Subcomplete Forcing and $\mathcal{L}$-Forcing

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ABSRACT

In his book **Proper Forcing** (1982) Shelah introduced three classes of forcings (complete, proper, and semi-proper) and proved a strong iteration theorem for each of them: The first two are closed under countable support iterations. The latter is closed under revised countable support iterations subject to certain standard restraints. These theorems have been heavily used in modern set theory. For instance using them, one can formulate "forcing axioms" and prove them consistent relative to a supercompact cardinal. Examples are PFA, which says that Martin's axiom holds for proper forcings, and MM, which says the same for semiproper forcings. Both these axioms imply the negation of CH. This is due to the fact that some proper forcings add new reals. Complete forcings, on the other hand, not only add no reals, but also no countable sets of ordinals. Hence they cannot change a cofinality to ω. Thus none of these theories enable us e.g. to show, assuming CH, that Namba forcing can be iterated without adding new reals.

More recently we discovered that the three forcing classes mentioned above have natural generalizations which we call "subcomplete", "subproper" and "semi-subproper". It turns out that each of these is closed under Revised Countable Support (RCS) iterations subject to the usual restraints.

The first part of our lecture deals with subcomplete forcings. These forcings do not add reals. Included among them, however, are Namba forcing, Prikry forcing, and many other forcings which change cofinalities. This gives a positive solution to the above mentioned iteration problem for Namba forcing. Using the iteration theorem one can also show that the **Subcomplete Forcing Axiom** (SCFA) is consistent relative to a supercompact cardinal. It has some of the more striking consequences of MM but is compatible with CH (and in fact with $\Diamond$).

(Note: Shelah was able to solve the above mentioned iteration problem for Namba forcing by using his ingenious and complex theory of "I-condition forcing". The relationship of I-condition forcing to subcomplete forcing remains a mystery. There are, however, many applications of subcomplete forcing which have not been replicated by I-condition forcing.)

In the second part of the lecture, we give an introduction to the theory of "$\mathcal{L}$-Forcings". We initially developed this theory more than twenty years ago in order to force the existence of new reals. More recently, we discovered that there is an interesting theory of $\mathcal{L}$-Forcings which do **not** add reals. (In fact, if we assume CH $+2^{\omega_1} = \omega_2$, then Namba forcing is among them.) Increasingly we came to feel that there should be a "natural" iteration theorem which would apply to a large class of these forcings. This led to the iteration theorem for subcomplete forcing.

Combining all our methods, we were then able to prove:

(1) Let κ be a strongly inaccessible cardinal. Assume CH. There is a subcomplete forcing extension in which κ becomes ω_2 and every regular cardinal $\tau \in (\omega_1, \kappa)$ acquires cofinality ω.
(2) Let κ be as above, where GCH holds below κ. Let $A \subset \kappa$. There is a subcomplete forcing extension in which:
   - κ becomes ω_2;
   - If $\tau \in (\omega_1, \kappa) \cap A$ is regular, then it acquires cofinality ω;
   - If $\tau \in (\omega_1, \kappa) \setminus A$ is regular, then it acquires cofinality ω_1.

We will not be able to fully prove these theorems in our lectures, but we hope to develop some of the basic methods involved.

Contents

Chapter 0

Preliminaries

ZF^- ("ZF without power set") consists of the axioms of extensionality and foundation together with:

(1) $\emptyset$, $\{x, y\}$, $\bigcup x$ are sets.
(2) (Axiom of Subsets or "Aussonderungsaxiom")
$x \cap \{z \mid \varphi(z)\}$ is a set.
(3) (Axiom of Collection)
$\bigwedge x \bigvee y\, \varphi(x, y) \to \bigwedge u \bigvee v \bigwedge x \in u\, \bigvee y \in v\, \varphi(x, y)$
(4) (Axiom of Infinity)
ω is a set.

Note (3) implies the usual replacement axiom, but cannot be derived from it without the power set axiom.

ZFC^- is ZF^- together with the strong form of the axiom of choice:

(5) Every set is enumerable by an ordinal.

Note The power set axiom is required to derive (5) from the weaker forms of choice.

The *Levy hierarchy* of formulae is defined in the usual way:
Σ_0 formulae are the formulae containing only bounded quantification – i.e. $\Sigma_0 =$ the smallest set of formulae containing the primitive formulae and closed under sentential operations and bounded quantification:

$$\bigwedge x \in y\, \varphi, \quad \bigvee x \in y\, \varphi$$

(where $\bigwedge x \in y\, \varphi = \bigwedge x(x \in y \to \varphi)$ and $\bigvee x \in y\, \varphi = \bigvee x(x \in y \wedge \varphi)$).
(In some contexts it is useful to introduce bounded quantifiers as primitive signs rather than defined operations.)

We set: $\Pi_0 = \Sigma_0$. Σ_{n+1} formulae are then the formulae of the form $\bigvee x\ \varphi$, where φ is Π_n. Similarly Π_{n+1} formulae have the form $\bigwedge x\, \varphi$, where φ is Σ_n.

A relation R on the model $\mathfrak{A}$ is called $\Sigma_n(\mathfrak{A})$ $(\Pi_n(\mathfrak{A}))$ iff it is definable over $\mathfrak{A}$ by a Σ_n (Π_n) formula.

R is $\Sigma_n(\mathfrak{A})$ $(\Pi_n(\mathfrak{A}))$ *in the parameters* $p_1, \ldots, p_m$ iff it is $\Sigma_n(\Pi_n)$ definable in the parameters $p_1, \ldots, p_n \in \mathfrak{A}$. It is $\underline{\Sigma}_n(\mathfrak{A})$ $(\underline{\Pi}_n(\mathfrak{A}))$ iff it is Σ_n (Π_n) definable in some parameters. It is $\Delta_n(\mathfrak{A})$ iff it is $\Sigma_n(\mathfrak{A})$ and $\Pi_n(\mathfrak{A})$.

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$\overline{\overline{x}}$ or card(x) denotes the cardinality of x. (We reserve the notation $|x|$ for other uses.)

If r is a well ordering or a set of ordinals, then otp(r) denotes its order type. crit(f) is the critical point of the function f (i.e. $\alpha = \text{crit}(f) \leftrightarrow (f \upharpoonright \alpha = \text{id} \wedge f(\alpha) > \alpha)$.

$F''A$ is the image of A under the function (or relation) F.

rng(R) is the range of the relation R.

dom(R) is the domain of the relation R.

TC(x) is the transitive closure of x, $H_\alpha = \{x \mid \overline{\overline{\text{TC}(x)}} < \alpha\}$.

Boolean Algebras and Forcing

The theory of forcing can be developed using "sets of conditions" or complete Boolean algebras. The former is most useful when we attempt to devise a forcing for a specific end. The latter is more useful when we deal with the general theory of forcing, as in the theory of iterated forcing. We adopt here an integrated approach which begins with Boolean algebras. By a Boolean algebra we mean a partial ordering $\mathbb{B} = \langle |\mathbb{B}|, c_{\mathbb{B}} \rangle$ with maximal and minimal elements 0, 1, lattice operations $\cap$, $\cup$ defined by:

$$a \subset (b \cap c) \longleftrightarrow (a \subset b \wedge a \subset c)$$
$$(b \cup c) \subset a \longleftrightarrow (b \subset a \wedge c \subset a)$$

and a complement operation $\neg$ defined by:

$$a \subset \neg b \longleftrightarrow a \cap b = 0,$$

satisfying the usual Boolean equalities. We call $\mathbb{B}$ a *complete Boolean algebra* if, in addition, for each $X \subset \mathbb{B}$ there are operations $\bigcap^{\mathbb{B}} X$, $\bigcup^{\mathbb{B}} X$ defined by:

$$a \subset \bigcap\nolimits^{\mathbb{B}} X \longleftrightarrow \bigwedge b \in X \; a \subset b,$$
$$\bigcup\nolimits^{\mathbb{B}} X \subset a \longleftrightarrow \bigwedge b \in X \; b \subset a,$$

s.t.

$$a \cap \bigcup_{b \in I} b = \bigcup_{b \in I} (a \cap b), \qquad a \cup \bigcap_{b \in I} = \bigcap_{b \in I} (a \cap b).$$

We shall generally write 'BA' for 'Boolean algebra'.

We write $\mathbb{A} \subseteq \mathbb{B}$ to mean that $\mathbb{A}$, $\mathbb{B}$ are BA's, $\mathbb{A}$ is complete, and $\mathbb{A}$ is completely contained in $\mathbb{B}$ – i.e.

$$\bigcap\nolimits^{\mathbb{B}} X = \bigcap\nolimits^{\mathbb{A}} X, \quad \bigcup\nolimits^{\mathbb{B}} X = \bigcup\nolimits^{\mathbb{A}} X \text{ for } X \subset \mathbb{A}.$$

If $\mathbb{A} \subseteq \mathbb{B}$ and $b \in \mathbb{B}$, we define $h(b) = h_{\mathbb{A},\mathbb{B}}(b)$ by: $h(b) = \bigcap\{a \in \mathbb{A} \mid b \subset a\}$. Thus:

- $h(\bigcup_i b_i) = \bigcup_i h(b_i)$ if $b_i \in \mathbb{B}$ for $i \in I$.
- $h(a \cap b) = a \cap h(b)$ if $a \in \mathbb{A}$.
- $b = 0 \leftrightarrow h(b) = 0$ for $b \in \mathbb{B}$.

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If $\mathbb{B}$ is a complete BA, we can form the canonical *maximal* $\mathbb{B}$*-valued model* $\mathbf{V}^{\mathbb{B}}$. The elements of $\mathbf{V}^{\mathbb{B}}$ are called *names* and there is a valuation function $\varphi \to [[\varphi]]_{\mathbb{B}}$ attaching to each statement $\varphi = \varphi'(t_1, \ldots, t_n)$ a truth value in $\mathbb{B}$. (Here φ is a ZFC formula and $t_1, \ldots, t_n$ are names.) All axioms of ZFC have truth value 1 (assuming ZFC). The sentential connectives are interpreted by:

$$[[\varphi \wedge \psi]] = [[\varphi]] \cap [[\psi]]; \quad [[\varphi \vee \psi]] = [[\varphi]] \cup [[\psi]];$$
$$[[\varphi \to \psi]] = [[\varphi]] \Rightarrow [[\psi]], \text{ where } (a \Rightarrow b) =_{\mathrm{Df}} \neg a \cup b;$$
$$[[\neg\varphi]] = \neg[[\varphi]].$$

The quantifiers are interpreted by:

$$[[\bigwedge v\, \varphi(v)]] = \bigcap_{x \in \mathbf{V}^{\mathbb{B}}} [[\varphi(x)]], \quad [[\bigvee v\, \varphi(v)]] = \bigcup_{x \in \mathbf{V}^{\mathbb{B}}} [[\varphi(x)]].$$

If $u \subset \mathbf{V}^{\mathbb{B}}$ is a set and $f : u \to \mathbb{B}$, then there is a name $x \in \mathbf{V}^{\mathbb{B}}$ s.t.

$$[[y \in x]] = \bigcup_{z \in u} [[y = z]] \cap f(\ \ )$$

for all $y \in \mathbf{V}^{\mathbb{B}}$. Conversely, for each $x \in \mathbf{V}^{\mathbb{B}}$ there is a set $u_x \subset \mathbf{V}^{\mathbb{B}}$ s.t.

$$[[y \in x]] = \bigcup_{z \in u_x} [[y = z]] \cap [[z \in x]].$$

We can, in fact, arrange things s.t. $\{\langle z, x\rangle \mid z \in u_x\}$ is a well founded relation. If $U \subset \mathbf{V}^{\mathbb{B}}$ is a class j and $A : U \to \mathbb{B}$, we may add to the language a predicate $\mathring{A}$ interpreted by: $[[\mathring{A}x]] = \bigcup_{z \in u} [[x = z]] \cap A(z)$. We inductively define for each $x \in \mathbf{V}$ a name $\check{x}$ by:

$$[[y \in \check{x}]] = \bigcup_{z \in x} [[y = \check{z}]],$$

and a predicate $\check{\mathbf{V}}$ by:

$$[[y \in \check{\mathbf{V}}]] = \bigcup_{\tau \in \mathbf{V}} [[y = \check{z}]].$$

