§3 Examples

§3.1 The transfer lemma for embeddings of ZFC$^-$ models

We recapitulate and expand upon some facts developed in [J] §5. *)

Def Let $M = \langle |M|, \in, \ldots \rangle$ be a transitive ZFC$^-$ model. Let $\pi : M \prec M'$, where M' is transitive. π is *cofinal* in M' iff $M' = \bigcup_{u \in M} \pi(u)$.

In the following, we suppose $M, N, \ldots$ to be transitive ZFC$^-$ models unless otherwise stated.

Fact 1 Let $\pi : M \prec M'$ and set $\tilde{M} = M' | \bigcup_{u \in M} \pi(u)$. Then $\tilde{M} \prec M'$ and $\pi : M \prec \tilde{M}$ cofinally.

(The proof uses: Let $x_1, \ldots, x_n \in \tilde{M}$, $x_i \in \pi(u_i)$. Then $\tilde{M} \models \varphi(x_1, \ldots, x_n) \leftrightarrow \langle x_1, \ldots, x_n \rangle \in X$, where $X = \{\langle \vec{z} \rangle \in u_1 \times \ldots \times u_n \mid M \models \varphi(\vec{z})\}$.)

Hence:

Fact 2 Let $\tau > \omega$ be regular in M, where $\pi : M \prec M'$. Set $\bar{H} = H_\tau^M$, $\tilde{H} = \bigcup_{u \in \bar{H}} \pi(u)$, $\bar{\pi} = \pi \upharpoonright \bar{H}$. Then $\bar{\pi} : \bar{H} \prec \tilde{H}$ cofinally.

Def Let $\pi : M \prec M'$. Let τ be regular in M. π is τ-*cofinal* iff
$$M' = \bigcup \{\pi(u) \mid u \in M \wedge \bar{\bar{u}} < \tau \text{ in } M\}$$
(Hence τ-cofinality implies cofinality.)

*) In ZFC$^-$ "the axiom of choice reads": Every set is enumerable by an ordinal.

Def Let $\tau > \omega$ be regular in M, $\bar{H} = H_\tau^M$. Let $\bar{\pi} : \bar{H} \prec H$ cofinally. By a liftup of $\langle M, \bar{\pi} \rangle$ we mean a pair $\langle M', \pi \rangle$ s.t. M' is transitive, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : M \prec M'$ τ-cofinally.

(We also say: "$\langle M', \pi \rangle$ is a liftup of M by $\bar{\pi}$".)

Fact 3 Let $\langle M, \bar{\pi} \rangle$ be as above. There is at most one liftup $\langle M', \pi \rangle$.

proof:

Clearly, every element of M' has the form $\pi(f)(x)$, where $f \in M$, $f : u \to M$ for a $u \in \bar{H}$, and $x \in \bar{\pi}(u)$. But

$M' \models \varphi(\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)) \longleftrightarrow$

$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(X)$, where

$X = \{\langle z_1, \ldots, z_n \rangle \mid \varphi(f_1(z_1), \ldots, f_n(z_n))\}$

(hence $X \in \bar{H}$).

This means that if $\langle M'', \pi' \rangle$ is a second liftup, we can define $\sigma : M' \leftrightarrow M''$ by $\sigma(\pi(f)(x)) = \pi'(f)(x)$. Hence $\sigma = \mathrm{id}$, $M' = M''$. But $\pi(z) = \pi(\mathrm{const}_z(0)) =$ $\sigma(\pi(\mathrm{const}_z(0))) = \pi'(\mathrm{const}_z(0)) = \pi'(z)$ for $z \in M$; where $\mathrm{const}_z = \{\langle z, 0 \rangle\}$. QED (Fact 3)

<u>Note</u> By this analysis it follows easily that, if $\langle M', \pi\rangle$ is the liftup of M by $\pi : \bar{H} \prec H$, where $\bar{H} = H_{\tau}^{M}$, and $\tau' = On \cap H$, then $\pi(\tau) = \tau'$ and $H = H_{\tau'}^{M'}$. $\bar{H}$ need not be an element of M, but if it is, it follows that $\pi(\bar{H}) = H$.

The proof of Fact 3 suggests a general method of constructing the liftup:

<u>Def</u> Let M be a transitive ZFC$^-$ model with predicates $A_1, \ldots, A_n$. Let $\bar{H} = H_{\bar{\tau}}^{M}$, where $\bar{\tau}$ is regular in M, and let $\pi : \bar{H} \prec H$ cofinally.

$$\mathbb{D} = \mathbb{D}_{M,\pi} = \langle D, E, I, \tilde{A}_1, \ldots, \tilde{A}_n \rangle$$

is defined by:

$$D = \{\langle x, f\rangle \mid f \in M \wedge f : u \to M \text{ for a } u \in \bar{H} \wedge x \in \pi(u)\}$$

$$\langle x, f\rangle E \langle y, g\rangle \longleftrightarrow \langle x, y\rangle \in \pi(\{\langle z, w\rangle \mid f(z) \in g(w)\})$$

$$\langle x, f\rangle I \langle y, g\rangle \longleftrightarrow \langle x, y\rangle \in \pi(\{\langle z, w\rangle \mid f(z) = g(w)\})$$

$$\tilde{A}_i(\langle x, f\rangle) \longleftrightarrow x \in \pi\{z \mid A_i f(z)\}.$$

We then get <u>Łoś Theorem</u> in the form

$$\mathbb{D} \models \varphi(\langle x_1, f_1\rangle, \ldots, \langle x_n, f_n\rangle) \longleftrightarrow$$

$$\longleftrightarrow \langle x_1, \ldots, x_n\rangle \in \pi(\{\langle z_1, \ldots, z_n\rangle \mid M \models \varphi(f_1(z_1), \ldots, f_n(z_n))\})$$

The proof is by induction on φ and is just like the proof of Łoś theorem for ultrapowers. Then $\mathbb{D} \models ZFC^-$ and $\mathbb{D}$ is an equality model with equality relation I.

This gives:

<u>Fact 4</u> The liftup of $\langle \bar{M}, \bar{\pi} \rangle$ exists iff iff E is well founded.

prf. (sketch)

($\rightarrow$) If $\langle M', \pi \rangle$ is the liftup, then $a E b \longleftrightarrow k(a) \in k(b)$ for $a, b \in \mathbb{D}$, where k is defined by $k(\langle x, f \rangle) = \pi(f)(x)$.

