B}' = \mathbb{B}_1 / G_0$. Since $\bar{G}'$ is $\bar{\mathbb{B}}'$ - generic over $\bar{N}[\bar{G}_0]$ and $\mathbb{B}'$ is subcomplete, there is $b' \in \mathbb{B}' \setminus \{0\}$ forcing the existence of σ' satisfying (a')–(d'). Let b, b_1 be as before.

Just as before, $b = b_0 \cap b_1 \neq 0$. We then let $G_1 \ni b$ be $\mathbb{B}_1$ - generic and finish the proof exactly as before. QED (Lemma 3)

Without proof we mention:

<u>Lemma 4</u> Lemma 2 holds with "semisubproper" in place of "subproper".

The proof is again essentially the same, but the modification is somewhat more complex.

(It facilitates proofs of this sort if we replace "full" by "weakly full" (in the sense of the next section §3), in the definition of "semi subproper". This is an inessential weakening.)

Theorem 5 Let $\mathbb{B} = \langle \mathbb{B}_i \mid i \leq \alpha \rangle$ be an RCS iteration s.t. $\Vdash_i (\check{\mathbb{B}}_{i+1}/\dot{G}_i$ is subproper$)$ for $i < \alpha$.
Suppose moreover that $\bar{\bar{\xi}} \leq \bar{\bar{\mathbb{B}}}_\xi$ and $\mathbb{B}_{\xi+1}$ collapses $\bar{\bar{\mathbb{B}}}_\xi$ to ω_1 for $\xi < \alpha$. Then each $\mathbb{B}_\xi$ is subproper.

proof

By induction on i we show:

(*) $\Vdash_h (\check{\mathbb{B}}_i/\dot{G}_h$ is subproper$)$ for $h \leq i$,

$h = 0$ then gives the desired result. We note that if G_h is $\mathbb{B}_h$ - generic and $\tilde{\mathbb{B}} = \mathbb{B}/G_h$ then $\tilde{\mathbb{B}}_{i-h} = \mathbb{B}_i/G_h$ for $h \leq i \leq \alpha$. Moreover

$\Vdash_{\tilde{\mathbb{B}}_{i-h}} (\tilde{\mathbb{B}}_{(i-h)+1}/\dot{G}_{i-h}$ is subproper$)$ for $h \leq i < \alpha$

holds in $V[G_h]$. (To see this, let $\tilde{G}$ be $\tilde{\mathbb{B}}_{i-h}$ - generic over $V[G_h]$. Then $G_h * \tilde{G} = G_i$ is $\mathbb{B}_i$ - generic over V. Moreover $\tilde{\mathbb{B}}_{i-h+1}/\tilde{G} = (\mathbb{B}_{i+1}/G_h)/\tilde{G} =$

$= \mathbb{B}_{i+1}/G_i$ is subproper in $V[G_i] =$

$= V[G_h][\tilde{G}]$.)

This means in practice that once we have shown $\mathbb{B}_i$ to be subproper, we can simply repeat the proof in $V[G_h]$ to show that $\mathbb{B}_i/G_h = \tilde{\mathbb{B}}_{i-h}$ is subproper, where G_h is $\mathbb{B}_h$ - generic.

Case 1 $i=0$. Trivial since $\mathbb{B}_0=\{0,1\}$ is subproper.

Case 2 $i=j+1$.

$h=i$ is trivial.

$h=0$: $\mathbb{B}_j$ is subproper by the ind. hyp and $\Vdash_j \mathbb{B}_{j+1}/\dot{G}_j$ is subproper. Hence $\mathbb{B}_{j+1}$ is subproper by Lemma 2. QED

To get the result for other $h\leq j$, simply repeat the proof in $V[G_h]$.

QED (Case 2)

Case 3 $i=\lambda$, $Lim(\lambda)$. For the usual reasons $h=\lambda$ is again trivial. For the usual reasons it will suffice to prove it for $h=0$.

it will suffice to prove it for $h=0$, for an $i<\lambda$.

Case 3.1 $cf(\lambda)\leq \max(\omega_1, \bar{\bar{\mathbb{B}}}_i)$ for an $i<\lambda$.

Then there is $\eta<\lambda$ s.t. $\Vdash_\eta cf(\check{\lambda})\leq\omega_1$. Since we know that $\mathbb{B}_\eta$ is subproper, it suffices by Lemma 2 to show:

Claim $\Vdash_\eta (\mathbb{B}_\lambda/\dot{G}_\eta$ is subproper)

But this says that, if G_η is $\mathbb{B}_\eta$-generic, then $\tilde{\mathbb{B}}_{\lambda-\eta}$ is subproper, where $\tilde{\mathbb{B}}=\mathbb{B}/G_\eta$. Hence we may assume w.l.o.g. that $cf(\lambda)\leq\omega_1$ in V (taking V as $V[G_\eta]$, $\mathbb{B}$ as $\tilde{\mathbb{B}}$ and λ as $\lambda-\eta$.)

Let θ be big enough that $\mathbb{B} \in H_\theta$ and θ verifies the subproperness of $\mathbb{B}_i$ for $i < \lambda$. Let $\tau > \theta$ be regular and set: $N = \langle L_\tau[A], A, \ldots \rangle$ where $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$, $\sigma(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}) = \theta, \mathbb{B}, \lambda$, where $\bar{N}$ is countable and full.

Claim σ witnesses the subproperness of $\mathbb{B}_\lambda$ w.r.t. N.

In other words, if $\sigma(\bar{s}) = s$ and $\sigma(\bar{\lambda}_i) = \lambda_i$ for $i = 1, \ldots, n$, where λ_i is regular with $\bar{\bar{\mathbb{B}}}_\lambda < \lambda_i$ $(i = 1, \ldots, m)$, and given $a \in \mathbb{B}_\lambda \setminus \{0\}$, there is $b \subset a$, $b \in \mathbb{B}_\lambda \setminus \{0\}$, s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$-generic, there is $\tilde{\sigma} \in V[G]$ s.t.

(a) $\tilde{\sigma} : \bar{N} \prec N$

(b) $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, \lambda, s, \lambda_i$ $(i = 1, \ldots, m)$

(c) $\sup \tilde{\sigma}''\bar{\lambda}_i = \tilde{\lambda}_i =_{\text{df}} \sup \sigma''\lambda_i$ for $i = 0, \ldots, m$, where $\bar{\lambda}_0 = On \cap \bar{N}$.

(d) $g = \tilde{\sigma}^{-1}{}''G$ is $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$.

proof.

Let $\langle x_i \mid i < \omega \rangle$ enumerate $\bar{N}$ and let $\langle \Delta_i \mid i < \omega \rangle$ enumerate the $\Delta \in \bar{N}$ s.t. Δ is strongly dense in $\bar{\mathbb{B}}_{\bar{\lambda}} \setminus \{0\}$. Since $\mathrm{cf}(\lambda) \leq \omega_1$ there is $\bar{f} \in \bar{N}$ s.t. $f = \sigma(\bar{f})$ maps ω_1 to λ s.t. $\sup f''\omega_1 = \lambda$. Pick $\bar{\xi}_i \in \mathrm{rng}(\bar{f})$ s.t. s.t. $\langle \bar{\xi}_i \mid i < \omega \rangle$ is monotone and cofinal in $\bar{\lambda}$ with $\bar{\xi}_0 = 0$. Set $\xi_i = \sigma(\bar{\xi}_i)$.