If $\sigma : \mathbb{A} \overset{\sim}{\leftrightarrow} \mathbb{B}$ is an isomorphism, then we can define an injection $\sigma^* : \mathbf{V}^{\mathbb{A}} \to \mathbf{V}^{\mathbb{B}}$ as follows: Let $R = \{\langle z, x\rangle \mid z \in u_x\}$ be the above mentioned well founded relation for $\mathbf{V}^{\mathbb{A}}$. By R-induction we define $\sigma^*(x)$, picking $\sigma^*(x)$ to be a $w \in \mathbf{V}^{\mathbb{B}}$ s.t.

$$[[y \in w]]_{\mathbb{B}} = \bigcup_{z \in u_x} [[y = \sigma^*(z)]]_{\mathbb{B}} \cap \sigma([[z \in x]]_{\mathbb{A}}).$$

Then:

(1) $\sigma([[\varphi(\vec{x})]]_{\mathbb{A}}) = [[\varphi(\sigma^*(\vec{x}))]]_{\mathbb{B}}$

for all ZFC formulae and all $x_1, \ldots, x_n \in \mathbf{V}^{\mathbb{A}}$. If $\sigma : \mathbb{A} \to \mathbb{B}$ is a complete embedding (i.e. $\sigma : \mathbb{A} \overset{\sim}{\leftrightarrow} \mathbb{A}' \subseteq \mathbb{B}$ for some $\mathbb{A}'$), then σ^* can be defined the same way, but (1) then holds only for Σ_0 formulae. In such contexts it is often useful to take $\mathbf{V}^{\mathbb{B}}$ as a $\mathbb{B}$-valued *identity model*, meaning that

$$[[x = y]] = 1 \longrightarrow x = y \quad \text{for } x, y \in \mathbf{V}^{\mathbb{B}}.$$

(If $\mathbf{V}^{\mathbb{B}}$ does not already have this property, we can attain it by factoring.) If $\sigma : \mathbb{A} \overset{\sim}{\leftrightarrow} \mathbb{B}$ and $\mathbf{V}^{\mathbb{A}}$, $\mathbf{V}^{\mathbb{B}}$ are identity models, then σ^* is bijective (and is, in fact, an isomorphism of $\langle \mathbf{V}^{\mathbb{A}}, I^{\mathbb{A}}, E^{\mathbb{A}} \rangle$ onto $\langle \mathbf{V}^{\mathbb{B}}, I^{\mathbb{B}}, E^{\mathbb{B}} \rangle$, where $I = (x, y) = [[x = y]]$, $E(x, y) = [[x \in y]]$). Another advantage of identity models in that $\{z \mid [[z \in x]] = 1\}$ is then a set, rather than a proper class.

There are many ways to construct a maximal $\mathbb{B}$-valued model $\mathbf{V}^{\mathbb{B}}$ and we can take its elements as being anything we want. Noting that $\mathbb{A} \subseteq \mathbb{B}$ means that id $\upharpoonright \mathbb{A}$ is a complete embedding, it is useful, when dealing with such a pair $\mathbb{A}$, $\mathbb{B}$, to arrange that $\mathbf{V}^{\mathbb{A}} \subset \mathbf{V}^{\mathbb{B}}$ and $(\mathrm{id} \upharpoonright \mathbb{A})^* = \mathrm{id} \upharpoonright \mathbf{V}^{\mathbb{A}}$. (We express this by: $\mathbf{V}^{\mathbb{A}} \subseteq \mathbf{V}^{\mathbb{B}}$.)

The *forcing relation* $\Vdash_{\mathbb{B}}$ is defined by:

$$b \Vdash \varphi \longleftrightarrow_{\mathrm{Df}} (b \neq 0 \wedge b \subset [[\varphi]]).$$

We also set: $\Vdash \varphi \leftrightarrow_{\mathrm{Df}} [[\varphi]] = 1$. Now suppose that W is an inner model of ZF and $\mathbb{B} \in W$ is complete in the sense of W. We can form $W^{\mathbb{B}}$ internally in W, and it turns out that all ZF axioms are true in $W^{\mathbb{B}}$. (If W satisfies ZFC, then ZFC holds in $W^{\mathbb{B}}$.) W could also be a set rather than a class. If W is only a model of ZF$^-$, we can still form $W^{\mathbb{B}}$, which will then model ZF$^-$ (or ZFC$^-$ if W models ZFC$^-$). (In this case, however, we may not be able – internally in W – to factor $W^{\mathbb{B}}$ to an identity model.)

We say that $G \subset \mathbb{B}$ is $\mathbb{B}$-*generic over* W iff G is an ultrafilter on $\mathbb{B}$ which respects all intersections and unions of $X \subset \mathbb{B}$ s.t. $X \in W$ – i.e.

$$\bigcap x \in G \longleftrightarrow \bigwedge b \in x \; b \in G, \quad \bigcup x \in G \longleftrightarrow \bigvee b \in X \; b \in G.$$

If G is generic, we can form the *generic extension* $W[G]$ of W by:

$$W[G] = \{x^G \mid x \in W^G\}, \text{ where } x^G = \{z^G \mid z \in u_x \wedge [[z \in x]] \in G\}.$$

Then $W \subset W[G]$, since $\check{x}^G = x$ (by G-induction on $x \in W$). Then:

$$W[G] \models \varphi(x_1^G, \ldots, x_n^G) \longleftrightarrow \bigvee b \in G \;\; b \Vdash \varphi(x_1, \ldots, x_n).$$

If we suppose, moreover, that for every $b \in \mathbb{B} \setminus \{0\}$ there is a generic $G \ni b$ (e.g. if $\varphi(\mathbb{B}) \cap W$ is countable), then:

$$b \Vdash \varphi(x_1, \ldots, x_n) \longleftrightarrow (W[G] \models \varphi(x_1^G, \ldots, x_n^G) \text{ for all generic } G \ni b).$$

If $\mathbb{B}$ is complete in $\mathbf{V}$ we shall often find it useful to work in a mythical universe in which:

(∗) $\mathbf{V}$ is an inner model and for every $b \in \mathbb{B} \setminus \{0\}$ there is a $G \ni b$ which is $\mathbb{B}$-generic over $\mathbf{V}$.

This is harmless, since if $\mathbb{C}$ collapsed $2^{\overline{\overline{\mathbb{B}}}}$ to ω, then (∗) holds of $\check{\mathbf{V}}$, $\check{\mathbb{B}}$ in $\mathbf{V}^{\mathbb{C}}$. We note that there is a $\mathring{G} \in \mathbf{V}^{\mathbb{B}}$ s.t. $\Vdash \mathring{G} \subset \check{\mathbb{B}}$ and $[[\check{b} \in \mathring{G}]] = b$ for $b \in \mathbb{B}$. ($\mathring{G}$ is in fact unique if $\mathbf{V}^{\mathbb{B}}$ is an identity model.) If then G is $\mathbb{B}$-generic over $\mathbf{V}$, we have $\mathring{G}^G = G$. Thus $\Vdash \mathring{G}$ is $\check{\mathbb{B}}$-generic over $\check{\mathbf{V}}$. We call $\mathring{G}$ the *canonical* $\mathbb{B}$-*generic name*.
If our language contains predicates $\mathring{A}$ other than 0, ∈, we set:

$$\mathring{A}^G = \{x^G \mid [[x \in \mathring{A}]] \in G\}.$$

Since $[[x \in \check{V}]] = \bigcup_{z \in \mathbf{V}} [[x = \check{z}]]$, we get:

$$\check{\mathbf{V}}^G = \{\check{z}^G \mid z \in \mathbf{V}\} = \mathbf{V}.$$

Sets of Conditions

By a *set of conditions* we mean $\mathbb{P} = \langle |\mathbb{P}|, \leq_{\mathbb{P}} \rangle$ s.t. $\leq = \leq_p$ is a transitive relation on $|\mathbb{P}|$. (Notationally we shall not distinguisch between $\mathbb{P}$ and $|\mathbb{P}|$.) We say that two conditions p, q are *compatible* ($p \| q$) if $\bigvee r\ r \leq p, q$. Otherwise they are *incompatible* ($p \perp q$). For each set of conditions $\mathbb{P}$ there is a *canonical complete* BA *over* $\mathbb{P}$ (BA($\mathbb{P}$)) defined as follows: For $X \subset \mathbb{P}$ set:

$$\neg X = \{q \mid \bigwedge p \in X \ \ p \perp q\}.$$

Then $X \subset \neg\neg X$ and $\neg\neg\neg X = \neg X$. Hence $\neg\neg$ is a hull operator on $\mathfrak{P}(\mathbb{P})$. Set $|\mathbb{B}| = \{X \subset \mathbb{P} \mid X = \neg\neg X\}$. Then $\text{BA}(\mathbb{P}) = \langle |\mathbb{B}|, c \rangle$, where c is the ordinary inclusion relation on $|\mathbb{B}|$. $\mathbb{B} = \text{BA}(\mathbb{P})$ is then a complete BA with the complement operation $\neg$ and intersection and union operations given by:

$$\bigcap^{\mathbb{B}} X = \bigcap X, \quad \bigcup^{\mathbb{B}} X = \neg\neg \bigcup X.$$

We say that $\Delta \subset \mathbb{P}$ is *dense in* $\mathbb{P}$ iff $\bigwedge p \in \mathbb{P} \bigvee q \subseteq p,\ q \in \Delta$. Δ is *predense in* $\mathbb{P}$ iff $\bigwedge p \in \mathbb{P}\ \bigvee q\ (q \| p$ and $q \in \Delta)$. (In other words, the closure of Δ under $\leq$ is dense in $\mathbb{P}$.)

Set: $[p] = \neg\neg\{p\}$ for $p \in \mathbb{P}$ (i.e. $[p] = \bigcap\{a \in \text{BA}(\mathbb{P}) \mid a \supset b\}$). The *forcing relation* for $\mathbb{P}$ is defined by:

$$p \Vdash \varphi \longleftrightarrow_{\text{Df}} [p] \subset [[\varphi]].$$

If $\mathbb{P} \in W$ and W is a transitive model of ZF, we say that G is $\mathbb{P}$-*generic over* W iff the following hold:

- If $p, q \in G$, then $p \| q$.
- If $p \in G$ and $p \leq q$, then $q \in G$.
- If $\Delta \in W$ is dense in $\mathbb{P}$, then $G \cap \Delta \neq \emptyset$.

If $\mathbb{B} = \mathrm{BA}(\mathbb{P})_W$ is the complete BA over $\mathbb{P}$ (as defined in W), then it follows that G is $\mathbb{P}$-generic over W iff $F = F_G = \{b \in \mathbb{B} \mid b \cap G \neq \emptyset\}$ is $\mathbb{B}$-generic over W. Conversely, if F is $\mathbb{B}$-generic, thus $G = G_F = \{p \mid [p] \in F\}$ is $\mathbb{P}$-generic.

We also note that if $\mathbb{B}$ is a complete BA, then $\langle \mathbb{B} \setminus \{0\}, \subset\rangle$ is a set of conditions, and there is an isomorphism $\sigma : \mathbb{B} \overset{\sim}{\leftrightarrow} \mathrm{BA}(\mathbb{B} \setminus \{0\})$ defined by: $\sigma(b) = \{a \mid a \supset b\}$. Moreover, G is a $\mathbb{B}$-generic filter iff it is a $\mathbb{B} \setminus \{0\}$-generic set. When dealing with Boolean algebras, we shall often write: "Δ is dense in $\mathbb{B}$" to mean "Δ is dense in $\mathbb{B} \setminus \{0\}$".