($\leftarrow$) Factor $\mathbb{D}$ by I to get $\mathbb{D}^* = \mathbb{D}/I$. Let $[u]$ be the equivalence class of u for $u \in \mathbb{D}$. Then $\mathbb{D}^*$ satisfies extensionality and has a well founded $\in$ - relation. Hence there is $\sigma^* : \mathbb{D}^* \overset{\sim}{\longleftrightarrow} M'$, where M' is transitive by Mostowski's isomorphism theorem. Set: $\sigma(u) = \sigma^*([u])$ for $u \in \mathbb{D}$. We can define $\pi : M \prec M'$ by $\pi(x) = \sigma(\langle 0, \mathrm{const}_x \rangle)$, where $\mathrm{const}_x = \{\langle x, 0 \rangle\}$ = the constant function x on $\{0\}$. Set: $\tilde{\mathbb{D}} = \{\langle x, f \rangle \in \mathbb{D} \mid f \in \bar{H}\}$; $H' = \{\sigma(u) \mid u \in \tilde{\mathbb{D}}\}$.

H' is easily seen to be transitive.

But $H = \{\bar{\pi}(f)(x) \mid \langle x, f \rangle \in \tilde{\mathbb{D}}\}$

Moreover:

$$\bar{\pi}(f)(x) \in \bar{\pi}(g)(y) \longleftrightarrow \langle x,y\rangle \in \bar{\pi}(\{\langle z,w\rangle \mid f(x)=g(y)\})$$
$$\longleftrightarrow \sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle)$$

for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. Hence there is an isomorphism $i' : H \overset{\sim}{\longleftrightarrow} H'$ defined by $i'(\bar{\pi}(f)(x)) = \sigma(\langle x,f\rangle)$. Hence $i' = \mathrm{id}$, $H = H'$ and $\sigma(\langle x,f\rangle) = \bar{\pi}(f)(x)$ for $\langle x,f\rangle \in \tilde{\mathbb{D}}$. In particular,

$$\pi(z) = \sigma(\langle 0, \mathrm{const}_z\rangle) = \bar{\pi}(\mathrm{const}_z)(0) =$$
$$= \mathrm{const}_{\bar{\pi}(z)}(0) = \bar{\pi}(z) \text{ for } z \in \bar{H}$$

Hence $\pi \upharpoonright \bar{H} = \bar{\pi}$. But then for $\langle x,f\rangle, \langle y,g\rangle \in \mathbb{D}$, we have:

$$\sigma(\langle x,f\rangle) \in \sigma(\langle y,g\rangle) \longleftrightarrow \langle x,y\rangle \in \bar{\pi}(\{\langle z,w\rangle \mid f(x) \in g(w)\})$$
$$\longleftrightarrow \pi(f)(x) \in \pi(g)(y).$$

Hence there is $i : M' \overset{\sim}{\longleftrightarrow} M'' \subset M''$ defined by $i(\sigma(\langle x,f\rangle)) = \pi(f)(x)$. M'' is easily seen to be transitive, however, so $i' = \mathrm{id}$ and each $z \in M'$ has the form $\pi(f)(x)$, where $\langle x,f\rangle \in \mathbb{D}$. It follows easily that $\langle M', \pi\rangle$ is the liftup of $\langle M, \bar{\pi}\rangle$. QED (Fact 4)

This gives us the <u>interpolation lemma</u>:

<u>Fact 5</u> Let $\pi' : \bar{M} \prec M'$. Let $\bar{\tau} \in \bar{M}$ be regular in $\bar{M}$ and set $\bar{H} = H^{\bar{M}}_{\bar{\tau}}$. Let $\pi : \bar{H} \prec H$ cofinally. Then:

(a) The liftup $\langle M, \pi \rangle$ of $\langle \bar{M}, \bar{\pi} \rangle$ exists

(b) There is a unique $\sigma : M \prec M'$ s.t. $\sigma\pi = \pi'$ and $\sigma \upharpoonright H = \mathrm{id}$.

prf.

To prove (a) we note that E is well founded, since $\langle x, f \rangle E \langle y, g \rangle \longleftrightarrow \pi'(f)(x) \in \pi'(g)(y)$. But for $\langle x_1, f_1 \rangle, \ldots, \langle x_n, f_n \rangle \in \mathbb{D}$ we have:

$M \models \varphi(\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)) \longleftrightarrow$

$\longleftrightarrow M' \models \varphi(\pi'(f_1)(x_1), \ldots, \pi'(f_n)(x_n))$

$\longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \bar{\pi}(\{\vec{z} \mid \bar{M} \models \varphi(f_1(z_1), \ldots, f_n(z_n))\})$.

Hence there is $\sigma : M \prec M'$ defined by $\sigma(\pi(f)(x)) = \pi'(f)(x)$ for $\langle x, f \rangle \in \mathbb{D}$. But this σ is characterized by the above conditions. QED (Fact 5)

The structure $\mathbb{D}^*$ will be of interest to us, however, even if it is ill founded. An embedding $\tilde{\pi} : M \prec \mathbb{D}^*$ is definable by $\tilde{\pi}(x) = [\langle 0, \mathrm{const}_x \rangle]$. This embedding is cofinal in the sense that for every $z \in \mathbb{D}^*$ there is $u \in M$ s.t. $\mathbb{D}^* \models z \in \tilde{\pi}(u)$.

In dealing with ill founded models of set theory it is useful to work with <u>solid</u> structures in the following sense:

<u>Def</u> Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, \ldots \rangle$ model the extensionality axiom. $\mathfrak{A}$ is <u>solid</u> iff the well founded core $wfc(\mathfrak{A})$ is transitive and $\in_{\mathfrak{A}} \cap wfc(\mathfrak{A})^2 = \in \cap wfc(\mathfrak{A})^2$.
($wfc(\mathfrak{A})$ is the set of $x \in \mathfrak{A}$ s.t. $\in_{\mathfrak{A}} \cap X^2$ is well founded, where X is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$).

Clearly, every model is isomorphic to a solid model.

We note the following facts about solid models of ZFC^-:

<u>Fact 6</u> Let $\mathfrak{A}$ be a solid model of ZFC^-. Let $H = wfc(\mathfrak{A})$. Then

(a) $\omega \subset H$; $\alpha \in H \to \alpha + 1 \in H$

(b) If $\alpha \in H$, $x \in \mathfrak{A}$, and $\mathfrak{A} \models \operatorname{rn}(x) \leq \alpha$, then $x \in H$

(c) H is admissible

prf.
(a), (b) are trivial. We prove (c).

(<u>Note</u> We take the replacement axiom of ZFC -
as reading:
$\bigwedge x \bigvee y\, \varphi(x, y, \vec{z}) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y, \vec{z})$
for arbitrary formulae φ. The theory KP
("Kripke - Platek set theory") is obtained
by restricting the fmla φ in this schema
- and in the separation schema - to
Σ_0 formulae. A transitive structure is called
<u>admissible</u> iff it satisfies KP.)
By (b), H is easily seen to satisfy Σ_0 -
separation, as well as the trivial
existence axioms: "$\emptyset$ is a set", "$\{x, y\}$ is a set",
"$\bigcup x$ is a set". We prove Σ_0 replacement,
Let $H \models \bigwedge x \bigvee y\, \varphi(x, y, \vec{z})$. Let $u \in H$.
Let $R(x,y)$ mean: "$\varphi(x, y, \vec{z})$ and y is
of minimal rank." Then there is
$v \in \mathfrak{A}$ s.t. $\mathfrak{A} \models \bigwedge x \in u \bigvee y \in v\, R(x,y)$.
But if we take v as being of minimal
rank in $\mathfrak{A}$, it must have rank
$\in H$. Hence $v \in H$. QED (Fact 6)

<u>Note</u> It follows that if $u \in H$ is transitive
and $\delta = On \cap H$, then $L_\delta(u)$ is
admissible.