We first define a sequence $\langle b_n \mid n < \omega \rangle$ s.t. $\Gamma(b_n, \bar{\mathbb{B}} \restriction \bar{\lambda})$ in $\bar{N}$, $b^*_{n+1} \subset b^*_n$, and $b_{n+1} \restriction \bar{\bar{\xi}}_{n+1} = b_n \restriction \bar{\bar{\xi}}_{n+1}$ as follows:

Pick b_0 s.t. $b_0^* \subset \bar{a}$ and $b_0^* \in \Delta_0$. Given b_n we construct b_{n+1} s.t.

$\{a \in \mathbb{B}_{\bar{\bar{\xi}}_{n+1}} \mid a \cap b^*_{n+1} \in \Delta_{n+1}\}$ is dense

below $\tilde{a} = h_{\bar{\bar{\xi}}_{n+1}}(b^*_n) = (b_n \restriction \bar{\bar{\xi}}_{n+1})^*$.

We accomplish this as follows:

Set $\Delta = \{b \mid \Vdash(b, \overline{\mathbb{B}}\restriction\bar\lambda) \wedge b^* \subset b_n^* \wedge b^* \in \Delta_{n+1}\}$. Then $\{b^* \mid b \in \Delta\}$ is dense below b_n^* in $\overline{\mathbb{B}}_{\bar\lambda} \setminus \{0\}$. Thus $\Delta' = \{h_{\bar{\bar\xi}_{n+1}}(b^*) \mid b^* \in \Delta\}$ is dense below $\tilde a =_{\mathrm{Df}} h_{\bar{\bar\xi}_{n+1}}(b_n^*)$ in $\overline{\mathbb{B}}_{\bar\lambda} \setminus \{0\}$.

Let A be a maximal antichain in Δ'. Then $\bigcup A = \tilde a$. For each $a \in A$ choose a $b_a \in \Delta$ s.t. $a = h_{\bar{\bar\xi}_{n+1}}(b_a^*) = (b_a \restriction \bar{\bar\xi}_{n+1})^*$ Set:

$$b_{n+1}(i) = \begin{cases} b_n(i) & \text{if } i < \bar{\bar\xi}_{n+1} \\ \bigcup\limits_{a \in A}(a \cap b_a(i)) \cup \neg\tilde a & \text{if } i \geq \bar{\bar\xi}_{n+1} \end{cases}$$

We claim that b_{n+1} has the desired properties. We first show that it is a good sequence for $\overline{\mathbb{B}}\restriction\bar\lambda$.

(1) $h_i(b_{n+1}(i)) = 1$

Trivial for $i < \bar{\bar\xi}_{n+1}$. Now let $i \geq \bar{\bar\xi}_{n+1}$. Then
$h_i(b_{n+1}(i)) = \bigcup\limits_{a \in A} a \cup \neg\tilde a = \tilde a \cup \neg\tilde a = 1$,
since $h_i(b_a(i)) = 1$. QED(1)

(2) $h_i(b_{n+1}^*) = (b_{n+1} \restriction i)^*$.

We recall the <u>disjoint distributive law</u> which holds in every complete BA:

(DDL) Let $b = \bigcup\limits_{i \in I} b_i$, where $b_i \cap b_j = 0$ for $i \neq j$. Let $a_i^j \subset b_i$ for $i \in I$, $j \in J$. Then

$$\bigcap_j \bigcup_i a_i^j = \bigcup_i \bigcap_j a_i^j.$$

pf.

$\bigcap\limits_j \bigcup\limits_i a_i^j = b \cap \bigcap\limits_j \bigcup\limits_i a_i^j = \bigcup\limits_i (b_i \cap \bigcap\limits_j \bigcup\limits_h a_h^j) =$
$= \bigcup\limits_i \bigcap\limits_j (b_i \cap \bigcup\limits_h a_h^j) = \bigcup\limits_i \bigcap\limits_j a_i^j$. QED(DDL)

As a step toward proving (2) we first note:

(3) If $j \geq \bar{\xi}_{n+1}$, then $(b_{n+1} \upharpoonright j)^* = \bigcup_{a \in A} (b_a \upharpoonright j)^*$.

prf.

$$(b_{n+1} \upharpoonright j)^* = \tilde{a} \cap \bigcap_{i \in [\bar{\xi}_{n+1}, j)} b_{n+1}(i)$$

$$= \tilde{a} \cap \bigcap_{i \in [\bar{\xi}_{n+1}, j)} \; \bigcup_{a \in A} (a \wedge b_a(i))$$

$$= \tilde{a} \cap \bigcup_{a \in A} \; \bigcap_{i \in [\bar{\xi}_{n+1}, j)} (a \cap b_a(i))$$

$$= \tilde{a} \cap \bigcup_{a \in A} (b_a \upharpoonright j)^* = \bigcup_{a \in A} (b_a \upharpoonright j)^*$$

since $a = (b_a \upharpoonright \bar{\xi}_{n+1})^*$ for $a \in A$. QED(3)

We now prove (2). For $j \in [\bar{\xi}_{n+1}, \lambda)$ have:

$$h_j(b^*_{n+1}) = h_j\Big(\bigcup_{a \in A} b^*_a\Big) = \bigcup_{a \in A} h_j(b^*_a) =$$

$$= \bigcup_{a \in A} (b_a \upharpoonright j)^* = (b_{n+1} \upharpoonright j)^*.$$

For $j < \bar{\xi}_{n+1}$ we have $h_{\bar{\xi}_{n+1}}(b^*_{n+1}) = \bigcup A = \tilde{a}$ by (3) and hence:

$$h_j(b^*_{n+1}) = h_j h_{\bar{\xi}_{n+1}}(b^*_{n+1}) = h_j(\tilde{a}) =$$

$$= h_j((b_n \upharpoonright \bar{\xi}_{n+1})^*) = (b_n \upharpoonright j)^* = (b_{n+1} \upharpoonright j)^*.$$

QED(2)

By (1) + (2) b_{n+1} is a good sequence:

(4) $GS(b_{n+1}, \mathbb{B} \upharpoonright \lambda)$.

But then:

(5) $\mathbb{R}(b_{n+1}, \overline{\mathbb{B}} \restriction \bar{\lambda})$

pf. Let $c \in U_{\bar{\lambda}}(b_{n+1})$. Then $c \in \mathbb{B}_i$ and $c \subset (b_{n+1} \restriction i)^*$ for some i. We may assume $i \geq \bar{\xi}_n$. Hence $c \subset \bigcup_{a \in A} (b_a \restriction i)^*$. Let $c' = c \cap (b_a \restriction i)^* \neq 0$. Then $c' \in U_{\bar{\lambda}}(b_a)$ and there is $d \subset c'$ s.t. $d \in S_{\bar{\lambda}}(b_a)$. It follows easily that $d \in S_{\bar{\lambda}}(b_{n+1})$. QED(5)

Finally we note:

(6) $\{a \in \mathbb{B}_{\bar{\xi}_{n+1}} \setminus \{0\} \mid a \cap b^*_{n+1} \in \Delta_{n+1}\}$ is dense below $\tilde{a} = h_{\bar{\xi}_{n+1}}(b_n)$ in $\mathbb{B}_{\bar{\xi}_{n+1}} \setminus \{0\}$.

pf. It suffices to show that it is predense. But $\bigcup A = \tilde{a}$ and $a \cap b^*_{n+1} = a \cap \bigcup_{a' \in A} b^*_{a'} = b^*_a \in \Delta_{n+1}$ for $a \in A$. QED(6)

This completes the construction of $\langle b_n \mid n < \omega \rangle$.