The Two Step Iteration

Let $\mathbb{A} \subseteq \mathbb{B}$, where $\mathbb{A}, \mathbb{B}$ are both complete. If (in some larger universe) G is $\mathbb{A}$-generic over $\mathbf{V}$, then $G' = \{b \in \mathbb{B} \mid \bigvee a \in G \ a \subset b\}$ is a complete filter on $\mathbb{B}$ and we can form the factor algebra $\mathbb{B}/G'$ (which we shall normally denote by $\mathbb{B}/G$). It is not hard to see that $\mathbb{B}/G$ is then complete in $\mathbf{V}[G]$. By the definition of the factor algebra there is a canonical homomorphism $\sigma : \mathbb{B} \to \mathbb{B}/G$ s.t. $\sigma(b) \subset \sigma(c) \leftrightarrow \neg b \cup c \in G'$. When the context permits we shall write b/G for $\sigma(b)$. We now list some basic facts about this situation.

Fact 1 Let $\mathbb{B}_0 \subseteq \mathbb{B}_1$, $\mathbb{B}_0$ and $\mathbb{B}_1$ being complete. Let G_0 be $\mathbb{B}_0$-generic over $\mathbf{V}$ and let $\tilde{G}$ be $\tilde{\mathbb{B}} = \mathbb{B}_1/G$-generic over $\mathbf{V}[G]$. Set $G_1 = G_0 * \tilde{G} =_{\mathrm{Df}} \{b \in \mathbb{B}_1 \mid b/G_0 \in \tilde{G}\}$. Then G_1 is $\mathbb{B}_1$-generic over $\mathbf{V}$ and $\mathbf{V}[G_1] = \mathbf{V}[G_0][\tilde{G}]$.

Conversely we have:

Fact 2 If G_1 is $\mathbb{B}_1$-generic over $\mathbf{V}$ and we set: $G_0 = \mathbb{B}_0 \cap G_1$, $\tilde{G} = \{b/G_0 \mid b \in G_1\}$. Then G_0 is $\mathbb{B}_0$-generic over $\mathbf{V}$, $\tilde{G}$ is $\mathbb{B}_1/G_0$-generic over $\mathbf{V}[G_0]$ and $G_1 = G_0 * \tilde{G}$.

Fact 3 Let $\mathbb{A} \subseteq \mathbb{B}$ and let $h = h_{\mathbb{A},\mathbb{B}}$ as defined above. Then

$$h(b) = [[\check{b}/\overset{\circ}{G} \neq 0]]_{\mathbb{A}},$$

$\overset{\circ}{G}$ being the $\mathbb{A}$-generic name.

Proof. $h(b) = \bigcap\{a \in \mathbb{A} \mid a \supset b\} = \bigcap_{a \in \mathbb{A}} ([[\check{a} \supset \check{b}]] \Rightarrow a)$ where $a = [[\check{a} \in \overset{\circ}{G}]]$

$= \bigcap_{a \in \mathbb{A}} [[\check{a} \supset \check{b} \to \check{a} \in \overset{\circ}{G}]] = [[\bigwedge a \in \check{\mathbb{A}}(a \supset \check{b} \to a \in \overset{\circ}{G})]] = [[\check{b}/\overset{\circ}{G} \neq 0]]$ QED(Fact 3)

Fact 4 Let $\mathbb{A} \subseteq \mathbb{B}$ and $\Vdash_{\mathbb{A}} \check{b} \in \check{\mathbb{B}}/\overset{\circ}{G}$, where $\hat{b} \in \mathbf{V}^{\mathbb{A}}$. There is a unique $b \in \mathbb{B}$ s.t. $\Vdash_{\mathbb{A}} \overset{\circ}{b} = \check{b}/\overset{\circ}{G}$.

Proof. To see uniqueness, let $\Vdash \check{b}/\overset{\circ}{G} = \check{b}'/\overset{\circ}{G}$. Then $\Vdash \check{b} \setminus \check{b}'/\overset{\circ}{G} = 0$. Hence $h(b \setminus b') = [[\check{b} \setminus \check{b}'/\overset{\circ}{G} \neq 0]] = 0$. Hence $b \setminus b' = 0$. Hence $b \subset b'$. Similarly $b' \subset b$.
To see the existence, note that $\Delta = \{a \in \mathbb{A} \mid \bigvee b \ a \Vdash \overset{\circ}{b} = \check{b}/\overset{\circ}{G}\}$ is dense in $\mathbb{A}$. Let

X be a maximal antichain in Δ. Let $a \Vdash \overset{\circ}{b} = \check{b}_a/\overset{\circ}{G}$ for $a \in X$. Set: $b = \bigcup_{a \in X} a \cap b_a$. Then $\Vdash \overset{\circ}{b} = \check{b}/\overset{\circ}{G}$, since if G is $\mathbb{A}$-generic there is $a \in X \cap G$ by genericity. Hence $\overset{\circ}{b}{}^G = b_a/G = b/G$. QED(Fact 4)

Fact 2 shows that, if $\mathbb{B}_0 \subseteq \mathbb{B}_1$, then forcing with $\mathbb{B}_1$ is equivalent to a *two step iteration*: Forcing first by $\mathbb{B}_0$ to get $\mathbf{V}[G_0]$ and then by a $\tilde{\mathbb{B}} \in \mathbf{V}[G_0]$.

We now show the converse: Forcing by $\mathbb{B}_0$ and then by some $\tilde{\mathbb{B}}$ is equivalent to forcing by a single $\mathbb{B}_1$:

Fact 5 Let $\mathbb{B}_0$ be complete and let $\Vdash_{\mathbb{B}_0} \overset{\circ}{\mathbb{B}}$ is complete. There is $\mathbb{B}_1 \supseteq \mathbb{B}_0$ s.t. $\Vdash_{\mathbb{B}_0}$ ($\overset{\circ}{\mathbb{B}}$ is isomorphic to $\check{\mathbb{B}}_1/\overset{\circ}{G}$). (Hence, whenever G_0 is $\mathbb{B}_0$-generic, we have $\mathbb{B}_1/G_0 \simeq \tilde{\mathbb{B}} =_{\mathrm{Df}} \overset{\circ}{\mathbb{B}}{}^{G_0}$.)

In order to prove this we first define:

Definition Let $\Vdash_{\mathbb{A}} \overset{\circ}{\mathbb{B}}$ is complete. $\mathbb{B} = \mathbb{A} * \overset{\circ}{\mathbb{B}}$ is the BA defined as follows: Assume $\mathbf{V}^{\mathbb{A}}$ to be an identity model and set:

$$|\mathbb{B}| =_{\mathrm{Df}} \{b \in \mathbf{V}^{\mathbb{A}} \mid \Vdash_{\mathbb{A}} b \in \overset{\circ}{\mathbb{B}}\}, \quad b \subset c \text{ in } \mathbb{B} \longleftrightarrow_{\mathrm{Df}} \Vdash_{\mathbb{A}} b \subset c.$$

This defines $\mathbb{B} = \langle |\mathbb{B}|, \subset \rangle$. $\mathbb{B}$ is easily seen to be a BA with the operations:

$$\begin{aligned} a \cap b = &\text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} c = a \cap b, \\ a \cup b = &\text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} c = a \cup b, \\ \neg b = &\text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} c = \neg b. \end{aligned}$$

Similarly, if $\langle b_i \mid i \in I \rangle$ is any sequence of elements of $\mathbb{B}$, there is a $\overset{\circ}{\mathbb{B}} \in \mathbf{V}^{\mathbb{A}}$ defined by:

$$\Vdash_{\mathbb{A}} \overset{\circ}{\mathbb{B}} : \check{I} \longrightarrow \overset{\circ}{\mathbb{B}}; \quad \Vdash_{\mathbb{A}} \overset{\circ}{\mathbb{B}}(\check{i}) = b_i \text{ for } i \in I.$$

We then have:

$$\begin{aligned} \bigcap_{i \in I} b_i = &\text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} c = \bigcap_{i \in \check{I}} \overset{\circ}{\mathbb{B}}(i), \\ \bigcup_{i \in I} b_i = &\text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} c = \bigcup_{i \in \check{I}} \overset{\circ}{\mathbb{B}}(i), \end{aligned}$$

showing that $\mathbb{B}$ is complete. Now define $\sigma : \mathbb{A} \to \mathbb{B}$ by:

$$\sigma(a) = \text{ that } c \text{ s.t. } \Vdash_{\mathbb{A}} (a \in \overset{\circ}{G} \wedge c = 1) \vee (a \notin \overset{\circ}{G} \wedge c = 0).$$

σ is easily shown to be a complete embedding.

Clearly, if G is $\mathbb{A}$-generic, then $\sigma''G$ is $\sigma''\mathbb{A}$-generic, and $\mathbf{V}[G] = \mathbf{V}[\sigma''G]$. Set $\tilde{G} = \sigma''G$, $\tilde{\mathbb{B}} = \mathbb{B}/\tilde{G}$. We then have for $b, c \in \mathbb{B}$:

$$\begin{aligned} b/\tilde{G} \subset c/\tilde{G} \longleftrightarrow &\bigvee a \in G\ \sigma(a) \subset (\neg b \cup c) \longleftrightarrow \\ &\longleftrightarrow (\neg b^G \cup c^G) = 1 \longleftarrow b^G \subset c^G \ \text{ (since } \sigma(a)^G = 1 \text{ for } a \in G). \end{aligned}$$

Hence there is $k : \tilde{\mathbb{B}} \overset{\sim}{\leftrightarrow} \overset{\circ}{\mathbb{B}}^G$ defined by: $k(b/\tilde{G}) = b^G$. Hence:

$$\Vdash_{\mathbb{A}} (\overset{\circ}{\mathbb{B}} \text{ is isomorphic to } \check{\mathbb{B}}/G).$$

If $\mathbb{A} = \mathbb{B}_0$ and we pick $\mathbb{B}_1, \pi : \mathbb{B} \overset{\sim}{\leftrightarrow} \mathbb{B}_1$ with $\pi\sigma = \text{id}$, then $\mathbb{B}_1$ satisfies Fact 5. QED

The algebra $\mathbb{A} * \overset{\circ}{\mathbb{B}}$ constructed above is often useful.

General Iterations

It is clear from the foregoing that an n-step iteration – i.e. the result of n successive generic extensions of $\mathbf{V}$ – can be adequately described by a sequence $\langle \mathbb{B}_i \mid i < n \rangle$ s.t. $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < n$. The final model is the result of forcing with $\mathbb{B}_{n-1}$. What about transfinite iterations? At first glance it might seem that there is no such notion, but in fact we can define the notion by turning the previous analysis on its head. We define:

Definition By an *iteration* of length $\alpha > 0$ we mean a sequence $\langle \mathbb{B}_i \mid i < \alpha \rangle$ of complete BA's s.t.

- $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$.
- If $\lambda < \alpha$ is a limit ordinal, then $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_i$, i.e. there is no proper $B \subset \mathbb{B}_\lambda$ s.t. $\bigcup_{i<\lambda} \mathbb{B}_i \subset B$ and $\bigcap X, \bigcup X \in B$ for all $X \subset B$.

If G_i is $\mathbb{B}_{i+1}$-generic and $G_i = G \cap \mathbb{B}_i$, then $\mathbf{V}[G] = \mathbf{V}[G_i][\tilde{G}_i]$ where $\tilde{G}_i = \{b/G_i \mid b \in G\}$ is $\tilde{\mathbb{B}}_i = \mathbb{B}_{i+1}/G_i$-generic. If G is λ-generic for a limit λ, then $\mathbf{V}[G]$ can be regarded as a "limit" of successive $\tilde{\mathbb{B}}_i$-generic extensions, where $G_i = G \cap \mathbb{B}_i$, $\tilde{\mathbb{B}}_i = \mathbb{B}_{i+1}/G_i$ for $i < \lambda$.