We now extend some of our definitions to solid models of ZFC$^-$.

Def Let $\mathfrak{A}$ be a solid model of ZFC$^-$; let $\tau \in \mathrm{wfc}(\mathfrak{A})$ be regular in $\mathfrak{A}$ and let $\bar{H} = H_\tau^{\mathfrak{A}}$ (hence $\bar{H} \subset \mathrm{wfc}(\mathfrak{A})$). Let $\bar{\pi} : \bar{H} \prec H$ cofinally, where H is transitive. $\langle \mathfrak{A}', \pi \rangle$ is a <u>liftup</u> of $\langle \mathfrak{A}, \bar{\pi} \rangle$ iff $\mathfrak{A}'$ is solid, $\pi \upharpoonright \bar{H} = \bar{\pi}$, and $\pi : \mathfrak{A} \prec \mathfrak{A}'$ is τ-cofinal (i.e. for each $x \in \mathfrak{A}'$ there is $u \in \mathfrak{A}$ s.t. $\bar{\bar{u}} < \tau$ in $\mathfrak{A}$ and $\mathfrak{A}' \models x \in \pi(u)$).

A virtual repetition of the proof of Fact 3 gives:

<u>Fact 7</u> Let $\langle \mathfrak{A}, \bar{\pi} \rangle$ be as above. Up to isomorphism there is at most one liftup $\langle \mathfrak{A}', \pi \rangle$.

<u>Note</u> As before, $\pi(\tau) = \sup\{\bar{\pi}(\nu) \mid \nu < \tau\}$ in $\mathfrak{A}'$; hence $\pi(\tau) \in \mathrm{wfc}(\mathfrak{A}')$ and $H = H^{\mathfrak{A}'}_{\pi(\tau)} \subset \mathrm{wfc}(\mathfrak{A}')$. If $\bar{H} \in \mathrm{wfc}(\mathfrak{A})$, then $H = \pi(\bar{H}) \in \mathrm{wfc}(\mathfrak{A}')$. But we can then form $\mathbb{D}$ as before [taking $\langle x, f \rangle \in \mathbb{D}$ iff ($\mathfrak{A} \models f : u \to V$) for a $u \in \bar{H}$ and $x \in \bar{\pi}(u)$)].

Repeating the proof of Fact 4 we get:

<u>Fact 8</u> Let $\langle \mathfrak{A}, \pi \rangle$ be as above. Then the liftup exists.

(<u>Note</u> The liftup $\langle \mathfrak{A}', \pi \rangle$ is unique only up to isomorphism. But then $wfc(\mathfrak{A}')$ is unique, by solidity.)

We now weaken our earlier definition of fullness to:

<u>Def</u> Let N be a transitive ZFC$^-$ model s.t. $N = \langle L_\tau[A], \in, A, \ldots \rangle$. N is <u>almost full</u> iff there is a solid model $\mathfrak{A}$ of ZFC$^-$ s.t. $N \in wfc(\mathfrak{A})$, N is regular in $\mathfrak{A}$, and $\mathfrak{A} \models V = L(N)$.

Then by the above we have:

<u>Fact 9</u> Let $N = \langle L_\tau[A], \in, A, \ldots \rangle$ be almost full. Let $\pi : N \prec N'$ cofinally. Then N' is almost full. (Moreover, if $\mathfrak{A}$ verifies the almost fullness of N and $\langle \mathfrak{A}', \pi' \rangle$ is the liftup of $\langle \mathfrak{A}, \pi \rangle$, then $\mathfrak{A}'$ verifies the almost fulness of N'.)

By Fact 6:

<u>Fact 10</u> Let N be almost full. There is δ s.t. $L_\delta(N)$ is admissible and N is regular in $L_\delta(N)$.

<u>Def</u> δ_N = the least δ s.t. $L_\delta(N)$ is admissible.

A major tool will be the following <u>transfer lemma</u>:

<u>Fact 11</u> Let $\bar{N}$ be almost full. Let $\pi : \bar{N} \prec N$ cofinally. Let $x_1, \ldots, x_n \in \bar{N}$ and let φ be a Π_1 formula. Then

$$L_{\delta_{\bar{N}}}(\bar{N}) \models \varphi(\bar{N}, \vec{x}) \rightarrow L_{\delta_N}(N) \models \varphi(N, \pi(\vec{x})).$$

proof.

Let $\bar{\Omega}$ witness the almost fulness of $\bar{N}$ and let $\pi' : \bar{\Omega} \prec \Omega$ be the liftup of $\langle \bar{\Omega}, \pi \rangle$. Obviously:

(1) $a \notin wfc(\bar{\Omega}) \rightarrow \pi'(a) \notin wfc(\Omega)$

(2) $L_{\delta_{\bar{N}}}(\bar{N}) \subset wfc(\bar{\Omega})$, $L_\delta(N) \subset wfc(\Omega)$ by Fact 6

Suppose not. Then there is a least $\alpha < \delta_N$ s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$. Since $L_{\delta_N}(N)$ is an initial segment of Ω, we have:

(3) $\Omega \models \alpha$ is least s.t. $L_\alpha(N) \models \neg\varphi(N, \pi(\vec{x}))$

(4) $\Omega \models \bigwedge \nu \leq \alpha \; L_\nu(N)$ is not admissible.

But then $\alpha = \pi'(\bar{\alpha})$, where in $\bar{\mathfrak{U}}$:
(5) $\bar{\alpha}$ is least s.t. $L_{\bar{\alpha}}(\bar{N}) \models \neg\varphi(N, \bar{x}^0)$
(6) $\bigwedge \nu \leq \bar{\alpha}$ $L_\nu(\bar{N})$ is not admissible.
But $\bar{\alpha} \in \mathrm{wfc}(\bar{\mathfrak{U}})$ by (1). Hence
$\bar{\alpha} < \beta = \mathrm{On} \cap \mathrm{wfc}(\bar{\mathfrak{U}})$ and $L_\beta(\bar{N})$ is admissible. Thus (5), (6) hold in $L_\beta(\bar{N})$, since $L_\beta(\bar{N})$ is an initial segment of $\bar{\mathfrak{U}}$. Hence (5), (6) hold outright and $\alpha < \delta_{\bar{N}}$. Contr!