By induction on $n < \omega$ we construct:

- $\langle c_n \mid n < \omega \rangle$ s.t. $R(c_n, \mathbb{B} \upharpoonright \xi_n)$

(Hence $c_0 = \emptyset$, $c_0^* = 1$, since $\xi_0 = 0$)

- $\langle \dot{\sigma}_n \mid n < \omega \rangle, \langle \dot{u}_n \mid n < \omega \rangle, \langle \dot{g}_n \mid n < \omega \rangle$

s.t. $\dot{\sigma}_n, \dot{u}_n, \dot{g}_n \in V^{\mathbb{B}_{\xi_n}}$

We inductively verify:

(a) $c_{n+1} \upharpoonright \xi_n = c_n$

(b) $\dot{\sigma}_0 = \check{\sigma}$, $g_0 = \{\check{1}\}$, $u_0 = \check{u}_0$, where

$u_0 = \langle \bar{s}, \bar{\lambda}_1, \ldots, \bar{\lambda}_m, \bar{f}, \bar{\mathbb{B}}, \bar{\theta}, \bar{\lambda} \rangle$

(c) c_n^* forces the following to hold in $V^{\mathbb{B}_{\xi_n}}$:

- $\dot{g}_n$ is $\check{\bar{\mathbb{B}}}_{\bar{\xi}_n}$ - generic over $\check{\bar{N}}$
- $(\check{b}_n \upharpoonright \bar{\xi}_n)^* \in \dot{g}_n$
- $\dot{\sigma}_n : \bar{N}[\dot{g}_n] \prec N[\dot{G}_n] \wedge \dot{\sigma}_n(\dot{g}_n) = \dot{G}_n$

(where $\dot{G}_n$ is the canonical generic name in $V^{\mathbb{B}_{\xi_n}}$)

(d) If $n > 0$, then c_n^* forces:

- $\dot{\sigma}_n(\dot{u}_{n-1}) = \dot{\sigma}_{n-1}(\dot{u}_{n-1})$
- $\dot{u}_n = \langle \check{x}_n, \dot{z}_n^0, \ldots, \dot{z}_n^m, \dot{u}_{n-1}, \dot{g}_n, \check{b}_n \rangle$,

where:

- $z_n^l =$ the least $z < \check{\bar{\lambda}}_l$ s.t. $\dot{\sigma}_n(z) \geq \check{\xi}_n^l$

(where $\langle x_i \mid i < \omega \rangle$ enumerates $\bar{N}$, $\langle \xi_n^l \mid n < \omega \rangle$ is a monotone, cofinal sequence in $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, \ldots, m)$, and $\bar{\lambda}_0 = On \cap \bar{N}$.

Since $c_n = c_{n+1} \upharpoonright \xi_n$ and $\tilde{\lambda} = \sup_n \xi_n$ is ω-
-cofinal in V, we conclude that
$\mathbb{P}(c, \mathbb{B} \upharpoonright \tilde{\lambda})$, where $c = \bigcup_n c_n$.
But then $c^* \in \mathbb{B}_\lambda \setminus \{0\}$ and $h_{\xi_n}(c^*) = c_n^*$.
Let $G \ni c^*$ be $\mathbb{B}_\lambda$-generic. Set:
$G_n = G \cap \mathbb{B}_{\xi_n} = \dot{G}_{\xi_n}^G$; $g_n = \dot{g}_n^{G_n}$, $\sigma_n = \dot{\sigma}_n^{G_n}$
Note that $\sigma_n(\bar{\xi}_h) = \xi_h$ for all $n, h < \omega$,
since $\sigma_n(\bar{f}) = f$ for all n. Then
$\sigma_n : \bar{N}[g_n] \prec \tilde{N}[G_n] \wedge \sigma_n(g_n) = G_n$ $(n < \omega)$.
Since $\sigma_n(x_h) = \sigma_h(x_h)$ for $h < n$, we can
define a new map $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ by:
$\tilde{\sigma}(x) =_{Df} \sigma_n(x)$ for suff. large n.
Since $\sigma_n(g_{n-1}) = \sigma_{n-1}(g_{n-1}) = G_{n-1} = G \cap \mathbb{B}_{\xi_{n-1}}$
for $n > 0$, it follows easily that:
$\sigma_n(g_h) = G_h = G \cap \mathbb{B}_{\xi_h}$ for $h \leq n < \omega$,
Thus $g_n = \sigma_{n+i}^{-1} {}'' G_n = \tilde{\sigma}^{-1} {}'' G_n$ $(n < \omega)$.
Set: $g = \tilde{\sigma}^{-1} {}'' G$. Then $g \cap \bar{\mathbb{B}}_{\bar{\xi}_n} = g_n$.

<u>Claim 1</u> g is $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$.
prf. $(b_n \upharpoonright \bar{\xi}_{n+1})^* \in g_{n+1} \subset g$ for $n < \omega$,
where $(b_n \upharpoonright \bar{\xi}_{n+i+1})^* \supset (b_{n+1} \upharpoonright \bar{\xi}_{n+i+1})^*$.

Hence $(b_n \restriction \bar{\xi}_{n+i})^* \in g$ for $n, i < \omega$. Hence

$$b_n^* = \bigcap_{i<\omega} (b_n \restriction \bar{\xi}_{n+i})^* \in g. \text{ But}$$

$\tilde{\Delta} = \{a \subset (b_n \restriction \bar{\xi}_n)^* \mid a \cap b_n^* \in \Delta_n\}$ is dense in $\bar{\mathbb{B}}_{\bar{\xi}_n} \setminus \{0\}$. Hence $g_n \cap \tilde{\Delta} \neq \emptyset$, since g_n is $\bar{\mathbb{B}}_{\bar{\xi}_n}$ - generic over $\tilde{N}$. Let $a \in g_n \cap \tilde{\Delta}$. Then $a \cap b_n^* \in g$, where $a \cap b_n^* \in \Delta_n$. Hence $g \cap \Delta_n \neq \emptyset$ for all n QED (Claim 1)

<u>Claim</u> 2 $\sup \tilde{\sigma}'' \bar{\lambda}_i = \tilde{\lambda}_i = \sup \sigma'' \lambda_i$ for $i = 0, \ldots, n$

($\geq$) $\tilde{\sigma}(z_n^i) = \sigma_n(z_n^i) \geq \xi_n^i$ where $\sup_n \xi_n^i = \tilde{\lambda}_i$

($\leq$) $\tilde{\sigma}(x) = \sigma_n(x) = \tilde{\lambda}_i$ for some n if $x < \bar{\lambda}_i$.