In practice, we usually at the i-th stage pick a $\overset{\circ}{\mathbb{B}}_i$ s.t. $\Vdash_{\mathbb{B}_i} (\overset{\circ}{\mathbb{B}}_i$ is a complete BA), and arrange that:

$$\Vdash_{\mathbb{B}_i} (\check{\mathbb{B}}_i/\overset{\circ}{G} \text{ is isomorphic to } \overset{\circ}{\mathbb{B}}).$$

If the construction of the $\mathbb{B}_i$'s is sufficiently canonical, then the iteration is completely characterized by the sequence of $\overset{\circ}{\mathbb{B}}_i$'s. However, our definition of "iteration" gives us great leeway in choosing $\mathbb{B}_\lambda$ for limit $\lambda < \alpha$. We shall make use of that freedom in these notes. Traditionally, however, a handfull of standard limiting procedures has been used. The *direct limit* takes $\mathbb{B}_\lambda$ as the minimal completion of the Boolean algebra $\bigcup_{i<\lambda} \mathbb{B}_i$. It is characterized up to isomorphism by the property that $\bigcup_{i<\lambda} \mathbb{B}_i \setminus \{0\}$ lies dense in $\mathbb{B}_\lambda$. (If $\mathbb{B}^* = \text{BA}(\bigcup_{i<\lambda} \mathbb{B}_i \setminus \{0\})$, there is then a unique isomorphism of $\mathbb{B}_\lambda$ onto $\mathbb{B}^*$ taking b to $[b]$ for $b \in \bigcup_{i<\lambda} \mathbb{B}_i \setminus \{0\}$.) Another frequently used variant is the *inverse limit*, which can be defined as follows: By a *thread* in

$\langle \mathbb{B}_i \mid i < \lambda \rangle$ we mean a $b = \langle b_i \mid i < \lambda \rangle$ s.t. $b_j \in \mathbb{B}_j \setminus \{0\}$ and $h_{\mathbb{B}_i \mathbb{B}_j}(b_j) = b_i$ for $i \leq j < \lambda$. We call $\mathbb{B}_\lambda$ an *inverse limit* of $\langle \mathbb{B}_i \mid i < \lambda \rangle$ iff

- If b is a thread, then $b^* = \bigcap_{i<\lambda} b_i \neq 0$ in $\mathbb{B}_\lambda$.
- The set of such b^* is dense in $\mathbb{B}_\lambda$.

$\mathbb{B}_\lambda$ is then characterized up to isomorphism by these conditions. (If T is the set of all threads, we can define a partial ordering of T by: $b \leq c$ iff $\bigwedge i < \lambda \; b_i \subset c_i$.) If we then set: $\mathbb{B}^* = \mathrm{BA}(T)$, there is a unique isomorphism of $\mathbb{B}_\lambda$ onto $\mathbb{B}^*$ taking b^* to $[b]$ for each thread b.)

By the *support* of a thread we mean the set of $j < \lambda$ s.t. $b_i \neq b_j$ for all $i < j$. The *countable support* (CS) limit is defined like the inverse limit using only those threads which have a countable support. A *CS iteration* is one in which $\mathbb{B}_\lambda$ is a CS limit for all limit $\lambda < \alpha$. (This is equivalent to taking the inverse limit at λ of cofinality ω and otherwise the direct limit.) Countable support iterations tend to work well if no cardinal has its cofinality changed to ω in the course of the iteration. Otherwise – e.g. if we are trying to iterate Namba forcing – we can use the *revised countable support* (RCS) *iteration*, which was invented by Shelah. The present definition is due to Donder: By an RCS *thread* we mean a thread b s.t. *either* there is $i < \lambda$ s.t. $b_i \Vdash_{\mathbb{B}_i} \mathrm{cf}(\check{\lambda}) = \omega$ *or* the support of b is bounded in λ. The RCS limit is then defined like the inverse limit, using only RCS threads. An RCS iteration is one which uses the RCS limit at all limit points.

Note Almost all iterations which have been employed to date make use of sublimits of the inverse limit – i.e. $\{b^* \mid b \text{ is a thread } \wedge b^* \neq 0\}$ is dense in $\mathbb{B}_\lambda$ for all limit λ. This means that $(\prod_{i<\lambda} \overline{\overline{\overline{\mathbb{B}}}}_i)^+$ remains regular. In these notes, however, we shall see that it is sometimes necessary to employ larger limits which do not have this consequence.

In dealing with iterations we shall employ the following conventions: If $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ is an iteration we assume the $\mathbf{V}^{\mathbb{B}_i}$ to be so constructed that $\mathbf{V}^{\mathbb{B}_i} \subseteq \mathbf{V}^{\mathbb{B}_j}$ (in the sense of our earlier definition). In particular $[[\varphi(\vec{x})]]_{\mathbb{B}_i} = [[\varphi(\vec{x}]]_{\mathbb{B}_j}$ for $x_1, \ldots, x_n \in \mathbf{V}^{\mathbb{B}_i}$, $i \leq j$, when φ is a Σ_0 formula. We shall also often simplify the notation by using the indices $i < \alpha$ as in: h_{ij} for $h_{\mathbb{B}_i \mathbb{B}_j}$, $\Vdash_i$ for $\Vdash_{\mathbb{B}_i}$, $[[\varphi]]_i$ for $[[\varphi]]_{\mathbb{B}_j}$. If $i_0 < \alpha$ and G is $\mathbb{B}_{i_0}$-generic, we set: $\mathbb{B}/G = \langle \mathbb{B}_{i_0+j}/G \mid j < \alpha - i_0 \rangle$. We can assume the factor algebras to be so defined that $\mathbb{B}_{i_0+h}/G \subseteq \mathbb{B}_{i_0+j}/G$ for $h \leq j < \alpha - i_0$. ($\tilde{\mathbb{B}} = \bigcup_{i<\alpha} \mathbb{B}_i$ is a BA. Hence we can form $\tilde{\mathbb{B}}/G$ and identify $\mathbb{B}_{i_0+j}/G$ with $\{b/G \mid b \in \mathbb{B}_{i_0+j}\}$.) It then follows easily that $\mathbb{B}/G$ is an iteration in $\mathbf{V}[G]$.

Chapter 1

Admissible sets

1.1 Introduction

Let $H = H_\omega$ be the collection of hereditarily finite sets. We use the usual Levy hierarchy of set theoretic formulae:

$$\Pi_0 = \Sigma_0 = \text{ fmlae in which all quantifiers are bounded.}$$
$$\Sigma_{n+1} = \text{ fmlae } \bigvee x\, \varphi \text{ where } \varphi \text{ is } \Pi_n.$$
$$\Pi_{n+1} = \text{ fmlae } \bigwedge x\, \varphi \text{ where } \varphi \text{ is } \Sigma_n.$$

The use of H offers an elegant way to develop ordinary recursion theory. Call a relation $R \subset H^n$ *r.e.* (or "H-r.e.") iff R is Σ_1-definable over H. We call R *recursive* (or *H-recursive*) iff it is Δ_1-definable (i.e. R and its complement $\neg R$ are Σ_1-definable). Then $R \subset \omega^n$ is rec (r.e.) in the usual sense iff it is the restriction of an H-rec. (H-r.e.) relation to ω. Moreover, there is an H-recursive function $\pi : \omega \leftrightarrow H$ s.t. $R \subset H^n$ is H-recursive iff $\{\langle x_1, \ldots, x_n\rangle \mid R(\pi(x_1), \ldots, \pi(x_n))\}$ is recursive. (Hence $\{\langle x, y\rangle \mid \pi(x) \in \pi(y)\}$ is recursive.)

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This suggests a way of relativizing the concepts of recursion theory to transfinite domains: Let $N = \langle |N|, \in, A_1, A_2, \ldots\rangle$ be a transitive structure (with finitely or infinitely many predicates). We define:

$$R \subset N^n \text{ is } N\text{-r.e. } (N\text{-rec.}) \text{ iff } R \text{ is } \underline{\Sigma}_1\ (\underline{\Delta}_1) \text{ definable over } N.$$

Since N may contain infinite sets, we must also relativize the notion "finite":

$$u \text{ is } N\textit{-finite} \text{ iff } u \in N.$$

There are, however, certain basic properties which we expect any recursion theory to possess. In particular:

- If A is recursive and u finite, then $A \cap u$ is finite.
- If u is finite and $F : u \to N$ is recursive, then $F''u$ is finite.

The transitive structures $N = \langle |N|, \in, A_1, A_2, \ldots\rangle$ which yield a satisfactory recursion theory are called *admissible*. They were characterized by Kripke and Platek as those which satisfy the following axioms:

(1) $\emptyset$, $\{x, y\}$, $\bigcup x$ are sets.
(2) The Σ_0-axiom of subsets (Aussonderung)
$x \cap \{z \mid \varphi(z)\}$ is a set, where φ is any Σ_0 formula.
(3) The Σ_0-axiom of collection
$\bigwedge x \bigvee y\, \varphi(x, y) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y)$ where φ is any Σ_0 formula.

Note Applying (3) to: $x \in u \to \varphi(x, y)$, we get:

$$\bigwedge x \in u \bigvee y\, \varphi(x, y) \longrightarrow \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y).$$

Note *Kripke-Platek set theory* (KP) consists of the above axioms together with the axiom of extensionality and the full axiom of foundation (i.e. for all formulae, not just Σ_0 ones). These latter axioms of course hold trivially in transitive domains. KPC (KP with choice) is KP augmented by: Every set is enumerable by an ordinal.

We now show that admissible structures satisfy the criteria stated above.

Lemma 1 *Let $u \in M$. Let A be $\underline{\Delta}_1(M)$. Then $A \cap u \in M$.*

Proof. Let $Ax \leftrightarrow \bigvee y\, A_0yx$, $\neg Ax \leftrightarrow \bigvee y\, A_1yx$, where A_0, A_1 are $\underline{\Sigma}_0$. Then $\bigwedge x \bigvee y(A_0yx \vee A_1yx)$. Hence there is $v \in M$ s.t. $\bigwedge x \in u \bigvee y \in v(A_0yx \vee A_1yx)$. Hence $u \cap A = u \cap \{x \mid \bigvee y \in v\, A_0yx\} \in M$. QED(Lemma 1)

Before verifying the second criterion we prove:

Lemma 2 *M satisfies:*

$$\bigwedge x \in u \bigvee y_1 \ldots y_n\, \varphi(x, \vec{y}\,) \longrightarrow \bigvee v \bigwedge x \in u \bigvee y_1 \ldots y_n \in v\, \varphi(x, \vec{y}\,)$$

for Σ_0 formulas φ.