QED (Fact 11)

<u>Note</u> Fact 11 is actually a special case of a more general theorem: If $\bar{N} = \langle L_\tau[A], \in, A, \ldots \rangle$ is a ZFC$^-$ model, $\pi : \bar{N} \prec N$ cofinally, and N is regular in $L_{\delta_N}(N)$, then the conclusion of Fact 11 holds (even if $\bar{N}$ is not regular in $L_{\delta_{\bar{N}}}(\bar{N})$).
We shall not need this, however, and do not prove it here, since our proof involves a modest application of fine structure theory.

§ 3.2 Barwise Theory

In addition to the transfer lemma we shall make use of Barwise' theory of infinitary language on admissible structures. In the following let M be an admissible structure satisfying choice in the form: Every set is enumerable by an ordinal. In admissibility theory the basic three notions of recursion theory are redefined as follows:

M-recursive $= \underline{\Delta}_1(M)$

M-recursively enumerable $= \underline{\Sigma}_1(M)$

M-finite $=$ element of M.

Barwise then developed an extension of first order logic involving formulae which are infinitely long but still M-finite. Thus a Barwise language on M is like predicate logic except that, whenever $\langle \varphi_i \mid i \in x \rangle \in M$ is a sequence of formulae, then $\bigwedge_{i \in x} \varphi_i$ and $\bigvee_{i \in x} \varphi_i$ are formulae. (Infinite blocks of quantifiers are not allowed, however.) The set of variables is M-infinite (i.e. we could have a variable v_ξ for each $\xi \in On \cap M$). A language is then specified by fixing its predicates, constants, and function symbols.

The syntax is developed internally in such a way that the basic syntactical notions (e.g. "formula", "term", "sentence") are $\Delta_1(M)$. A <u>mathematical theory</u> $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ then consists of a language $\mathcal{L}_0$ and a set $\mathcal{L}_1$ of axioms (all of which are sentences). $\mathcal{L}_1$ should be $\Sigma_1(M)$, if we wish to make use of the admissibility of M. We augment the usual predicate logical rules of inference by two infinitary rules:

$$\frac{\psi \to \varphi_i \quad (i \in x)}{\psi \to \bigwedge\!\!\!\bigwedge_{i \in x} \varphi_i} \qquad\qquad \frac{\varphi_i \to \psi \quad (i \in x)}{\bigvee\!\!\!\bigvee_{i \in x} \varphi_i \to \psi}$$

for $\langle \varphi_i \mid i \in x \rangle \in M$.

A <u>proof</u> is then a (possibly infinite) sequence of formulae, each of which is an axiom or follows from the previous formulae by a rule of inference. <u>If</u> the axiom set $\mathcal{L}_1$ is $\Sigma_1(M)$, it turns out that every provable formula has a proof p which is M-finite (i.e. $p \in M$). From this we get the

M-finiteness lemma: If φ is provable in $\mathcal{L}$, then it is provable from an M-finite $u \subset \mathcal{L}_1$.

A model $\mathfrak{A}$ of the language $\mathcal{L}_0$ is described by fixing its domain of individuals $|\mathfrak{A}|$ and the interpretation $s^{\mathfrak{A}}$ of each predicate symbol, constant, or function symbols, just as in finitary predicate logic. We can then straightforwardly define truth ($\mathfrak{A} \models \varphi$) for $\mathcal{L}_0$-sentences φ and satisfaction ($\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$) for $\mathcal{L}_0$-formulae containing only finitely many free variables. We say that $\mathfrak{A}$ models the theory $\mathcal{L} = \langle \mathcal{L}_0, \mathcal{L}_1 \rangle$ iff all axioms in $\mathcal{L}_1$ are true in $\mathfrak{A}$. The notion of proof is correct in the sense that, if $\mathfrak{A}$ models $\mathcal{L}$, then sentence provable in $\mathcal{L}$ is true in $\mathfrak{A}$.

The final stone in this mosaic is the completeness theorem for countable M: If M is countable, then $\mathcal{L}$ is consistent iff $\mathcal{L}$ has a model.

This means that for any admissible M, we can make the completeness theorem true in a generic extension of V simply by collapsing M to ω. In many cases we can then use this to prove properties of V.

We note that if $\mathcal{L}_1$ is $\Sigma_1(M)$ in parameters $\vec{p}$, then the statement "$\mathcal{L}$ is consistent" is uniformly $\Pi_1(M)$ in $\vec{p}$, since it says that M contains no proof of a contradiction. (But by the foregoing, "$\mathcal{L}$ is consistent" is equivalent to:
$\Vdash_{\mathbb{P}}$ "$\mathcal{L}$ has a model", where $\mathbb{P}$ is any set of conditions which collapses M to ω.) It is in this context that we will apply the transfer lemma: Let N be almost full and let $\pi : N \prec N'$ cofinally. Let $\mathcal{L}$ be a theory on $L_{\delta_N}(N)$ s.t. the set of axioms $\mathcal{L}_1$ is $\Sigma_1(N)$ in parameters N and $\vec{p} \in N$. Let $\mathcal{L}'$ have the same definition in N', $\pi(\vec{p})$ over $L_{\delta_{N'}}(N')$. Then:
$\mathcal{L}$ is consistent $\longrightarrow$ $\mathcal{L}'$ is consistent.

In this paper we shall deal only with languages $\mathcal{L}$ on M which contain a binary "$\in$-predicate" $\dot{\in}$ and a designated constant $\underline{x}$ for each $x \in M$. (We suppose $\dot{\in}$ and $\langle \underline{x} \mid x \in M \rangle$ to have a uniform $\Delta_1(M)$ definition over any admissible M.) We also suppose that the set of axioms $\mathcal{L}_0$ contains a base theory consisting of:

- ZFC$^-$ (including the schemata of separation and replacement for all finite formulae of $\mathcal{L}_0$)
- the "defining" axioms for the constants $\underline{x}$ $(x \in M)$: $\bigwedge v \, (v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$.

We note that if $\mathfrak{A}$ is a solid model of $\mathcal{L}$, we then have $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$ for all $x \in M$.

§ 3.4 The forcing $\mathbb{P}_A$.

Let $A \subset \omega_2$ be a stationary set of points of cofinality ω. We define:

Def $\mathbb{P}_A$ = the set of $p : \alpha+1 \to A$ s.t. $\alpha < \omega_1$ and p is a normal function.

$p \leq q$ in $\mathbb{P}_A$ $\longleftrightarrow$ $p \supset q$.

Hence if G is $\mathbb{P}_A$-generic, $f = \bigcup G$ is a cofinal normal function $f : \omega_1 \to A$. It is easily established that f adds no reals.

Lemma 1 $\mathbb{P}_A$ is subcomplete.

proof.

Let $\mathbb{P}_A \in H_\theta$. Let $\tau > \theta$ be regular. Let $N = \langle L_\tau[A], A, \ldots \rangle$ where $H_\tau \subset N$. Let $\sigma : \bar{N} \prec N$ s.t. $\sigma(\bar{\theta}, \bar{\mathbb{P}}) = \theta, \mathbb{P}_A$ and $\bar{N}$ is countable and full.