QED (Claim 2)

(Note that by Claim 2, we have: $\tilde{\sigma} : \bar{N} \prec \tilde{N}$ cofinally, since $\lambda_0 = On \cap \tilde{N}$.)

Trivially

<u>Claim 3</u> $\tilde{\sigma}(\bar{\theta}, \bar{\mathbb{B}}, \bar{\lambda}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, \lambda, s, \lambda_i$

This proves that σ witnesses the subproperness of $\mathbb{B}_\lambda$.

It remains only to carry out the construction of c_n $(n < \omega)$.

The construction of $c_0 = \emptyset$, $\dot\sigma_0 = \hat\sigma$, $\dot u_0 = \check u_0$, $\dot g_0 = \{\check 1\}$ has already been given. Now let $c_n, \dot\sigma_n, \dot u_n, \dot g_n$ be given s.t. (a)–(d) hold. Let $G_n \ni c_n^*$ be $\mathbb{B}_{\overline{\xi}_n}$-generic. Set:

$\sigma_n = \dot\sigma_n^{G_n}$, $u_n = \dot u_n^{G_n}$, $g_n = \dot g_n^{G_n}$. Then

$\sigma_n : \overline{N}[g_n] \prec N[G_n]$ and $\sigma_n(g_n) = G_n$.

Hence $X_n = \operatorname{rng}(\sigma_n) \prec N[G_n]$ is full in $V[G_n]$. But $\tilde{\mathbb{B}} = \mathbb{B}_{\overline{\xi}_{n+1}} / G_n$ is subproper in $V[G_n]$. Clearly $\sigma_n(\overline{\mathbb{B}}_{\overline{\xi}_{n+1}} / g_n) = \tilde{\mathbb{B}}$.

Moreover $\sup \sigma_n{}''\overline{\lambda}_\ell = \tilde\lambda_\ell =_{\mathrm{df}} \sup \sigma''\lambda_\ell$ for $\ell = 0, \dots, m$. Since:

$$[\![(b_m \upharpoonright \check{\overline{\xi}}_{n+1})^* / \dot g_n \neq 0]\!]_{\mathbb{B}_{\overline{\xi}_n}} = h_{\overline{\xi}_n}(b_n \upharpoonright \overline{\xi}_{n+1})^* =$$

$$= h_{\overline{\xi}_n} h_{\overline{\xi}_{n+1}}(b_n^*) = (b_n \upharpoonright \overline{\xi}_n)^* \in g_n,$$

we have: $(b_n \upharpoonright \overline{\xi}_{n+1})^* / g_n \neq 0$.

Set: $\overline{b} = (b_n \upharpoonright \overline{\xi}_{n+1})^* / g_n$ and

$b \doteq \sigma_n(\overline{b}) = (\sigma_n(b_n) \upharpoonright \xi_{n+1})^* / G_n$.

Then $b \in \tilde{\mathbb{B}} \cap X$ and $b \neq 0$. Hence there is a condition $\tilde c \in \tilde{\mathbb{B}} \setminus \{0\}$ s.t. $\tilde c \subset b$ and $\tilde c$ forces the following to hold:

(*) Let $\tilde{G} \ni \tilde{c}$ be $\tilde{\mathbb{B}}$-generic over $V[G_n]$. Then there is $\tilde{\sigma} \in V[G_n][\tilde{G}]$ s.t.

(a) $\tilde{\sigma} : \bar{N}[g_n] \prec N[G_n]$ s.t. $\tilde{\sigma}(g_n) = G_n$

(b) $\tilde{\sigma}(u_n) = \sigma_n(u_n)$

(c) $\sup \tilde{\sigma}'' \bar{\lambda}_i = \tilde{\lambda}_i \quad (i = 0, \ldots, m)$

(d) $\tilde{g} = \tilde{\sigma}^{-1}{}'' \tilde{G}$ is $\bar{\mathbb{B}}_{\bar{\xi}_{n+1}} / g_n$ - generic over $\bar{N}[g_n]$.

<u>Note</u> that $\tilde{\sigma}(b_n) = \sigma_n(b_n)$ by (b). Hence $\tilde{\sigma}(\bar{b}) = \sigma_n(\bar{b}) = b$. Since $\tilde{c} \subset b$, we then have:
$(b_n \upharpoonright \bar{\xi}_{n+1}) / g_n \in \tilde{g}$.

W.l.o.g we may assume $\tilde{c} = c^*$, where $R(c, (\mathbb{B} \upharpoonright \xi_{n+1}) / G_n)$. We may also assume $c = \dot{c}^{G_n}$, where $\Vdash_{\xi_n} R(\dot{c}, (\mathbb{B} \upharpoonright \check{\xi}_{n+1}) / \dot{G}_n)$ and c_n forces the statements about $\dot{c}$ which ensure (*) for $c_n \in G_n$.

For $\xi_n \leq i < \xi_{n+1}$ set:
$c'(i) =$ that $b \in \mathbb{B}_{i+1}$ s.t. $\Vdash_{\xi_n} \check{b} / \dot{G}_n = \dot{c}(\check{i})$

Set: $c'(i) = 1$ for $i < \xi_n$. Then $R(c', \mathbb{B} \upharpoonright \xi_{n+1})$, since $\Vdash_{\xi_n} R(\check{c}' / \dot{G}_n, \mathbb{B} \upharpoonright \check{\xi}_{n+1} / \dot{G}_n)$.

Moreover $h_{\xi_n}(c'^*) = [\![\check{c}'^* / \dot{G}_n \neq 0]\!]_{\mathbb{B}_{\xi_n}} =$
$= [\![c'^* \neq 0]\!]_{\mathbb{B}_{\xi_n}} = 1$. If we then set:

$$c_{n+1}(i) = \begin{cases} c_n(i) & \text{if } i < \xi_n \\ c'(i) & \text{if } \xi_n \leq i < \xi_{n+1} \end{cases}$$

we have:

$c^*_{n+1} = c^*_n \wedge c'^* \neq 0$, since $h_{\xi_n}(c^*_n \wedge c'^*) =$
$= c^*_n \wedge h_{\xi}(c'^*) = c^*_n \neq 0$. Since
$\mathbb{R}(c_{n+1} \upharpoonright \xi_n, \mathbb{B} \upharpoonright \xi_n)$ and
$$(c_{n+1} \upharpoonright \xi_n)^* \Vdash_{\xi_n} \mathbb{R}(c_{n+1}/\dot{G}_n, \mathbb{B} \upharpoonright \breve{\xi}_{n+1}/\dot{G}_n),$$
we conclude: $\mathbb{R}(c_{n+1}, \mathbb{B} \upharpoonright \xi_{n+1})$.
Hence $c^*_{n+1} \neq 0$ in $\mathbb{B}_{\xi_{n+1}}$, since $\mathbb{B}$ is an RCS iteration. Now let $G_{n+1} \ni c^*_{n+1}$ be $\mathbb{B}_{\xi_{n+1}}$ - generic. Then $G_n = G_{n+1} \cap \mathbb{B}_{\xi_n}$ is $\mathbb{B}_{\xi_n}$ - generic and $c^*_n \in G_n$. Set:
$$\tilde{\mathbb{B}} = \mathbb{B}_{\xi_{n+1}}/G_n, \quad c = c_{n+1}/G_n,$$
$$\tilde{G} = \{b/G_n \mid b \in G_{n+1}\},$$
Then $\tilde{G}$ is $\tilde{\mathbb{B}}$ - generic over $N[G_n]$ and $\tilde{c} = c^* \in \tilde{G}$. Thus $(*)$ holds. Set:
$\hat{g} = \tilde{\sigma}^{-1}{}''\tilde{G}$, where $\hat{g}$ is given by $(*)$,
$\tilde{\sigma}$ extends uniquely to a
$$\sigma_{n+1} : \bar{N}[g_n][\tilde{g}] \prec N[G_n][\tilde{G}]$$
s.t. σ_{n+1}
$$g_{n+1} = g_n * \tilde{g} = \{b \in \bar{\mathbb{B}}_{\bar{\xi}_{n+1}} \mid b/g_n \in \tilde{g}\}.$$
We know: $G_n * \tilde{G} = G_{n+1}$. Since
$\sigma_{n+1}(u_n) = \sigma_n(u_n)$, we have $\sigma_{n+1}(g_n) = G_n$