Proof. Assume $\bigwedge x \in u \bigvee y_1 \ldots y_n\, \varphi(x, \vec{y}\,)$. Then
$\bigwedge x \bigvee w(\underbrace{x \in u \to \bigvee y_1 \ldots y_n \in w\, \varphi(x, \vec{y})}_{\Sigma_0})$. Hence there is $v' \in M$ s.t.
$\bigwedge x \in u \bigvee w \in v' \bigvee y_1 \ldots y_n \in w\, \varphi(x, \vec{y}\,)$. Take $v = \bigcup v'$. QED(Lemma 2)

Finally we get:

Lemma 3 *Let $u \in M$, $u \subset \mathrm{dom}(F)$, where F is $\underline{\Sigma}_1(M)$. Then $F''u \in M$.*

Proof. Let $y = F(x) \leftrightarrow \bigvee z\, F'zyx$, where F' is $\underline{\Sigma}_0(M)$. Since $\bigwedge x \in u \bigvee y\; y = F(x)$, there is v s.t. $\bigwedge x \in u \bigvee y, z \in v\, F'zyx$. Hence $F''u = v \cap \{y \mid \bigvee x \in u \bigvee z \in v\, F'zyx\}$. QED(Lemma 3)

By similarly straightforward proofs we get:

Lemma 4 *If $Ry\vec{x}$ is Σ_1, so is $\bigvee y\, Ry\vec{x}$.*

Lemma 5 *If $Ry\vec{x}$ is Σ_1, so is $\bigwedge y \in u\, Ry\vec{x}$ (since $\bigwedge y \in u \bigvee z\, \varphi(y, z) \leftrightarrow \bigvee v \underbrace{\bigwedge y \in v \bigvee z \in v\, \varphi(y, z)}_{\Sigma_0}$).*

Lemma 6 *If $R, Q \subset M^n$ are Σ_1, then so are $R \cup Q$, $R \cap Q$.*

Lemma 7 *If $R'(y_1, \ldots, y_n)$ is Σ_1 and $f(x_1, \ldots, x_m)$ is a Σ_1 function for $i = 1, \ldots, n$, then $R(f_1(\vec{x}), \ldots, f_n(\vec{x}))$ is Σ_1.*

Proof. $R(\vec{f}(\vec{x})) \leftrightarrow \bigvee y_1 \ldots y_n (\bigwedge_{i=1}^{n} y_i = f_i(\vec{x}) \wedge R(\vec{y}))$. QED(Lemma 7)

Note The boldface versions of Lemmas 4–7 follow immediately.

Corollary 8 *If the functions $f(z_1, \ldots, z_n)$, $g_i(\vec{x})$ $(i = 1, \ldots, n)$ are Σ_1 in a parameter p, then so is $h(\vec{x}) \simeq f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$.*

Lemma 9 *The following functions are Δ_1: $\bigcup x$, $x \cup y$, $x \cap y$, $x \setminus y$ (set difference), $\{x_1, \ldots, x_n\}$, $\langle x_1, \ldots, x_n \rangle$, $\mathrm{dom}(x)$, $\mathrm{rng}(x)$, $x''y$, $x \restriction y$, x^{-1}, $x \times y$, $(x)_i^n$, where: $(\langle z_0, \ldots, z_{n-1} \rangle)_i = z_i$; $(u)_i^n = \emptyset$ otherwise;*

$$x[z] = \begin{cases} x(z) & \text{if } x \text{ is a function and } z \in \mathrm{dom}(x), \\ \emptyset & \text{if not.} \end{cases}$$

Note As a corollary of Lemma 3 we have: If f is $\underline{\Sigma}_1$, $u \in M$, $u \subset \mathrm{dom}(f)$. Then $f \restriction u \in M$, since $f \restriction u = g''u$, where $g(x) \simeq \langle f(x), x \rangle$.

Lemma 10 *If $f : M^{n+1} \to M$ is Σ_1 in the parameter p, then so are:*

$$F(u, \vec{x}) = \{f(z, \vec{x}) \mid z \in u\}, \quad F'(u, \vec{x}) = \langle f(z, \vec{x}) \mid z \in u \rangle.$$

Proof. $y = F(u, \vec{x}) \leftrightarrow \bigwedge z \in y \bigvee v \in u\ z = f(y, \vec{z}) \wedge \bigwedge v \in u \bigvee z \in y\ z = f(y, \vec{x})$.
But $F'(u, \vec{x}) = \{f'(z, \vec{x}) \mid z \in u\}$, where $f'(y, \vec{x}) = \langle f(y, \vec{x}), \vec{x} \rangle$. QED(Lemma 10)

(**Note** The proof of Lemma 10 shows that, even if f is not defined everywhere, F is Σ_1 in p, where:

$$F(u, \vec{x}) \simeq \{f(y, \vec{x}) \mid y \in u\},$$

where this equation means that $F(u, \vec{x})$ is defined and has the displayed value iff $f(y, \vec{x})$ is defined for all $y \in u$. Similarly for F'.)

Lemma 11 (Set Recursion Theorem)
Let G be an $n+2$-ary Σ_1 function in the parameter p. Then there is F which is also Σ_1 in p s.t.

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) \mid z \in y \rangle)$$

(where this equation means that F is defined with the displayed value iff $F(z, \vec{x})$ is defined for all $z \in y$ and G is defined at $\langle y, \vec{x}, \langle F(z, \vec{x}) \mid z \in y \rangle \rangle$.)

Proof. Set $u = F(y, \vec{x}) \leftrightarrow \bigvee f (\varphi(f, \vec{x}) \wedge \langle u, y \rangle \in f)$, where

$$\varphi(f, \vec{x}) \longleftrightarrow (f \text{ is a function } \wedge \bigcup \mathrm{dom}(f) \subset \mathrm{dom}(f) \wedge \\ \wedge \bigwedge y \in \mathrm{dom}(f)\ f(y) = G(y, \vec{x}, f \restriction y)).$$

The equation is verified by $\in$-induction on y. QED(Lemma 11)

Corollary 12 *TC, rn are Δ_1 functions, where*

$$\mathrm{TC}(x) = \text{ the transitive closure of } x = x \cup \bigcup_{z \in x} \mathrm{TC}(z),$$
$$\mathrm{rn}(x) = \text{ the rank of } \mathrm{lub}\{\mathrm{rn}(z) \mid z \in x\}.$$

Lemma 13 *ω, $On \cap M$ are Σ_0 classes.*

Proof. $x \in On \leftrightarrow (\bigcup x \subset x \wedge \bigwedge z, w \in x(z \in w \vee w \in z))$,
$x \in \omega \leftrightarrow (x \in On \wedge \neg\mathrm{Lim}(x) \wedge \bigwedge y \in x \neg\mathrm{Lim}(y))$, where $\mathrm{Lim}(x) \leftrightarrow (x \neq 0 \wedge x \in On \wedge x = \bigcup x)$.

Corollary 14 *The ordinal functions $\alpha + 1, \alpha + \beta, \alpha \cdot \beta, \alpha^\beta, \ldots$ are Δ_1.*

An even more useful version of Lemma 11 is

Lemma 15 *Let G be as in Lemma 11. Let $h : M \to M$ be Σ_1 in p s.t. $\{\langle x, z\rangle \mid x \in h(z)\}$ is well founded. There is F which is Σ_1 in p s.t.,*

$$F(y, \vec{x}\,) \simeq G(y, \vec{x}, \langle F(z, \vec{x}\,) \mid z \in h(y)\rangle).$$

The proof is just as before. We also note:

Lemma 16.1 *Let $u \in H_\omega$. Then the class u and the constant function $f(x) = u$ are Σ_0.*

Proof. $\in$-induction on u: $x \in u \leftrightarrow \bigvee_{z \in u} x = z,\ x = u \leftrightarrow (\bigwedge z \in x\ z \in u \wedge \bigwedge_{z \in u} z \in x)$.
QED

Lemma 16.2 *If $\omega \in M$, then the constant function $x = \omega$ is Σ_0.*

Proof. $x = \omega \leftrightarrow (\bigwedge z \in x\ z \in \omega \wedge \emptyset \in x \wedge \bigwedge z \in x\ z \cup \{z\} \in x)$.

Lemma 16.3 *If $\omega \in M$, the constant for $x = H_\omega$ is Σ_1 (hence Δ_1).*

Proof. $x = H_\omega \leftrightarrow (\bigwedge z \in x \bigvee u \bigvee f \bigvee n \in \omega(\bigcup n \subset u \wedge x \subset u \wedge f : n \leftrightarrow x)) \wedge \emptyset \in x \wedge \bigwedge z, w \in x(\{z, w\}, z \cup w \in x)$.

Lemma 17 *Fin, $\mathfrak{P}_\omega(x)$ are Δ_1, where* $\mathrm{Fin} = \{x \in M \mid \overline{\overline{x}} < \omega\}$, $\mathfrak{P}_\omega(x) = \mathrm{Fin} \cap \mathfrak{P}(x)$.

Proof. $x \in \mathrm{Fin} \leftrightarrow \bigvee n \in \omega \bigvee f \text{ fin} \leftrightarrow x$,
$x \notin \mathrm{Fin} \leftrightarrow \bigvee y(y = \omega \wedge \bigwedge n \in y \bigvee f \bigvee n \subset x \text{ fin} \leftrightarrow n)$,
$y = \mathfrak{P}_\omega(x) \leftrightarrow \bigwedge u \in y(u \in \mathrm{Fin} \wedge u \subset x) \wedge \bigwedge z \in x\,(\{z\} \in y \wedge \bigwedge u, v \in y\ u \cup v \in y)$
QED

The constructible hierarchy relative to a class A is defined by:

$$L_0[A] = \emptyset; \quad L_{\nu+1}[A]d = \mathrm{Def}(\langle L_\nu[A], A \cap L_\nu[A]\rangle)$$
$$L_\lambda[A] = \bigcup_{\nu < \lambda} L_\nu[A] \quad \text{ for limit } \lambda,$$

where $\text{Def}(\mathfrak{A})$ is the set of $B \subset \mathfrak{A}$ which are $\mathfrak{A}$-definable in parameters from $\mathfrak{A}$. We also define $L_\nu = L_\nu[\emptyset]$.

The constructible hierarchy *over a set* u is defined by:

$$L_0(u) = \text{TC}(\{u\}), \quad L_{\nu+1}(u) = \text{Def}(L_\nu(u)),$$
$$L_\lambda(u) = \bigcup_{\nu<\lambda} L_\nu(u) \quad \text{for limit } \lambda.$$

It is easily seen that:

Lemma 18 *If $A \subset M$ is $\Delta_1(M)$ in p, then $\langle L_\nu[A] \mid \nu \in M\rangle$ is $\Delta_1(M)$ in p.*

- *If $u \in M$, then $\langle L_\nu(u) \mid \nu \in M\rangle$ is $\Delta_1(M)$ in u.*

By set recursion we can also define a sequence $\langle <^A_\nu \mid \nu < \infty\rangle$ s.t.

- *$<^A_\nu$ well orders $L_\nu[A]$.*
- *$<^A_\mu$ end extends $<^A_\nu$ for $\nu \le \mu$.*

Then:

Lemma 19 *If $A \in M$ is $\Delta_1(M)$ in p, then $\langle <^A_\nu \mid \nu \in M\rangle$ is $\Delta_1(M)$ in p.*

Definition $L^A_\nu = \langle L_\nu[A], A \cap L_\nu[A]\rangle$.

$$\langle L^A_\nu, B_1, B_2, \ldots\rangle = \langle L_\nu[A], A \cap L_\nu[A], B_1, B_2, \ldots\rangle.$$

It follows easily that:

Lemma 20 *Let $M = \langle L^A_\alpha, B_1, \ldots\rangle$ be admissible. Then $<_M =_{\text{Df}} \bigcup_{\nu<\alpha} <^A$ is a $\Delta_1(M)$ well ordering of M. Moreover, there is a $\Delta_1(M)$ map $h : M \to M$ s.t. $h(x) = \{z \mid z <_M x\}$.*

Using this, it follows easily that every $\Sigma_1(M)$ relation is uniformizable by a $\Sigma_1(M)$ function.

Thus the KP axioms give us a "reasonable" recursion theory. They do not suffice, however, to get Σ_1-uniformization. In fact, since we have not posited the axiom of choice, we do not even have N-finite uniformization. However, the admissible structures dealt with in these notes will almost always satisfy Σ_1-uniformization. This can happen in different ways. If $N = L^A_\tau =_{\text{Df}} \langle L_\tau[A], A\rangle$, there is a well ordering $<$ of N s.t. the function $h(x) = \{z \mid z < x\}$ is Σ_1. We can then uniformize $R(y, \vec{x}\,)$ as follows: Let $R(y, \vec{x}\,) \leftrightarrow \bigvee z\, R'(y, z, \vec{x}\,)$, where R' is Σ_0. R is then uniformized by:

$$\bigvee z(R'(y, z, \vec{x}\,) \wedge \bigwedge \langle y', z'\rangle \in h(\langle y, z\rangle) \neg R(u', z'\vec{x}\,)).$$

The same holds for $N = L_\tau(a)$ where a is a transitive set with a well ordering constructible from a below τ. If N is a ZFC^- model with a definable well ordering $<$, then every definable relation has a definable uniformization. If $N^* = \langle N, A_1, A_2, \ldots\rangle$ is the result of adding all N-definable predicates to N, then the $\Sigma_1(N^*)$ relations are exactly the N-definable relations, so uniformization holds trivially.