Claim σ witnesses the subcompleteness of $\mathbb{P}_A$.

Let $\bar{\lambda}_0 = On \cap \bar{N}$, $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$ $(i = 1, \ldots, m)$ where $\mathbb{P}_A \in H_{\lambda_i}$ + $\lambda_i \in (\omega_1, \theta)$ is regular $(i = 1, \ldots, m)$.

Let $\tilde{\lambda}_i = \sup \sigma'' \bar{\lambda}_i$ $(i = 0, n, m)$

Let $\sigma(\bar{s}) = s$

Claim 1 There is $\sigma_0 : \bar{N} \prec N$ with

(a) $\sup \sigma_0 {}'' \omega_2^{\bar{N}} \in A$

(b) $\sigma_0(\bar{s}, \bar{\lambda}_i, \bar{\mathbb{P}}) = s, \lambda_i, \mathbb{P}$

(c) $\sup \sigma_0 {}'' \bar{\lambda}_i = \tilde{\lambda}_i$ $(i = 0, n, m)$

prf.

For $\alpha < \omega_2$ set X_α = the smallest $X \prec N$ s.t. $\alpha \cup \mathrm{rng}(\sigma) \subset X$. Set:

$C = \{\alpha < \omega_2 \mid \alpha = \omega_2 \cap X\}$. Then C is club in ω_2.

For $\alpha \in C$ set $\pi_\alpha : N_\alpha \xleftrightarrow{\sim} X_\alpha$. Then

(1) $\alpha = \mathrm{crit}(\pi_\alpha)$, $\pi_\alpha(\alpha) = \omega_2$

(2) Set $\sigma_\alpha = \pi_\alpha^{-1}\sigma$. Then

$\langle N_\alpha, \sigma_\alpha \rangle$ = the liftup of $\langle \bar{N}, \sigma_\alpha \upharpoonright H_{\omega_3}^{\bar{N}} \rangle$

prf.

Form $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar{N}, \sigma_\alpha \upharpoonright H_{\omega_3}^{\bar{N}} \rangle$.

Then there is $\pi' : N' \prec N_\alpha$ s.t. $\pi' \sigma' = \sigma_\alpha$ and $\pi' \upharpoonright H_{\omega_3}^{N'} = \mathrm{id}$. But then $\pi' \upharpoonright \alpha = \mathrm{id}$, since $\alpha < \omega_3^{N_\alpha}$. Hence

$\alpha \cup \mathrm{rng}(\sigma) \subset \mathrm{rng}(\pi')$. Hence

$\mathrm{rng}(\pi') = \mathrm{rng}(\pi_\alpha)$, $\pi' = \pi_\alpha$, QED (2)

Now let $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar{N}, \sigma \upharpoonright H_{\omega_2}^{\bar{N}} \rangle$. Since

$\pi_\alpha \restriction H_{\omega_2}^{N_\alpha} = \mathrm{id}$ and $\pi_\alpha \sigma_\alpha = \sigma$, we have $\sigma_\alpha \restriction H_{\omega_2}^{\bar N} = \sigma \restriction H_{\omega_2}^{\bar N}$. Hence:

(3) $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar N, \sigma_\alpha \restriction H_{\omega_2}^{\bar N} \rangle$.

Hence there is $\pi' : N' \prec N_\alpha$ s.t. $\pi' \sigma' = \sigma_\alpha$ and $\pi' \restriction H_{\omega_2}^{N'} = \mathrm{id}$.

(Note It is in fact easily seen that if $\alpha_0 = \min C$, then $\alpha_0 = \omega_2^{N'}$, $N' = N_{\alpha_0}$, and $\pi' = \pi_\alpha^{-1} \pi_{\alpha_0}$.)

Clearly $\pi' : N' \prec N_\alpha$ cofinally, since $\sigma' : \bar N \prec N'$ cofinally and $\sigma_\alpha : \bar N \prec N_\alpha$ cofinally.

Since $\sigma' : \bar N \prec N$ is the liftup of $\langle \bar N, \sigma \restriction H_{\omega_2}^{\bar N} \rangle$, we have:

(4) $\sigma'(\tau) = \sup \sigma'{}''\tau$ whenever $\tau \geq \omega_2^{\bar N}$ is regular in $\bar N$.

Similarly:

(5) $\sigma_\alpha(\tau) = \sup \sigma_\alpha{}''\tau$ whenever $\tau \geq \omega_3^{\bar N}$ is regular in $\bar N$.

Now let $\delta' = \delta_{N'}$. Let $\mathcal{L}'$ be the language on $L_{\delta'}(N')$ containing the base theory and with a new constant $\dot\sigma$ and the axioms:

- $\dot{\sigma} : \underline{\bar{N}} \prec \underline{N}'$ cofinally
- $\dot{\sigma}(\bar{\alpha}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \underline{\sigma'(\bar{\alpha})}, \underline{\sigma'(\bar{\mathbb{P}})}, \underline{\sigma'(\bar{\lambda}_i)}$ $(i = 1, \ldots, m)$
- $\sup \dot{\sigma}''\tau = \dot{\sigma}(\tau)$ whenever τ is regular in $\bar{N}$.

$\mathcal{L}'$ is consistent, since it is modeled by $\langle H_{\omega_1}, \sigma' \rangle$. Moreover the theory $\mathcal{L}'$ is $\Sigma_1(L_{\delta'}(N'))$ in the parameters

$N', \bar{N}, \bar{\alpha}, \bar{\lambda}_i, \sigma'(\bar{\alpha}), \sigma'(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$.

Now let $\delta_\alpha = \delta_{N_\alpha}$ and let $\mathcal{L}^\alpha$ be $\Sigma_1(L_{\delta_\alpha}(N_\alpha))$ by the same definition in the parameters:

$N_\alpha, \bar{N}, \bar{\alpha}, \bar{\lambda}_i, \sigma_\alpha(\bar{\alpha}), \sigma'(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$

Then $\mathcal{L}^\alpha$ is consistent by the transfer lemma, since $\pi' : N' \prec N_\alpha$ is cofinal and $\pi'\sigma' = \sigma_\alpha$ and $\pi' \upharpoonright H_{\omega_1} = \mathrm{id}$.

By this we get:

(6) Let $\mathrm{cf}(\alpha) = \omega$. Then in V there is a map σ_1 s.t.