Hence $\sigma_{n+1} : \bar{N}[g_{n+1}] \prec N[G_{n+1}]$ and $\sigma_{n+1}(g_{n+1}) = G_{n+1}$. We also note that $(b_n \upharpoonright \bar{\xi}_{n+1})/g_n \in \tilde{g}$, so;

- $b_{n+1} \upharpoonright \bar{\xi}_{n+1} = b_n \upharpoonright \bar{\xi}_{n+1} \in g_{n+1}$.

By our construction:

- g_{n+1} is $\mathbb{B}_{\bar{\xi}_{n+1}}$ - generic over $\bar{N}$
- $\sigma_{n+1} : \bar{N}[g_{n+1}] \prec N[G_{n+1}]$ s.t. $\sigma_{n+1}(g_{n+1}) = G_{n+1}$
- $\sigma_{n+1}(u_n) = \sigma_n(u_n)$

Now set:

$$u_{n+1} = \langle x_{n+1}, z^0, \ldots, z^m, u_n, g_n, b_n \rangle$$

where z^ℓ = the least $z < \bar{\lambda}_\ell$ s.t. $\dot{\sigma}_{n+1}(z) \geq \xi^\ell_{n+1}$ $(\ell = 0, \ldots, m)$.

All of this is forced by c^*_{n+1}, so there are terms $\dot{\sigma}_{n+1}, \dot{g}_{n+1}, \dot{u}_{n+1}$ s.t.

$$\sigma_{n+1} = \dot{\sigma}_{n+1}^{G_{n+1}}, \quad g_{n+1} = \dot{g}_{n+1}^{G_{n+1}}, \quad u_{n+1} = \dot{u}_{n+1}^{G_{n+1}}$$

and the above statements are forced by c^*_{n+1}.

This completes the construction.

QED (Case 3.1)

Case 3.2 Case 3.1 fails

Then $\lambda > \omega_1$ is regular and $\overline{\overline{\mathbb{B}}}_i < \lambda$ for $i < \lambda$. Hence λ is regular in $V^{\mathbb{B}_i}$ for $i < \lambda$. Thus, be the definition of RCS-iteration, we have: $\mathbb{B}_\lambda$ is the minimal completion of $\bigcup_{i<\lambda} \mathbb{B}_i$ (i.e. $\bigcup_{i<\lambda} \mathbb{B}_i \setminus \{0\}$ is dense in $\mathbb{B}_\lambda \setminus \{0\}$).

We again let θ s.t. $\mathbb{B} \in H_\theta$ and θ verifies the subproperness of $\mathbb{B}_i$ for $i < \lambda$. We claim that θ verifies the subproperness of $\mathbb{B}_\lambda$. Let $\kappa > \theta$ be regular and let $N = \langle L_\tau[A], A, \ldots \rangle$ s.t. $H_\theta \subset N$. Again let $\sigma : \bar{N} \prec N$, $\sigma(\bar\theta, \bar{\mathbb{B}}, \bar\lambda) = \theta, \mathbb{B}, \lambda$, where $\bar N$ is countable and full.

Claim σ witnesses the subproperness of $\mathbb{B}_\lambda$ wrt. N.

We again let $\sigma(\bar s) = s$ and suppose that $\sigma(\bar\lambda_i) = \lambda_i$ for $i = 1, \ldots, m$, where λ_i is regular and $\overline{\overline{\mathbb{B}}}_\lambda < \lambda_i$ for $i = 1, \ldots, m-1$. We set: $\lambda_m = \lambda$. We are given $a \in \mathbb{B}_\lambda \setminus \{0\}$ and claim that there is $b \subset a$, $b \in \mathbb{B}_\lambda \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$-generic, then there is $\tilde\sigma \in V[G]$ s.t. (a)-(d) hold as before.

Note We are of course not constrained to prove $\sup \tilde\sigma '' \bar\lambda_m = \sup \sigma '' \lambda_m$, since we do not have $\overline{\overline{\mathbb{B}}}_\lambda < \bar\lambda_m$. However, this will come out of the proof, and including $\lambda = \lambda_m$ in our list of regular cardinals facilitates our proof. We shall, of course, exploit the fact that $\overline{\overline{\mathbb{B}_i}} < \lambda$ for $i < \lambda$.

To prove this we again pick a cofinal monotone sequence $\langle \bar{\xi}_i \mid i < \omega \rangle$ in $\bar{\lambda}$ with $\bar{\xi}_0 = 0$. However, we are no longer able to enforce that $\sigma_h(\bar{\xi}_n) = \sigma(\bar{\xi}_n)$, where $\langle \sigma_h \mid h < \omega \rangle$ is the sequence of maps we intend to add, converging to $\tilde{\sigma}$. This will make our construction more complex. We will be able to enforce $\sup \sigma_h{}''\bar{\lambda} = \sup \sigma''\lambda$. We again let $\langle x_i \mid i < \omega \rangle$ enumerate $\bar{N}$ and $\langle \Delta_i \mid i < \omega \rangle$ enumerate the strongly dense subsets of $\bar{\mathbb{B}}_{\bar{\lambda}} = \sigma^{-1}(\mathbb{B}_\lambda)$ in $\bar{N}$. We define the sequence $\langle b_i \mid i < \omega \rangle$ exactly as before. Set: $\tilde{\lambda} = \tilde{\lambda}_m = \sup \sigma''\bar{\lambda}$. Then $\tilde{\lambda} < \lambda$. $\mathbb{B}_{\tilde{\lambda}}$ will now in large part play the role that $\mathbb{B}_\lambda$ played in Case 3.1.

We again choose a monotone cofinal sequence $\langle \xi_i^l \mid i < \omega \rangle$ in $\tilde{\lambda}_l = \sup \sigma''\bar{\lambda}_l$ for $l = 0, \ldots, m$. We take $\xi_i^m = \xi_i$, where $\langle \xi_i \mid i < \omega \rangle$ is defined as above.

We wish to construct c_n ($n<\omega$) which will play the same role as in Case 3.1. In particular c_n^* should force the existence of σ_n, g_n s.t.:

(1) • g_n is $\bar{\mathbb{B}}_{\bar{\xi}_n}$ - generic over $\bar{N}$

• $b_n \upharpoonright \bar{\xi}_n \in g_n$

• $\sigma_n : \bar{N}[g_n] \prec N[G_n]$ s.t. $\sigma(g_n) \neq G_n$.