1.2 Ill founded ZF$^-$ models

We now prove a lemma about arbitrary (possibly ill founded) models of ZF$^-$ (where the language of ZF$^-$ may contain predicates other than '$\in$'). Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \ldots\rangle$ be such a model. For $X \subset A$ we of course write $\mathfrak{A}|X = \langle X, \in_{\mathfrak{A}} \cap X^2, \ldots\rangle$. By the *well founded core* of $\mathfrak{A}$ we mean the set of all $x \in A$ s.t. $\in_{\mathfrak{A}} \cap\, \mathcal{C}(x)^2$ is well founded, where $\mathcal{C}(x)$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$. Let wfc($\mathfrak{A}$) denote the restriction of $\mathfrak{A}$ to its well founded core. Then wfc($\mathfrak{A}$) is a well founded structure satisfying the axiom of extensionality, and is, therefore, isomorphic to a transitive structure. Hence there is $\mathfrak{A}'$ s.t. $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ and wfc($\mathfrak{A}'$) is transitive. We say that a model $\mathfrak{A}$ of ZF$^-$ is *solid* iff wfc($\mathfrak{A}$) is transitive and $\in_{\mathrm{wfc}(\mathfrak{A})} = \in \cap \mathrm{wfc}(\mathfrak{A})^2$. Thus every consistent set of sentences in ZF$^-$ has a solid model. Note that if $\mathfrak{A}$ is solid, then $\omega \subset \mathrm{wfc}(\mathfrak{A})$. By Σ_0-absoluteness we of course have:

(1) $\qquad \mathrm{wfc}(\mathfrak{A}) \models \varphi(\vec{x}\,) \longleftrightarrow \mathfrak{A} \models \varphi(\vec{x}\,)$

if $x_1, \ldots, x_n \in \mathrm{wfc}(\mathfrak{A})$ and φ is a Σ_0-formula. By $\in$-induction on $x \in \mathrm{wfc}(\mathfrak{A})$ it follows that the rank function is absolute:

(2) $\qquad \mathrm{rn}(x) = \mathrm{rn}^{\mathfrak{A}}(x) \quad \text{for} \quad x \in \mathrm{wfc}(\mathfrak{A})$.

Using this we prove:

Lemma 21 *Let $\mathfrak{A}$ be a solid model of* ZF$^-$. *Then* wfc($\mathfrak{A}$) *is admissible.*

Proof. Let φ be Σ_0 and let

(3) $\qquad \mathrm{wfc}(\mathfrak{A})) \models \bigwedge x \bigvee y\, \varphi(x, y, \vec{z}\,)$

where $z_1, \ldots, z_n \in \mathrm{wfc}(\mathfrak{A})$. Let $u \in \mathrm{wfc}(\mathfrak{A})$. By (3) and Σ_0 absoluteness:

(4) $\qquad \mathfrak{A} \models \bigwedge x \in u \bigvee y\, \varphi(x, y, \vec{z}\,)$.

Since $\mathfrak{A}$ is a ZFC$^-$ model, there must then be $v \in \mathfrak{A}$ of minimal $\mathfrak{A}$-rank $\mathrm{rn}^{\mathfrak{A}}(v)$ s.t.

(5) $\qquad \mathfrak{A} \models \bigwedge x \in u \bigvee y \in v\ \varphi(x, y, \vec{z}\,)$.

It suffices to note that $\mathrm{rn}^{\mathfrak{A}}(v) \in \mathrm{wfc}(\mathfrak{A})$, hence $\mathrm{rn}^{\mathfrak{A}}(v) = \mathrm{rn}(v)$ and $v \in \mathrm{wfc}(\mathfrak{A})$. (Otherwise there is $r \in \mathfrak{A}$ s.t. $\mathfrak{A} \models r < \mathrm{rn}(v)$ and there is $v' \in \mathfrak{A}$ s.t. $\mathfrak{A} \models v' = \{x \in v \mid \mathrm{rn}(x) < r\}$. Hence v' satisfies (5) and $\mathrm{rn}^{\mathfrak{A}}(v') < \mathrm{rn}^{\mathfrak{A}}(v)$. Contradiction!) By Σ_0 absoluteness, then:

(6) $\qquad \mathrm{wfc}(\mathfrak{A}) \models \bigwedge x \in u \bigvee y \in v\ \varphi(x, y, \vec{z}\,)$.

QED (Lemma 21)

As immediate corollaries we have:

Corollary 21.1 *Let $\delta = On \cap \mathrm{wfc}(\mathfrak{A})$. Then $L_\delta(a)$ is admissible for $a \in \mathrm{wfc}(\mathfrak{A})$.*

Corollary 21.2 *$L^A_\delta = \langle L_\delta[A], A \cap L_\delta[A]\rangle$ admissible whenever A is $\mathfrak{A}$-definable.*

(*Proof.* We may suppose w.l.o.g. that A is one of the predicates of $\mathfrak{A}$.)

Note In Lemma 21 we can replace ZF^- by KP. In this form it is known as *Ville's Lemma*. However, a form of Lemma 21 was first employed in our paper [NA] with Harvey Friedman. If memory serves us, the idea was due to Friedman.

1.3 Primitive Recursive Set Functions

A function $f : \mathbf{V} \to \mathbf{V}$ is called *primitive recursive* (pr) iff it is generated by successive applications of the following schemata:

(i) $f(\vec{x}\,) = x_i$ (here $\vec{x}$ is $x_1, \ldots, x_n$)
(ii) $f(\vec{x}\,) = \{x_i, x_j\}$
(iii) $f(\vec{x}\,) = x_i \setminus x_j$
(iv) $f(\vec{x}\,) = g(h_1(\vec{x}\,), \ldots, h_m(\vec{x}\,))$
(v) $f(y, \vec{x}\,) = \bigcup\limits_{z \in y} g(z, \vec{x}\,)$
(vi) $f(y, \vec{x}\,) = g(y, \vec{x}, \langle f(z, \vec{x}\,) \mid z \in y \rangle)$

We call $A \subset \mathbf{V}^n$ a pr *relation* iff its characteristic function is a pr function. (*However*, a function can be a pr relation without being a pr function.) pr functions are ubiquitous. It is easily seen for instance that the functions listed in Lemma 9 are pr. Lemmas 4–7 hold with 'Σ_1' replaced by 'pr'. The functions $\mathrm{TC}(x)$, $\mathrm{rn}(x)$ are easily seen to be pr. We call $f : On^n \to \mathbf{V}$ a pr function if it is the restriction of a pr function to On. The functions $\alpha + 1, \alpha + \beta, \alpha \cdot \beta, \alpha^\beta, \ldots$ etc. are then pr.

Since the pr functions are proper classes, the above discussion is carried out in second order set theory. However, all that needs to be said about pr functions can, in fact, be adequately expressed in ZFC. To do this we talk about pr *definitions*:

By a pr definition we mean a finite list of schemata of the form (i)–(vi) s.t.

- the function variable on the left side does not occur in a previous equation in the list.
- every function variable on the right side occurs previously on the left side.

Clearly, every pr definition s defines a pr function F_s. Moreover, for each s, F_s has a *canonical* Σ_1 *definition* $\varphi_s(y, x_1, \ldots, x_n)$. (Indeed, the relation $\{\langle x, s\rangle \mid x \in F_s\}$ is Σ_1.) The canonical definition has some remarkable absoluteness properties. If u is transitive, let F_s^u be the function obtained by relativizing the canonical definition to u. Hence $F_s^u \subset F_s$ is a partial map on u. Then:

- If u is pr closed, then $F_s^u = F_s \cap u$.
- If α is closed under the functions $\nu + 1, \nu \cdot \tau, \nu^\tau, \ldots$ etc., then $L_\alpha[A]$ is pr closed for every $A \subset \mathbf{V}$.

These facts are provable in ZFC^-. The proofs can be found in [AS] or [PR] As corollaries we get:

(1) *Let* $\mathbf{V}[G]$ *be a generic extension of* $\mathbf{V}$. *Then* $\mathbf{V} \cap F_s^{\mathbf{V}[G]} = F_s^{\mathbf{V}}$.
(2) *Let* $\mathfrak{A}$ *be a solid model of* ZFC^-. *Let* $A = \mathrm{wfc}(\mathfrak{A})$. *Then*

$$F_s^{\mathfrak{A}} \cap A = F_s^A = F_s.$$

Proof. We prove (2). Clearly $F_s^A = F_s$, since A being admissible, is pr closed. But each $x \in A$ is an element of a transitive pr closed $u \in A$, since A is admissible. Hence $y = F_s^{\mathfrak{A}}(x) \leftrightarrow y = F_s^u(x) \leftrightarrow y = F_s^A(x)$. QED

Chapter 2

Barwise Theory

Jon Barwise worked out the syntax and model theory of certain infinitary (but M-finitary) languages on countable admissible structures M. In so doing, he created a powerful and flexible tool for set theorists, which enables us to construct transitive structures using elementary model theory. In this chapter we give an introduction to Barwise' work, whose potential for set theory has, we feel, been unduly neglected.

Let M be admissible. Barwise develops a first order theory in which arbitrary M-finite conjunctions and disjunctions are allowed. The predicates, however, have only a (genuinely) finite number of argument places and there are no infinite strings of quantifiers. If we wish to make use of the notion of M-finiteness, we must "arithmetize" the language – i.e. identify its symbols with objects in M. A typical arithmetization is:

Predicates: $P^n_x = \langle 0, \langle n, x\rangle\rangle$ $(x \in M,\ 1 \leq n < \omega)$
($P^n_x =$ the x-th n-place predicate)

Constants: $c_x = \langle 1, x\rangle$ $(x \in M)$

Variables: $v_x = \langle 2, x\rangle$ $(x \in M)$

Note The set of variables must be M-infinite, since otherwise a single formula could exhaust all the variables. We let P^2_0 be the identity predicate ($\dot{=}$) and also reserve P^2_1 as the $\in$-predicate ($\dot{\in}$), which will be a part of most interesting languages.

By a *primitive formula* we mean $Pt_1 \ldots t_n = \langle 3, \langle P, t_1, \ldots, t_n\rangle\rangle$, where P is an n-place predicate and $t_1, \ldots, t_n$ are variables and constants. We then define:

$$\neg\varphi = \langle 4, \varphi\rangle, \quad (\varphi \vee \psi) = \langle 5, \langle \varphi, \psi\rangle\rangle, \quad (\varphi \wedge \psi) = \langle 6, \langle \varphi, \psi\rangle\rangle,$$
$$(\varphi \to \psi) = \langle 7, \langle \varphi, \psi\rangle\rangle, \quad (\varphi \leftrightarrow \psi) = \langle 8, \langle \varphi, \psi\rangle\rangle, \quad \bigwedge v\, \varphi = \langle 9, \langle v, \varphi\rangle\rangle,$$
$$\bigvee v\, \varphi = \langle 10, \langle v, \varphi\rangle\rangle, \quad \text{and: } \mathbb{W} f = \langle 11, f\rangle, \quad \mathbb{M} f = \langle 12, f\rangle.$$

The set Fml of 1-st order *M-formulas* is the smallest set X which contains all primitive formulae, is closed under $\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$, and s.t.

- If v is a variable and $\varphi \in X$, then $\bigwedge v\, \varphi, \bigvee v\, \varphi \in X$.