- $\sigma_1 : \bar{N} \prec N'$ cofinally
- $\sigma_1(\bar{\alpha}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \sigma_\alpha(\bar{\alpha}), \sigma_\alpha(\bar{\mathbb{P}}), \sigma_\alpha(\bar{\lambda}_i)$ $(i = 1, \ldots, m)$
- $\sup \sigma_1''\tau = \sigma_1(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

<u>Note</u> If $\alpha > \sup \sigma'' \omega_2^{\bar{N}}$, then we cannot have $\sigma_1 = \sigma_\alpha$, since $\sigma_\alpha \upharpoonright \omega_2^{\bar{N}} = \sigma \upharpoonright \omega_2^{\bar{N}}$.

proof of (6)

Let $Y \prec H_{\omega_2}$ be countable s.t. $N_\alpha, \sigma_\alpha \in Y$. $\alpha = \sigma_\alpha(\omega_2^{\bar{N}})$ is ω-cofinal and $\sigma_\alpha(\tau)$ is ω-cofinal whenever $\tau > \omega_2^{\bar{N}}$ is regular in $\bar{N}$ by (5). Hence $Y \cap \sigma_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

Let $k : \bar{H} \overset{\sim}{\longleftrightarrow} Y$, $k(\bar{N}_\alpha) = N_\alpha$, $k(\bar{\sigma}_\alpha) = \sigma_\alpha$, $k(\bar{\delta}_\alpha) = \delta_\alpha$, $k(\bar{\mathcal{L}}^\alpha) = \mathcal{L}^\alpha$. Then $k \upharpoonright \bar{N}_\alpha : \bar{N}_\alpha \prec N_\alpha$ cofinally (since $On \cap N_\alpha$ has cofinality ω) and $k'' \bar{\sigma}_\alpha(\tau)$ is cofinal in $\sigma_\alpha(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$. $\bar{\mathcal{L}}^\alpha$ is consistent and therefore, by countability, has a solid model $\mathfrak{A}$.

Let $\bar{\sigma}_1 = \dot{\sigma}^{\mathfrak{A}}$. Then $\bar{\sigma}_1 \in wfc(\mathfrak{A})$ and !

- $\bar{\sigma}_1 : \bar{N} \prec \bar{N}_\alpha$ cofinally
- $\bar{\sigma}_1(\bar{\imath}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \bar{\sigma}_\alpha(\bar{\imath}), \bar{\sigma}_\alpha(\bar{\mathbb{P}}), \bar{\sigma}_\alpha(\bar{\lambda}_i)$.
- $\sup \bar{\sigma}_1{}''\tau = \bar{\sigma}_1(\tau)$ whenever $\tau \geq \omega_2^{\bar{N}}$ is regular in $\bar{N}$.

But then $\sigma_1 = k\bar{\sigma}_1$ has the desired properties,

QED (6).

Now let $\alpha \in A \cap C$. Then $\mathrm{cf}(\alpha) = \omega$. Let σ_1 be as in (6) and set $\sigma_0 = \pi_\alpha \sigma_1$. Then $\alpha = \sup \sigma_0{}''\omega_2^{\bar{N}} \in A$, since $\pi_\alpha \upharpoonright \alpha = \mathrm{id}$.
$\sigma_0(\bar{\imath}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \pi\sigma_\alpha(\bar{\imath}, \bar{\mathbb{P}}, \bar{\lambda}_i) = \imath, \mathbb{P}_A, \lambda_i$.
But $\sup \sigma_1{}''\bar{\lambda}_i = \sigma_1(\bar{\lambda}_i) = \sigma_\alpha(\bar{\lambda}_i) = \sup \sigma_\alpha{}''\bar{\lambda}_i$.
Hence $\sup \sigma_0{}''\bar{\lambda}_i = \sup \pi{}''\sigma_1(\bar{\lambda}_i) =$
$= \sup \pi{}''\sigma_\alpha(\bar{\lambda}_i) = \sup \sigma{}''\bar{\lambda}_i = \tilde{\lambda}_i$

QED (Claim 1)

Now let σ_0 be as in Claim 1 and let $\alpha = \sup \sigma_0{}''\omega_2^{\bar{N}} \in A$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. Set $\bar{g} = \bigcup \bar{G}$. Then $\bar{g} : \omega_1^{\bar{N}} \to \bar{A}$ is normal and cofinal, where $\sigma(\bar{A}) = A$. But $\sigma_0(\bar{A}) = A$, since $A = \bigcup_{p \in \mathbb{P}} \mathrm{dom}(p)$. Set $g = \sigma_0 \circ \bar{g}$

Then $g : \omega_1^{\bar N} \to A$ is normal with $\sup g''\omega_1^{\bar N} = \alpha \in A$. Set $p = g \cup \{\langle \alpha, \omega_1^{\bar N} \rangle\}$. Then $p \in \mathbb{P}_A$ and $\sigma_0(p)$

$$p \leq \sigma_0(q) \longleftrightarrow q \subset \bar{g} \longleftrightarrow q \in \bar{G}$$

for $q \in \bar{\mathbb{P}}$. Hence, if $G \ni p$ is $\mathbb{P}_A$-generic, then $\bar{G} = \sigma_0^{-1}{}''G$.

QED (Lemma 1)

Note $\mathbb{P}_A$ is provably not semiproper. Hence we have shown that not every subcomplete forcing is semiproper.

Note This example is rather special in the sense that σ_0 lies in V. That will not hold if - as in the next example - a regular cardinal becomes ω-cofinal.

§3.5 Prikry forcing

Let U be a normal measure on κ. Let $\mathbb{P} = \mathbb{P}_U$ be the set of Prikry conditions for adding a cofinal ω-sequence to κ.

Lemma 2 $\mathbb{P}$ is subcomplete.

We remember that $\mathbb{P}$ consists of all pairs $\langle s, X\rangle$ s.t. $X \in U$ and $s: n \to \kappa$ is monotone for some n.

$$\langle s', X'\rangle \leq \langle s, X\rangle \underset{\text{df}}{\longleftrightarrow} (s' \supset s \wedge X' \subset X \wedge$$
$$\wedge\, \mathrm{rng}(s') \setminus \mathrm{rng}(s) \subset X)$$

If G is $\mathbb{P}$-generic, then

$$S = \bigcup\{s \mid \bigvee X\ \langle s, X\rangle \in G\}$$

is called a Prikry sequence. G is then recoverable from S by:

$$G = \{\langle s, X\rangle \mid s \subset S \wedge \mathrm{rng}(S) \setminus \mathrm{rng}(s) \subset X\}.$$

It can be shown that $S: \omega \to \kappa$ is a Prikry sequence iff $\mathrm{rng}(S)$ is almost contained in every $X \in U$, Now let $\mathbb{P} \in H_\theta$. Let $\tau > \theta$ be regular and $N = \langle L_\tau[A], A, \in \rangle$ s.t. $H_\theta \subset N$. Let $\sigma : \bar{N} \prec N$ be countable and full s.t. $\sigma(\bar{\mathbb{P}}) = \mathbb{P}$.

<u>Claim</u> σ witnesses the subcompleteness of $\mathbb{P}$.