However, c_n^* cannot itself fix the value of $\sigma_n(\bar{\xi}_n)$, so it makes no sense to require $R(c_n, \mathbb{B}_{\sigma_n(\bar{\xi}_n)})$. The best we can require is that $R(c_n, \mathbb{B}_{\tilde{\lambda}})$ and that if $G_{\tilde{\lambda}} \ni c_n^*$ is $\mathbb{B}_{\tilde{\lambda}}$ - generic, then (1) holds with $G_n = G_{\tilde{\lambda}} \cap \mathbb{B}_{\sigma_n(\bar{\xi}_n)}$.

If $s = \langle \nu_0, \ldots, \nu_{n-1} \rangle$ is any monotone sequence in $\tilde{\lambda}$ we simultaneously construct an $e_s \in \mathbb{B}_{\nu_{n-1}}$ ($e_s = 1$ if $n=0$) s.t. $e_s \wedge c_n^*$ fixes the value of $\sigma_n(\bar{\xi}_h)$ as ν_h for $h<n$. We will then have $e_s \wedge c_n^* \in \mathbb{B}_{\nu_{n-1}}$.

Moreover, if $n > 0$, then $e_s \subset C^*_{n-1}$.
(We can, of course, have $e_s = 0$, but not for all s.)

Def For $n < \omega$ let S_n = the set of monotone $s : n \to \tilde{\lambda}$. Set $S = \bigcup_n S_n$. For $s \in S$ set $|s| = \mathrm{dom}(s) = n$ if $s \in S_n$. We also set $\max(s) = \sup \mathrm{rng}(s)$ (hence $\max(\emptyset) = 0$).

By induction on n we construct:

- c_n s.t. $\mathbb{P}(c_n, \mathbb{B}_{\tilde{\lambda}})$
- $\langle e_s \mid s \in S_n \rangle$ s.t. $e_s \in \mathbb{B}_{\max(s)}$
- $\langle \dot{\sigma}_s \mid e_s \neq 0 \rangle, \langle \dot{u}_s \mid e_s \neq 0 \rangle, \langle \dot{g}_s \mid e_s \neq 0 \rangle$ s.t. $\dot{\sigma}_s, \dot{g}_s, \dot{u}_s \in V^{\mathbb{B}_{\max(s)}}$.

We inductively verify:

(a). $e_\emptyset = 1 \in \mathbb{B}_0$, $e_{s\nu} \in \mathbb{B}_{\max(s)}$
(where $s\nu = s^\frown\langle \nu \rangle$)

- $e_{s\nu} \cap e_{s\nu'} = 0$ if $\nu \neq \nu'$.
- $\bigcup_\nu e_{s\nu} = e_s \cap C^*_{|s|}$ (hence $e_{s\nu} \subset C^*_{|s|}$)
- $e_s \subset C_{|s|}(i)$ for $i \geq \max(s)$
(hence $e_s \cap C^*_{|s|} = e_s \cap (C_s \restriction \max(s))^* \in \mathbb{B}_{\max(s)}$)
- $C_n(i) \subset C_m(i)$ for $m < n$
(hence $C^*_n \subset C^*_m$)
- $e_s \cap C_{|s|}(i) = e_s \cap C_n(i)$ for $n \geq |s|$, $i < \max(s)$

(hence $e_s \cap (c_{|s|} \upharpoonright \max(s))^* = e_s \cap (c_n \upharpoonright \max(s))^*$ for $|s| \leq n$.)

(b) $\dot\sigma_\emptyset = \check\sigma$, $\dot g_\emptyset = \check{\{1\}}$, $\dot u_\emptyset = \check u_\emptyset$, where

$u_\emptyset = \langle x_0, \bar\imath, \bar\lambda_1, \ldots, \bar\lambda_m, \bar{\mathbb{B}}, b_0 \rangle$, where $\langle x_i \mid i < \omega \rangle$ enumerates $\bar N$.

(c) $e_s \cap c^*_{|s|}$ forces the following in $V^{\mathbb{B}_{\max(s)}}$

- $\dot g_s$ is $\check{\mathbb{B}}_{\bar\xi_{|s|}}$ – generic over $\check{\bar N}$
- $(b_{|s|} \upharpoonright \bar\xi_{|s|})^* \in \dot g_s$
- $\dot\sigma_s : \check{\bar N}[\dot g_s] \prec \check N[\dot G]$ and $\dot\sigma_s(\dot g_s) = \dot G$, where $\dot G$ is the canonical generic name;

(d) If $s = \bar s\nu$, then $e_s \cap c^*_{|s|}$ forces:

- $\dot\sigma_s(\dot u_{\bar s}) = \dot\sigma_{\bar s}(\dot u_{\bar s})$
- $\dot u_s = \langle \check x_{|s|}, \dot z^0_s, \ldots, \dot z^m_s, \dot g_s, \check b_{|s|}, \bar\xi_{|s|} \rangle$, where
- $\dot z^\ell_s =$ the least $z < \check{\bar\lambda}_\ell$ s.t. $\dot\sigma_s(z) \geq \check{\bar\xi}^\ell_{|s|}$ ($\ell = 0, \ldots, m$)

(e) $e_{s\nu} \Vdash_{\max(s)} \dot\sigma_s(\check{\bar\xi}_{|s|}) = \check\nu$

<u>Note</u> By (a) we easily have: $s(i) \neq s'(i) \to e_s \cap e_{s'} = 0$ for $s, s' \in S$.

We shall delay the construction of $c_m, e_s, \dot\sigma_s, \dot u_s, \dot g_s$ and the verification of (a)–(e) until later

We note:

(1) Let $n > 0$. Then $\bigcup_{|s|=n} e_s = c^*_{n-1}$

pf. Ind. on n

$n=1$: $\bigcup_s e_s = \bigcup_\nu e_{\emptyset\nu} = e_\emptyset \cap c_0^* = 1 = c_0^*$

$n = m+1$: $\bigcup_s e_s = \bigcup_{|s|=m} \bigcup_\nu e_{s\nu} =$

$= \bigcup_{|s|=m} (e_s \cap c^*_m) = c^*_{m-1} \cap c^*_m = c^*_m$ QED

Our intention is to fuse the c_n $(n<\omega)$ into a c s.t. $\mathbb{R}(c, \mathbb{B}\restriction\tilde{\lambda})$ just as in Case 3.1, but it will be somewhat trickier in the present case. We set:

$\bar{c}(i) = \bigcap_{n<\omega} c_n(i)$; $c(i) = \bar{c}(i) \cup \mathrm{Th}_i(\bar{c}(i))$.

<u>Claim</u> (a) $h_\ell((\bar{c}\restriction i)^*) = (\bar{c}\restriction\ell)^*$ for $\ell \le i \le \tilde{\lambda}$

(Hence $(\bar{c}\restriction i)^* = (c\restriction i)^*$)

(b) $\mathbb{R}(c\restriction i, \mathbb{B}\restriction\tilde{\lambda})$ for $i \le \tilde{\lambda}$.

The proof will be by induction on i.