- If $f = \langle \varphi_i \mid i \in I \rangle \in M$ and $\varphi_i \in X$ for $i \in I$, then $\bigvee\!\!\!\bigvee_{i \in I} \varphi_i =_{\mathrm{Df}} \bigvee\!\!\!\bigvee f$ and $\bigwedge\!\!\!\bigwedge_{i \in I} \varphi_i =_{\mathrm{Df}} \bigwedge\!\!\!\bigwedge f$ are in I.

Then the usual syntactical notions are Δ_1, including: Fml, Cnst (set of constants), Vbl (set of variables), Sent (set of all sentences), $\mathrm{Fr}(\varphi) =$ the set of free variables in φ, and: $\varphi({}^{v_1,\ldots,v_n}/{}_{t_1,\ldots,t_n}) \simeq$ the result of replacing all free occurences of the vbl v_i by t_i (where $t_i \in \mathrm{Vbl} \cup \mathrm{Const}$), as long as this can be done without any new occurence of a variable t_i being bound; otherwise undefined.

That Vbl, Const are Δ_1 (in fact Σ_0) is immediate. The characteristic function $\mathcal{X}$ of Fml is definable by a recursion of the form:

$$\mathcal{X}(x) = G(x, \langle \mathcal{X}(z) \mid z \in \mathrm{TC}(x) \rangle).$$

Similarly for the functions $\mathrm{Fr}(\varphi)$ and $\varphi({}^{\vec{v}}/{}_{\vec{t}})$. Then $\mathrm{Sent} = \{\varphi \mid \mathrm{Fr}(\varphi) = \emptyset\}$.

Note We of course employ the usual notation, writing $\varphi(t_1, \ldots, t_n)$ for $\varphi({}^{v_1,\ldots,v_n}/{}_{t_1,\ldots,t_n})$, where the sequence $v_1, \ldots, v_n$ is taken as known.

M-finite predicate logic has as axioms all instances of the usual predicate logical axiom schemata together with:

$$\bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i \longrightarrow \varphi_j, \qquad \varphi_j \longrightarrow \bigvee\!\!\!\bigvee_{i \in u} \varphi_i \quad \text{for } j \in u \in M.$$

The *rules of inference* are:

$$\frac{\varphi, \varphi \to \psi}{\psi} \quad \text{(modus ponens)},$$
$$\frac{\varphi \to \psi}{\varphi \to \bigwedge x\, \psi}, \quad \frac{\psi \to \varphi}{\bigvee x\, \psi \to \varphi} \quad \text{for } x \notin \mathrm{Fr}(\varphi),$$
$$\frac{\varphi \to \psi_i \ \ (i \in u)}{\varphi \to \bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i}, \quad \frac{\psi_i \to \varphi \ \ (i \in u)}{\bigvee\!\!\!\bigvee_{i \in u} \psi_i \to \varphi}.$$

We say that φ is *provable* from a set of statements A if φ is in the smallest set which contains A and the axioms and is closed under the rules of inference. We write $A \vdash \varphi$ to mean that φ is provable from A. (Note: By the last rule, $\bigvee\!\!\!\bigvee \emptyset \to \varphi$ for every φ, hence $\vdash \neg \bigvee\!\!\!\bigvee \emptyset$. Similarly $\vdash \bigwedge\!\!\!\bigwedge \emptyset$.)

A formula is provable if and only if it has a proof. Because we have not assumed choice to hold in our admissible structure M, we must use a somewhat unorthodox concept of proof, however.

Definition By a *proof from A* we mean a sequence $\langle p_i \mid i < \alpha \rangle$ s.t. $\alpha \in On$ and for each $i < \alpha$, if $\psi \in p_i$, then either $\psi \in A$ or ψ is an axiom or ψ follows from $\bigcup_{h<i} p_h$ by a single application of one of the rules.
$p = \langle p_i \mid i < \alpha \rangle$ is a *proof of* φ iff $\varphi \in \bigcup_{i<\alpha} p_i$.

If A is $\Sigma_1(M)$ in a parameter q it follows easily that $\{p \in M \mid p$ is a proof from $A\}$ is $\Sigma_1(M)$ in the same parameter. It is also easily seen that $A \vdash \varphi$ iff there exists a proof of φ from A. A more interesting conclusion is:

Lemma 1 *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is an M-finite proof of φ from A.*

Proof. $(\leftarrow)$ is trivial. We prove $(\rightarrow)$.
Let $X =$ the set of φ s.t. there exists a $p \in M$ which proves φ from A.

Claim $\{\varphi \mid A \vdash \varphi\} \subset X$.

Proof. We know that $A \subset X$ and all axioms lie in X. Hence it suffices to show that X is closed under the rules of proof. This must be demonstrated rule by rule. As an example we show:

Claim Let $\varphi \rightarrow \psi_i \in X$ for $i \in u$, where $u \in M$. Then $\varphi \rightarrow \bigwedge\!\!\!\bigwedge_{i \in u} \psi_i \in X$.

Proof. Let $P(p, \psi)$ mean: p is a proof of ψ from A. Then P is $\underline{\Sigma}_1(M)$. By our assumption:

(1) $\quad \bigwedge i \in u \ \bigvee p\, P(p, \varphi \rightarrow \psi_i)$.

Now let $P(p, \psi) \leftrightarrow \bigvee z\, P'(z, p, \psi)$, where P' is Σ_0. We then have:

(2) $\quad \bigwedge i \in u \ \bigvee z \ \bigvee p\, P'(z, p, \varphi \rightarrow \psi_i)$

whence follows easily that there is $v \in M$ with:

(3) $\quad \bigwedge i \in u \ \bigvee z \in v \ \bigvee p \in v\, P'(z, p, \varphi \rightarrow \psi_i)$.

Set $w = \{p \in v \mid \bigvee i \in u \ \bigvee z \in v\, P'(z, p, \psi)\}$. Then

(4) $\quad \bigwedge i \in u \ \bigvee p \in w\, P(p, \varphi \rightarrow \psi_i)$ and w consists of proofs from A.

Let $\alpha \in M$, $\alpha \geq \mathrm{dom}(p)$ for all $p \in w$. Define a proof p^* of length $\alpha + 1$ by:

$$p^*(i) = \begin{cases} \bigcup\{p_i \mid p \in w \wedge i \in \mathrm{dom}(p)\} & \text{for } i < \alpha, \\ \{\varphi \rightarrow \bigwedge\!\!\!\bigwedge_{i \in u} \psi_i\} & \text{for } i = \alpha. \end{cases}$$

Then $p^* \in M$ proves $\varphi \rightarrow \bigwedge\!\!\!\bigwedge_{i \in u} \psi_i$ from A. QED(Lemma 1)

From this we get the *M-finiteness lemma*:

Lemma 2 *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is $u \in M$ s.t. $u \subset A$ and $u \vdash \varphi$.*

Proof. $(\leftarrow)$ is trivial. We prove $(\rightarrow)$.
Let $p \in M$ be a proof of φ from A. Let $u =$ the set of ψ s.t. for some $i \in \mathrm{dom}(p)$, $\psi \in p_i$, but ψ is not an axiom and does not follow from $\bigcup_{h<i} p_h$ by a single application of a rule. Then $u \in M$, $u \subset A$, and p is a proof from u. Hence $u \vdash \varphi$.QED(Lemma 2)

Another consequence of Lemma 1 is

Lemma 3 *Let A be $\Sigma_1(M)$ in q. Then $\{\varphi \mid A \vdash \varphi\}$ is $\Sigma_1(M)$ in the same parameter q (uniformly in the Σ_1 definition of A from q).*

Proof. $\{\varphi \mid A \vdash \varphi\} = \{\varphi \mid \bigvee p \in M \ \ p \text{ proves } \varphi \text{ from } A\}$. QED

Corollary 4 *Let A be $\Sigma_1(M)$ in q. Then "A is consistent" is $\Pi_1(M)$ in the same parameter q (uniformly in the Σ_1 definition of A from q).*

Note that, since $u \in M$ is uniformly $\Sigma_1(M)$ in itself, we have:

Corollary 5 *$\{\langle u, \varphi\rangle \mid u \in M \wedge u \vdash \varphi\}$ is $\Sigma_1(M)$.*

Similarly:

Corollary 6 *$\{u \in M \mid u$ is consistent$\}$ is $\Pi_1(M)$.*

Note Call a proof p *strict* iff $\overline{\overline{p}}_i = 1$ for $i \in \text{dom}(p)$. This corresponds to the more usual notion of proof. If M satisfies the axiom of choice in the form: Every set is enumerable by an ordinal, then Lemma 1 holds with "strict proof" in place of "proof". We leave this to the reader.

Languages

We will normally not employ all of the predicates and constants in our M-finitary first order logic, but cut down to a smaller set of symbols which we intend to interpret in a model. Thus we define a *language* to be a set $\mathcal{L}$ of predicates and constants. By a *model* of $\mathcal{L}$ we mean a structure

$$\mathfrak{A} = \langle|\mathfrak{A}|, \langle t^{\mathfrak{A}} \mid t \in \mathcal{L}\rangle\rangle$$

s.t. $|\mathfrak{A}| \neq \emptyset$, $P^{\mathfrak{A}} \subset |\mathfrak{A}|^n$ whenever P is an n-place predicate, and $c^{\mathfrak{A}} \in |\mathfrak{A}|$ whenever $|\mathfrak{A}|$ is a constant. By a *variable assignment* we mean a map $f : \text{Vbl} \to \mathfrak{A}$ (Vbl being the set of all variables). The *satisfaction relation* $\mathfrak{A} \models \varphi[f]$ is defined in the usual way, where $\mathfrak{A} \models \varphi[f]$ means that the formula φ becomes true in $\mathfrak{A}$ if the free variables in φ are interpreted by f. We leave the definition to the reader, remarking only that:

$$\mathfrak{A} \models \bigwedge\!\!\!\bigwedge_{i\in u} \varphi_i[f] \quad \text{iff} \quad \bigwedge i \in u \ \ \mathfrak{A} \models \varphi_i[f],$$
$$\mathfrak{A} \models \bigvee\!\!\!\bigvee_{i\in u} \varphi_i[f] \quad \text{iff} \quad \bigvee i \in u \ \ \mathfrak{A} \models \varphi_i[f].$$

We adopt the ususal conventions of model theory, writing $\mathfrak{A} = \langle|\mathfrak{A}|, t_1^{\mathfrak{A}}, \ldots\rangle$ if we think of the predicates and constants of $\mathcal{L}$ as being arranged in a fixed sequence $t_1, t_2, \ldots$ Similarly, if $\varphi = \varphi(v_1, \ldots, v_n)$ is a formula in which at most the variables $v_1, \ldots, v_n$ occur free, we write: $\mathfrak{A} \models \varphi[x_1, \ldots, x_n]$ for: $\mathfrak{A} \models \varphi[f]$ where $f(v_i) = x_i$ $(i = 1, \ldots, n)$. If φ is a statement, we write: $\mathfrak{A} \models \varphi$. If A is a set of statements we write: $\mathfrak{A} \models A$ to mean: $\mathfrak{A} \models \varphi$ for all $\varphi \in A$.

The *correctness theorem* says that if A is a set of $\mathcal{L}$-statements and $\mathfrak{A} \models A$, then A is consistent. (We leave this to the reader.)