Let $\lambda_i \in \operatorname{rng}(\sigma)$ s.t. $\mathbb{P} \in H_{\lambda_i}$ and $\lambda_i \in (\omega_1, \theta)$ is regular ($i = 1, \dots, m$).

Set: $\bar{\lambda}_i = \sigma^{-1}(\lambda_i)$.

Let $\sigma(\bar{s}) = s$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over $\bar{N}$. We must show:

<u>Claim</u> There is $p \in \mathbb{P}$ which forces that whenever $G \ni p$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ s.t.

(a) $\sigma_0 : \bar{N} \prec N$ cofinally

(b) $\sigma_0(\bar{s}, \bar{\mathbb{P}}, \bar{\lambda}_i) = s, \mathbb{P}, \lambda_i$ $(i = 1, \dots, m)$

(c) $\sup \sigma_0{}''\bar{\lambda}_i = \tilde{\lambda}_i =_{\mathrm{Df}} \sup \sigma{}''\bar{\lambda}_i$

for $i = 1, \dots, m$.

(d) $\bar{G} = \sigma_0^{-1}{}''G$.

Let $\langle N', \sigma' \rangle$ = the liftup of $\langle \bar{N}, \sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}} \rangle$ where $\sigma(\bar{\kappa}) = \kappa$. Then $\sigma' : \bar{N} \prec N'$ cofinally and $\sup \sigma'{}''\tau = \sigma'(\tau)$ for all $\tau \geq \bar{\kappa}$ s.t. τ is regular in $\bar{N}$.

Let $\bar{g} : \omega \to \bar{\kappa}$ be the Prikry sequence engendered by $\bar{G}$. Set $g' = \sigma' \circ \bar{g}$. Then $g' : \omega \to \kappa' = \sigma'(\bar{\kappa})$ cofinally.

(1) g' is a Prikry sequence for N' (w.r.t. $\mathcal{U}' = \sigma'(\bar{\mathcal{U}})$)

prf.

We must show that $\mathrm{rng}(g')$ is almost contained in X for every $X \in \mathcal{U}'$. But $X = \sigma'(f)(\xi)$, where $f \in \bar{N}$, $f : \alpha \to \bar{\mathcal{U}}$ for an $\alpha < \kappa$, and $\xi < \sigma(\alpha) = \sigma'(\alpha)$. Hence $\bar{Y} = \bigcap f''\alpha \in \bar{\mathcal{U}}$ and $Y = \sigma'(\bar{Y}) = \bigcap \sigma'(f)''\sigma(\alpha) \in \mathcal{U}'$. Hence $\bar{g}$ is almost contained in $\bar{Y}$ and g' is almost contained in $Y \subset X$.

QED(1)

Now let $\langle N'', \sigma'' \rangle$ be the liftup of $\langle \bar{N}, \sigma \upharpoonright$ [illegible] $\rangle$, where $\mu = \bar{\kappa}^{++\bar{N}}$.

Then $\sigma''(\bar{\kappa}+\bar{N}) = \kappa^+$, $\sigma''(H^{\bar{N}}_{\bar{\kappa}+\bar{N}}) = H_{\kappa^+}$ and $\sigma''(\tau) = \sup \sigma''``\tau$ whenever $\tau \geq \bar{\kappa}^{+\bar{N}}$ is regular in $\bar{N}$.

Let $\delta' = \delta_{N'}$, $\delta'' = \delta_{N''}$. Let $\mathcal{L}'$ be the infinitary language on $L_{\delta'}(N')$ comprising the basic theory, a new constant $\mathring{\sigma}$ and the further axioms:

- $\mathring{\sigma} : \underline{\bar{N}} \prec \underline{N'}$ cofinally
- $\mathring{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{P}}}, \underline{\bar{\lambda}_i}, \underline{\bar{\kappa}}) = \underline{\sigma'(\bar{s})}, \underline{\sigma'(\bar{\mathbb{P}})}, \underline{\sigma'(\bar{\lambda}_i)}, \underline{\sigma'(\bar{\kappa})}$
- $\sup \mathring{\sigma}''\tau = \mathring{\sigma}(\tau)$ whenever $\tau \geq \bar{\kappa}$ is regular in $\bar{N}$
- $\mathring{\sigma} \circ \underset{\sigma}{\bar{G}}$ is Prikry generic over N'

Then $\mathcal{L}'$ is consistent, since $\langle H_{\kappa^+}, \sigma' \rangle$ models $\mathcal{L}'$.

But $\langle N', \sigma' \rangle$ is the liftup of $\langle \bar{N}, \sigma'' \upharpoonright H^{\bar{N}}_{\bar{\kappa}} \rangle$ since $\sigma'' \upharpoonright H^{\bar{N}}_{\bar{\kappa}} = \sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}}$.

Hence there is unique $\pi : N' \prec N''$ cofinally, s.t. $\pi \upharpoonright H^{N'}_{\sigma'(\bar{\kappa})} = \mathrm{id}$ and $\pi\sigma' = \sigma''$. But then $\mathcal{L}''$

is consistent, where $\mathcal{L}''$ has the same definition over $L_{\delta''}(N'')$ in the parameters $\bar{\tau}, \bar{\mathbb{P}}, \bar{\lambda}_i, \bar{g}, \bar{u}, \sigma''(\bar{\tau}), \sigma''(\bar{\mathbb{P}}), \sigma''(\bar{\lambda}_i), \sigma''(\bar{u})$. (Note that $\sigma'(\bar{u}) = U \cap H^{N'}_{\sigma'(\bar{\tau}+\bar{N})}$ and $\sigma''(\bar{u}) = U$, since $\bar{u} = \{x \mid V_\lambda \langle \lambda, x\rangle \in \bar{\mathbb{P}}\}$.)

Now generically collapse δ'' to ω. In the resulting model $V[\tilde{G}]$ let $\mathfrak{A}$ be a solid model of $\mathcal{L}''$. Set $\sigma_1 = \mathring{\sigma}^{\mathfrak{A}}$, $g = \sigma_1 \circ \bar{g}$. Then

(2) g is Prikry generic over V since $\sigma''(\bar{u}) = U$,

Since g is Prikry generic over N'' and N'' is regular in $L_{\delta''}(N'')$, g is also Prikry generic over $L_{\delta''}(N'')$; hence

(3) $L_{\delta''}(N''[g])$ is admissible.

Let $\mathcal{L}^*$ be the language on $L_{\delta''}(N''[g])$ with the base, a constant $\mathring{\sigma}$, the axioms of $\mathcal{L}''$, and the axiom: $\underline{g} = \mathring{\sigma}''\underline{\bar{g}}$, Then $\mathcal{L}^*$ is consistent, since $\langle H^{V[G]}_{\kappa^{++}}, \sigma'' \rangle$ is a model. But $\mathcal{L}^* \in V[g]$. We now virtually repeat the proof of (6) in §3.4 to get:

(4) ~~In~~ $V[g]$ there is σ^* s.t.