In the following suppose that $j < \tilde{\lambda}$ and that (a), (b) hold for $i<j$. Suppose moreover that $a \in U_i(\bar{c}\restriction i)$ – i.e. there is $h<i$ s.t.

$a \subset (\bar{c}\restriction h)^*$ and $a \in \mathbb{B}_h$.

(2) $a \cap c^*_m \neq 0$ for all $m<\omega$.

pf.

Clearly $a \in U_i(c_m\restriction i)$. Let $a \subset (\bar{c}\restriction h)^*$, $a \in \mathbb{B}_h$, $h<i$. Then $a \cap c^*_m \neq 0$ since $h_i(a \cap c^*_m) = a \cap (c_m\restriction i)^* = a \neq 0$.

QED

(3) For $m<\omega$ there is $s \in S_m$ s.t. $a \cap e_s \cap c^*_m \neq 0$.

pf. $\bigcup_{|s'|=m+1} e_{s'} = c^*_m$. Hence $a \cap e_{s'} \neq 0$ for an $s' = s\nu$, $|s|=m$. Hence $a \cap e_{s'} \subset e_s \cap c^*_m$. QED.

(4) Let $a \cap e_s \cap c_m^* \neq 0$ s.t. $\max(s) < i$.
Then $a \cap e_s \cap c_m^* \in U_i(\bar{c} \restriction i)$ (Moreover $a \cap e_s \cap c_m^* \in \mathbb{B}_h$, $a \cap e_s \cap c_m^* \subset (\bar{c} \restriction h)^*$ for an $h < i$ s.t. $h \geq \max(s)$.

proof

Let $\mu = \max(s)$. Then $e_s \cap c_n^* = e_s \cap (c_n \restriction \mu)^* =$
$= e_s \cap (c_m \restriction \mu)^*$ for all $m \geq n$ by (a). Hence
$e_s \cap c_m^* = e_s \cap (\bar{c} \restriction \mu)^* = e_s \cap (c \restriction \mu)^*$ by the ind. hyp.
Let $a \in \mathbb{B}_h$, $a \subset (c \restriction h)^*$, $h < i$. Then
$a \cap e_s \cap c_m^* \subset (c \restriction i)^* \cap (c \restriction \mu)^* = (c \restriction j)^*$ where
$j = \max(\mu, h)$. Obviously $a \cap e_s \cap c_m^* \in \mathbb{B}_j$.

QED (4)

(5) There is s s.t. $\max(s) \geq i$ and
$a \cap e_s \cap c_{|s|}^* \neq 0$.

proof.

Let $n < \omega$ be big enough that $\xi_n^1 = \sigma(\xi_n) \geq \eta$
(hence $n > 0$). Set $a' = a \cap e_s \cap c_m^* \neq 0$ where
$|s| = m$. If $\max(s) \geq i$, we are done.
If not, let $\mu = \max(s) < i$. By (4) we have:
$a' \in U_i(\bar{c} \restriction i)$, $a' \in \mathbb{B}_j$, $a' \subset (c \restriction j)^*$ where
$\mu \leq j < i$. Note that

$$e_s \cap c_m^* \Vdash_\mu \dot{\sigma}_s(\dot{z}_s^1) \geq \check{\xi}_m^1 \geq \check{\eta}.$$

For each $\zeta < \bar{\lambda}$ set: $d_\zeta = e_s \cap c_m^* \cap [\![\dot{z}_s^1 = \check{\zeta}]\!]_{\mathbb{B}_\mu}$
Then $d_\zeta \in \mathbb{B}_\mu$ and $e \cap c_m^* = \bigcup_\zeta d_\zeta$.
Hence $\tilde{a} = a' \cap d_\zeta \neq 0$ for some $\zeta < \bar{\lambda}$.
Hence $\tilde{a} \in \mathbb{B}_j$, $\tilde{a} \subset (\bar{c} \restriction j)^*$. Hence
$\tilde{a} \in U_i(\bar{c} \restriction i)$.

Now let $\zeta < \bar{\xi}_p$, where $n < p$. Let $\tilde{a} \wedge e_s \wedge c^*_{p+1} \neq 0$, where $|s'| = p+1$.

<u>Claim</u> $\max(s') \geq \eta$

proof.
Let $\tilde{a} \wedge e_{s'} \wedge c^*_{p+1} \in G$, where G is $\mathbb{B}_{\tilde{\lambda}}$-generic.
Then $e_s \wedge c^*_n \in G$. Hence $z^1_n = \zeta$, where $z^1_n = \dot{z}^1_n{}^G$, and $\sigma_n(\zeta) \geq \xi^1_n \geq \eta$, where $\sigma_n = \dot{\sigma}_n^G$. But $e_s \wedge e_{s'} \neq 0$; hence $s = s' \upharpoonright n$.
Hence $\sigma_p(z^1_n) = \sigma_n(z^1_n)$ and
$\eta \leq \sigma_p(\zeta) \leq \sigma_p(\bar{\xi}_p) = \max(s')$. QED(5)

We now prove the Claim by induction on $i \leq \tilde{\lambda}$.
<u>Case 1</u> $i = 0$. Trivial since $\bar{c} \upharpoonright i = \emptyset$
<u>Case 2</u> $i = j+1$ (hence $i < \tilde{\lambda}$).
Then $(c \upharpoonright j)^* = (\bar{c} \upharpoonright j)^*$ and $R(c \upharpoonright j, \mathbb{B} \upharpoonright \tilde{\lambda})$.
We must show:
<u>Claim</u> $h_j(\bar{c}(j)) \supset (\bar{c} \upharpoonright j)^*$.
(Hence $(c \upharpoonright i)^* = (\bar{c} \upharpoonright i)^*$, $h_j(c(j)) = 1$ and hence $R(c \upharpoonright i, \mathbb{B} \upharpoonright \tilde{\lambda})$)
Suppose not. Let $a = (\bar{c} \upharpoonright j)^* \setminus h_j(\bar{c}(j))$.
Then $a \in U_i(\bar{c} \upharpoonright i)$, since $a \in \mathbb{B}_j$, $a \subset (\bar{c} \upharpoonright j)^*$.
By (5) we can find $n < \omega$, $s \in S_n$ s.t. $\max(s) \geq i$ and $e_s \wedge c^*_n \wedge a \neq 0$.
Choose n minimal for this property.

Then $s = \bar{s}\nu$, where $\max(\bar{s}) \leq j$. Hence $e_s \in \mathbb{B}_j$. By (a) we have

$$e_s \cap c_n(h) = e_s \cap c_m(h) \text{ for } m \geq n,\ h \leq j.$$

Hence $e_s \cap a \cap \bar{c}(j) \neq 0$. But $h_j(e_s \cap a \cap \bar{c}(j)) = (e_s \cap a) \cap h_j(\bar{c}(j)) = 0$

Contr! QED (Case 2)

<u>Case 3</u> $i = \eta < \tilde{\lambda}$, $\lim(\eta)$.

By the ind. hypothesis (a) holds below η.

Hence $GS(c \upharpoonright \eta, \mathbb{B} \upharpoonright \eta)$.

<u>Claim</u> $R(c \upharpoonright \eta, \mathbb{B} \upharpoonright \eta)$

(Hence $R(c \upharpoonright \eta, \mathbb{B})$ since $\mathbb{B}$ is an RCS iteration.)