- $\sigma^* : \bar{N} \prec N''$ cofinally
- $\sigma^*(\bar{\kappa}, \bar{\mathbb{P}}, \bar{x}_i) = \sigma''(\bar{\kappa}), \sigma''(\bar{\mathbb{P}}), \sigma''(\bar{x}_i)$ $(i = 1, \dots, m)$
- $\sigma^* \circ \bar{g} = g$
- $\sup \sigma^{*''}\tau = \sigma^*(\tau)$ whenever $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$.

proof (sketch)

We work in $V[g]$. As before let $Y \prec H_{\kappa^{++}}$ be countable s.t. $\bar{N}, N'', \sigma'' \in Y$. As before $Y \cap \sigma''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$. Let $k : \bar{H} \overset{\sim}{\longleftrightarrow} Y$, $k(\bar{N}'') = N''$, $k(\bar{\sigma}'') = \sigma''$, $k(\bar{\delta}'') = \delta''$, $k(\bar{\mathcal{L}}^*) = \mathcal{L}^*$.

Then $k \upharpoonright \bar{N}_\alpha : \bar{N}_\alpha \prec N_\alpha$ cofinally. Moreover $k''\bar{\sigma}''(\tau)$ is cofinal in $\sigma''(\tau)$ whenever $\tau = \bar{\kappa}$ or $\tau \geq \bar{\kappa}^{++\bar{N}}$ is regular in $\bar{N}$. $\bar{\mathcal{L}}^*$ is consistent + hence has a solid model $\mathfrak{A}$. Let $\sigma^* = \dot{\sigma}^{\mathfrak{A}}$. Then

$\bar\sigma^* \in wfc(\mathcal{O}\!\iota)$ and:

- $\bar\sigma^*: \bar N \prec \bar N''$ cofinally
- $\bar\sigma^*(\bar\kappa, \bar{\mathbb{P}}, \bar\lambda_i) = \bar\sigma''(\bar\kappa), \bar\sigma''(\bar{\mathbb{P}}), \bar\sigma''(\bar\lambda_i)$
- $\bar g^* = \bar\sigma^* \circ \bar g$, where $k(\bar g^*) = g$
- $\sup \bar\sigma^{*}{}'' \tau = \bar\sigma^*(\tau)$ if $\tau = \bar\kappa$ or $\tau \geq \bar\kappa^{++\bar N}$ is regular in $\bar N$,

But then $\sigma^* = k\bar\sigma^*$ has the desired properties.

QED(4)

Since $\langle N'', \sigma''\rangle$ = the liftup of $\langle \bar N, \sigma \restriction H^{\bar N}_{\bar\kappa^{++}}\rangle$, there is $\pi_0: N'' \prec N$ s.t. $\pi_0 \restriction H^{\bar N}_{\bar\kappa^{++}} = id$ and $\pi_0\sigma'' = \sigma$. Set: $\sigma_0 = \pi_0\sigma^*$.

It follows easily that:

- $\sigma_0: \bar N \prec N$ cofinally
- $\sigma_0(\bar\kappa, \bar{\mathbb{P}}, \bar\lambda_i) = \kappa, \mathbb{P}, \lambda_i$
- $\sup \sigma_0{}'' \bar\lambda_i = \tilde\lambda_i$
- $g = \sigma_0 \circ \bar g = \sigma^* \circ \bar g$

But g, G are interdefinable in $V[g] = V[G]$, where G is a $\mathbb{P}$-generic set. Similarly for $\bar g, \bar G$ in $\bar N[\bar G]$, Hence

- $\bar G = \sigma_0^{-1}{}'' G$.

Since G is $\mathbb{P}$-generic, there must be a $p \in G$ which forces all of this.

QED(Lemma 2)

This proof can easily be modified to show that $\mathbb{P}$ is subproper above μ for each $\mu < \kappa$, in the sense of the definition at the end of § 2: Letting $\sigma \upharpoonright H^{\bar{N}}_{\bar{\kappa}} : H^{\bar{N}}_{\bar{\kappa}} \prec \tilde{H}$ cofinally, we have $\tilde{H} = H_{\tilde{\kappa}}$, where $\tilde{\kappa} = \sup \sigma''\bar{\kappa} = \sigma'(\kappa)$, where $\langle N', \sigma' \rangle$ is defined as above. But then $\sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}} \in \tilde{H}$ for $\bar{\mu} < \bar{\kappa}$. We can thus add to $\mathcal{L}'$ the axiom: $\dot{\sigma} \upharpoonright H^{\bar{N}}_{\bar{\mu}} = \underline{\sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}}}$. Carrying this axiom with us through the rest of the proof, we arrive at σ_0 s.t. $\sigma_0 \upharpoonright H^{\bar{N}}_{\bar{\mu}} = \sigma \upharpoonright H^{\bar{N}}_{\bar{\mu}}$. QED

Our further examples will all be reversable $\mathcal{L}$-forcings in the sense of [J] § 3. This will be true even of Namba forcing, since we have shown in [J] § 6 that Namba forcing is equivalent to such an $\mathcal{L}$-forcing.

From now on we assume a knowledge of [J] § 3.

§ 3.6 Namba Forcing

Lemma 3 The forcing $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of [J] §5, Example 1 (p. 7) is subcomplete.

In this forcing we start with a regular $\beta \geq \omega_2$ s.t. $2^{\omega} = \omega_1$ and $2^{\beta} = \beta$. $\mathbb{P}$ then collapses each regular $\gamma \in (\omega_1, \beta]$ to ω_1, making it ω-cofinal without collapsing reals. In [J] §6 we show that if $\beta = \omega_2$, then $BA(\mathbb{P}) = B.A(\mathbb{N})$, where $\mathbb{N}$ is Namba forcing. Hence:

Corollary 3.1 If $2^{\omega} = \omega_1$ and $2^{\omega_1} = \omega_2$, then Namba forcing is subcomplete.

. . . .

We assume that the reader has a good understanding of [J] §3. We shall also make use of Corollary 2.8 in [J] §4, which says that if G is $\mathbb{P}$-generic over V, then G is definable from $\langle M^G, \pi^G, B^G \rangle$ by:

$$p \in G \leftrightarrow (M^p = M^G \upharpoonright (|p|+1) \wedge \pi^p = \pi^G \upharpoonright (|p|+1)^2 \wedge$$
$$\wedge\, b^p = (\pi^G_{|p|,\omega_1})^{-1} {}'' B^G \wedge$$
$$\wedge \bigwedge_{\langle \bar{a}, a \rangle \in F^p} \pi^G_{|p|,\omega_1} : \langle M^p_{|p|}, \bar{a} \rangle \prec \langle M, a \rangle).$$

We set: $M = L^A_\beta = \langle L_\beta[A], A \rangle$, where $L_\beta[A] = H_\beta$. Set: $N = \langle H_{\beta^+}, M, <, \ldots \rangle$,