Let $a \in U_\eta(c \upharpoonright \eta)$, $a \in \mathbb{B}_i \setminus \{0\}$, $a \subset (c \upharpoonright i)^*$ where $i < \eta$. We must find $a' \subset a$ s.t. $a' \in S_\eta(c \upharpoonright \eta)$.

<u>Case 3.1</u> There is $a' \subset a$ s.t. $a' \in U_\eta(c \upharpoonright \eta)$ and $\forall j < \eta\ a' \Vdash_j cf(\check{\lambda}) = \omega$.

Then trivially $a' \in S_\eta(c \upharpoonright \eta)$.

<u>Case 3.2</u> Case 3.1 fails.

Let $s \in S_n$ s.t. $\max(s) \geq \eta$ and $e_s \cap c_n \cap a \neq 0$, with n chosen minimally. Then $s = \bar{s}\nu$, with $\nu \geq \eta$ and $\max(\bar{s}) < \eta$. Set:

$a' = a \cap e_{\bar{s}} \cap c^*_{n-1}$. Then $a \in U_\eta(c \upharpoonright \eta)$ with $a' \subset (c \upharpoonright j)^*$, $a' \in \mathbb{B}_j$ for a $j \geq \max(\bar{s})$ by (6).

But $a \cap e_r \subset a'$ and $a \cap e_r \in \mathbb{B}_i$, since $e_r \in \mathbb{B}_{max(\bar{r})}$. Hence $a \cap e_r \in U_\eta(c \restriction \eta)$. But then $a \cap e_r \in U_\eta(c_m \restriction \eta)$. Hence there is $\tilde{a} \subset a$ s.t. $\tilde{a} \in S_\eta(c_m \restriction \eta)$. Hence $\tilde{a} \subset (c_m \restriction i)^*$ for all $i < \eta$, since Case 3.1 fails. Moreover $\tilde{a} \in \mathbb{B}_i$ for an $i < \eta$. By (a), $e_r \cap (c_m \restriction i)^* = e_r \cap (c_m \restriction i)^*$ for all $m \geq |r|$, $i < \eta$, since $max(u) \geq \eta$. Hence $\tilde{a} \subset (c \restriction i)^*$ for $i < \eta$, since $a \subset e_r$. Hence $\tilde{a} \in S_\eta(c \restriction \eta)$. QED (Case 3.2)

<u>Case 4</u> $i = \tilde{\lambda}$.

Then $R(c \restriction j, \mathbb{B} \restriction \tilde{\lambda})$ for $j < \tilde{\lambda}$ and $cf(\tilde{\lambda}) = \omega$ in V. The conclusion is trivial.

QED (<u>Claim</u>)

Since $R(c, \mathbb{B} \restriction \tilde{\lambda})$ and $\mathbb{B}$ is an RCS iteration, we conclude: $R(c, \mathbb{B})$. (In particular, $R(c, \mathbb{B} \restriction \lambda)$!)

We now show that c has the desired properties. Let G be $\mathbb{B}_\lambda$-generic with $c^* \in G$. We claim that there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \bar{N} \prec N$

(b) $\sigma_0(\bar{\theta}, \bar{\mathbb{B}}, \bar{s}, \bar{\lambda}_i) = \theta, \mathbb{B}, s, \lambda_i$ $(i = 1, \ldots, m)$

(c) $\sup \sigma_0 '' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, \ldots, m)$

(d) $\bar{a} \in g = \sigma_0^{-1} '' G$ and g is $\bar{\mathbb{B}}$-generic over $\bar{N}$.

We know that $c_n^* \in G$ for $n < \omega$. Since $c_n^* = \bigcup_{|s| = n+1} e_s$, we know: For each n there is exactly one s_n s.t. $|s_n| = n$ and $e_{s_n} \in G$. Set $s = \bigcup_n s_n = \langle \tilde{\xi}_i \mid i < \omega \rangle$. Since $c_n^* \cap e_{s_n} \Vdash \dot{\sigma}_{s_n} : \check{\bar{N}}[\dot{g}_{s_n}] \prec \check{N}[\dot{G}_n]$, we have: $\sigma_n : \bar{N}[g_n] \prec N[G_n]$, where $g_n = \dot{g}_n^G = \sigma_n^{-1} '' G_n$, $G_n = \dot{G}_n^G = G \cap \mathbb{B}_{\tilde{\xi}_n}$. Clearly $\sigma_n(\bar{\xi}_i) = \tilde{\xi}_i$ for $i \leq n$, and $\sigma_n(x_i) = \sigma_i(x_i)$ for $i \leq n$. and $\sigma_n(g_n) = G_n$. Hence we can define $\tilde{\sigma} : \bar{N} \prec N$ exactly as before and get:

$\sup \tilde{\sigma} '' \lambda_l = \tilde{\lambda}_l$ $(l = 0, \ldots, m)$

exactly as before. Finally, note that $e_{s_n} \cap c_n^* \Vdash (b_n \upharpoonright \bar{\xi}_n) \in \dot{g}_{s_n}$; hence $b_n \in g$, whence follows — exactly as before — that g is $\bar{\mathbb{B}}_{\bar{\lambda}}$-generic over $\bar{N}$. But $b_n \subset \bar{a}$, Hence $\bar{a} \in g$. QED

We have thus shown that $\mathbb{B}_\lambda$ is subproper. All that remains is to define $c_n, e_s, \dot{\sigma}_s$ etc. and verify (a) – (e). By recursion on n we define

$$\Gamma_n = \langle c_n, \langle e_s \mid |s| = n\rangle, \langle \dot{\sigma}_s \mid |s| = n \wedge e_s \neq 0\rangle,$$

$$\langle \dot{u}_s \mid |s| = n \wedge e_s \neq 0\rangle, \langle \dot{g}_s \mid |s| = n \wedge e_s \neq 0\rangle\rangle$$

and verify (a) – (e) (e.g. (e) will then be verified for $|s\nu| \leq n$). Γ_n is defined by (a), (b) for $n = 0$. The verifications are trivial. Now let Γ_n be given s.t. (a) – (e) hold. Before proceeding further, we note that by the disjoint distributive law used in the construction of $\langle b_n \mid n < \omega\rangle$, the following holds in all complete BA's:

(1) Let $b = \bigcup_{i \in I} b_i$, $b_i \cap b_j = 0$ for $i \neq j$, and $a_i \subset b_i$. Then $\bigcap_j (a_j \cup (b \setminus b_j)) = \bigcup_j a_j$.

pf.

Set $a_i^j = \begin{cases} a_i & \text{if } i = j \\ b_i & \text{if } i \neq j \end{cases}$. Then

$$\bigcap_j (a_j \cup (b \setminus b_j)) = \bigcap_j \bigcup_i a_i^j = \bigcup_i \bigcap_j a_i^j = \bigcup_i a_i.$$

QED (1)

We now define e_s for $s = n+1$:

(2) $e_{s\nu} =_{\text{Df}} c_n^* \cap e_s \cap [\![\check{\nu} = \dot{\sigma}_s(\check{\xi}_{n+1})]\!]_{\mathbb{B}_{\max(s)}}$

for $|s| = n$.