Barwise' Completeness Theorem says that the converse holds if our admissible structure M is countable:

Theorem 7 *Let M be a countable admissible structure. Let A be a set of statements in the M-language $\mathcal{L}$. If A is consistent in M-finite predicate logic, then A has a model $\mathfrak{A}$.*

Proof (sketch). We make use of the following theorem of Rasiowa and Sikorski: Let $\mathbb{B}$ be a Boolean algebra. Let $X_i \subset \mathbb{B}$ $(i < \omega)$ s.t. the Boolean union $\bigcup X_i = b_i$ exists in the sense of $\mathbb{B}$. Then $\mathbb{B}$ has an ultrafilter U s.t.

$$b_i \in U \longleftrightarrow X_i \cap U \neq \emptyset \quad \text{for} \;\; i < \omega.$$

(*Proof*. Successively choose c_i $(i < \omega)$ by $c_0 = 1$, $c_{i+1} = c_i \cap b \neq 0$, where $b \in X_i \cup \{\neg b_i\}$. Let $\overline{U} = \{a \in \mathbb{B} \mid V_i \, c_i \subset a\}$. Then $\overline{U}$ is a filter and extends to an ultrafilter on $\mathbb{B}$.)

Extend the language $\mathcal{L}$ by adding an M-infinite set C of new constants. Call the extended language $\mathcal{L}^*$ and set:

$$[\varphi] = \{\psi \mid A \vdash \psi \leftrightarrow \varphi\}$$

for $\mathcal{L}^*$-statements φ. Then

$$\mathbb{B} = \{[\varphi] \mid \varphi \in \mathrm{St}_{\mathcal{L}^*}\}$$

in the Lindenbaum algebra of $\mathcal{L}^*$ with the operations:

$$\begin{aligned}
&[\varphi] \cup [\psi] = [\varphi \vee \psi], \quad [\varphi] \cap [\psi] = [\varphi \wedge \psi], \quad \neg[\varphi] = [\neg\varphi],\\
&\bigcup_{i \in u} [\varphi_i] = \Big[\bigvee\!\!\!\bigvee_{i \in u} \varphi_i \Big] \quad (u \in M), \quad \bigcap_{i \in u} [\varphi_i] = \Big[\bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i \Big] \quad (u \in M),\\
&\bigcup_{c \in C} [\varphi(c)] = [\bigvee v \; \varphi(v)], \quad \bigcap_{c \in C} [\varphi(c)] = [\bigwedge v \; \varphi(v)].
\end{aligned}$$

The last two equations hold because the constants in C, which do not occur in the axioms A, behave like free variables. By Rasiowa and Sikorski there is then an ultrafilter U on $\mathbb{B}$ which respects the above operations. We define a model $\mathfrak{A} = \langle |\mathfrak{A}|, \langle t^{\mathfrak{A}} \mid t \in \mathcal{L} \rangle \rangle$ as follows: For $c \in C$ set $[c] = \{c' \in C \mid [c = c'] \in U\}$. If $P \in \mathcal{L}$ is an n-place predicate, set:

$$P^{\mathfrak{A}}([c_1], \ldots, [c_n]) \longleftrightarrow [Pc_1 \ldots c_n] \in U.$$

If $t \in \mathcal{L}$ is a constant set:

$$t^{\mathfrak{A}} = [c], \text{ where } \; c \in C, \quad [t = c] \in U.$$

A straighforward induction then shows:

$$\mathfrak{A} \models \varphi[[c_1], \ldots, [c_n]] \longleftrightarrow [\varphi(c_1, \ldots, c_n)] \in U$$

for formulae $\varphi = \varphi(v_1, \ldots, v_n)$ with at most the free variables $v_1, \ldots, v_n$. In particular $\mathfrak{A} \models \varphi \leftrightarrow [\varphi] \in U$ for $\mathcal{L}^*$-statements φ. Hence $\mathfrak{A} \models A$, since $[\varphi] = 1$ for all $\varphi \in A$. QED(Theorem 7)

Combining the completeness theorem with the M-finiteness lemma, we get the well known *Barwise compactness theorem*:

Corollary 8 *Let M be countable. Let $\mathcal{L}$ be $\underline{\Delta}_1$ and A be $\underline{\Sigma}_1$. If every M-finite subset of A has a model, then so does A.*

By a *theory* or *axiomatized language* we mean a pair $\mathcal{L} = \langle \mathcal{L}_0, A \rangle$ s.t. $\mathcal{L}_0$ is a language and A a set of $\mathcal{L}_0$-statements. We say that $\mathfrak{A}$ *models* $\mathcal{L}$ iff $\mathfrak{A}$ is a model of $\mathcal{L}_0$ and $\mathfrak{A} \models A$. We also write: $\mathcal{L} \vdash \varphi$ for $(A \vdash \varphi \wedge \varphi \in \mathrm{Fml}_{\mathcal{L}_0})$. We say that $\mathcal{L} = \langle \mathcal{L}_0, A \rangle$ is $\Sigma_1(M)$ (in the parameter p) iff $\mathcal{L}_0$ is $\Delta_1(M)$ (in p) and A is $\Sigma_1(M)$ (in p). Similarly for: $\mathcal{L}$ is $\Delta_1(M)$ (in p).

⋆ ⋆ ⋆ ⋆ ⋆

We now consider the class of axiomatized languages containing a fixed predicate $\dot{\in}$, the special constants $\underline{x}$ $(x \in M)$ (We can set e.g. $\underline{x} = \langle 1, \langle 0, x \rangle \rangle$.) and the *basic axioms*

- Extensionality
- $\bigwedge v(v \mathrel{\dot{\in}} \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z}) \quad (x \in M)$

(Further predicates, constants, and axioms are allowed, of course.) We call any such theory an "$\in$-theory". Then:

Lemma 9 *Let $\mathfrak{A}$ be a solid model of the $\in$-theory $\mathcal{L}$. Then $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$ for $x \in M$.*

Proof. $\in$-induction on x.

Definition Let $\mathcal{L}$ be an $\in$-theory. $\mathrm{ZF}^-_{\mathcal{L}}$ is the set of (really) finite $\mathcal{L}$-statements which are axioms of $\mathcal{L}$. (Similarly for $\mathrm{ZFC}^-_{\mathcal{L}}$.)
We write $\mathcal{L} \vdash \mathrm{ZF}^-$ for $\mathcal{L} \vdash \mathrm{ZF}^-_{\mathcal{L}}$. (Similarly for $\mathcal{L} \vdash \mathrm{ZFC}^-$.)

⋆ ⋆ ⋆ ⋆ ⋆

$\in$-theories are a suseful tool in set theory. We now bring some typical applications. We recall that an ordinal α is called *admissible* if L_α is admissible and *admissible in* $a \subset \alpha$ if $L^a_\alpha = \langle L_\alpha[a], a \rangle$ is admissible.

Lemma 10 *Let $\alpha > \omega$ be a countable admissible ordinal. There is $a \subset \omega$ s.t. α is the least ordinal admissible in a.*

This follows straightforwardly from:

Lemma 11 *Let M be a countable admissible structure. Let $\mathcal{L}$ be a consistent $\Sigma_1(M)$ $\in$-theory s.t. $\mathcal{L} \vdash \mathrm{ZF}^-$. Then $\mathcal{L}$ has a solid model $\mathfrak{A}$ s.t. $On \cap \mathrm{wfc}(\mathfrak{A}) = On \cap M$.*

We first show that Lemma 11 implies Lemma 10, and then prove Lemma 11. Take $M = L_\alpha$, where α is as in Lemma 10. Let $\mathcal{L}$ be the M-theory with:
Predicate: $\dot\in$
Constants: $\underline{x}$ $(x \in M)$, $\mathring{a}$
Axioms: Basic axioms + ZF$^-$, and $\underline{\beta}$ is not admissible in $\mathring{a}$ $(\beta < \alpha)$.

Then $\mathcal{L}$ is consistent, since $\langle H_{\omega_1}, \in, a\rangle$ is a model, where a is any $a \subset \omega$ which codes a well ordering of type $\geq \alpha$ (and $\underline{x}$ is interpretedly x for $x \in M$). Now let $\mathfrak{A}$ be a solid model of $\mathcal{L}$ s.t. $On \cap \text{wfc}(\mathfrak{A}) = \alpha$. Then wfc($\mathfrak{A}$) is admissible by Chapter 1, Lemma 21. Hence so is L^a_α, where $a = \mathring{a}^{\mathfrak{A}}$. But β is not admissible in a for $\omega < \beta < \alpha$, since "β is admissible in a" is $\Sigma_1(L^a_\alpha)$; hence the same Σ_1 statement would hold of β in $\mathfrak{A}$. Contradiction! QED(Lemma 10)

Note Pursuing this method a bit further we can prove: Let $\omega < \alpha_0 < \ldots < \alpha_{n-1}$ be a sequence of countable admissible ordinals. There is $a \subset \omega$ s.t. α_i = the i-th $\alpha > \omega$ which is admissible in a $(i < n)$. A similar theorem holds for countable infinite sequences, but the proof requires forcing and is much more complex. It is given in §5 and §6 [AS]

We now prove Lemma 11 by modifying the proof of the completeness theorem. Let $\Gamma(v)$ be the set of formulae $v \in On$, $v > \underline{\beta}$ $(\beta \in M)$. Add an M-infinite (but $\Delta_1(M)$) set E of new constants to $\mathcal{L}$. Let $\mathcal{L}'$ be $\mathcal{L}$ with the new constants and the new axioms $\Gamma(e)$ $(e \in E)$. Then $\mathcal{L}'$ is consistent, since any M-finite subset of the axioms can be modeled by interpreting the new constants as ordinals in wfc($\mathfrak{A}$), $\mathfrak{A}$ being any solid model of $\mathcal{L}$. As in the proof of completeness we then add a new class C of constants which is not M-finite. We assume, however, that C is $\Delta_1(M)$. We add no further axioms, so the elements of C behave like free variables. The so extended language $\mathcal{L}''$ is clearly $\underline{\Sigma}_1(M)$. Now set:

$$\Delta(v) = \{v \notin On\} \cup \bigcup_{\beta \in M} \{v \leq \underline{\beta}\} \cup \bigcup_{e \in E} \{e < v\}.$$

Claim Let $c \in C$. Then $\bigcup\{[\varphi] \mid \varphi \in \Delta(c)\} = 1$ in the Lindenbaum algebra of $\mathcal{L}''$.

Proof. Suppose not. Set $\Delta' = \{\neg\varphi \mid \varphi \in \Delta(c\}$. Then there is an $\mathcal{L}''$ statement ψ s.t. $A \cup \{\psi\}$ is consistent, where $\mathcal{L}'' = \langle \mathcal{L}''_0, A\rangle$ and $A \cup \{\psi\} \vdash \Delta'$. Pick an $e \in E$ which does not occur in ψ. Let A^* be the result of omitting the axioms $\Gamma(e)$ from A. Then $A^* \cup \{\psi\} \cup \Gamma(e) \vdash c \leq e$. By the M-finiteness lemma there is $\beta \in M$ s.t. $A^* \cup \{\psi\} \cup \{\underline{\beta} \leq e\} \vdash c \leq e$. But e behaves here like a free variable, so $A^* \cup \{\psi\} \vdash c \leq \underline{\beta}$. But $A \supset A^*$ and $A \cup \{\psi\} \vdash \underline{\beta} < c$. Thus $A \cup \{\psi\}$ is inconsistent. Contradiction! QED(Claim)

Now let U be an ultrafilter on the Lindenbaum algebra of $\mathcal{L}''$ which respects both the operations listed in the proof of the completeness theorem and the unions $\bigcup\{[\varphi] \mid \varphi \in \Delta(c)\}$ for $c \in C$. Let $X = \{\varphi \mid [\varphi] \in U\}$. Then as before, $\mathcal{L}''$ has a model $\mathfrak{A}$, all of whose elements have the form $c^{\mathfrak{A}}$ for a $c \in C$ and such that $\mathfrak{A} \models \varphi \leftrightarrow \varphi \in X$ for $\mathcal{L}''$-statements φ. We assume w.l.o.g. that $\mathfrak{A}$ is solid. It suffices to show that

$Y = \{x \in \mathfrak{A} \mid x > \underline{v} \text{ in } \mathfrak{A} \text{ for all } v \in m\}$ has no minimal element in $\mathfrak{A}$. Let $x \in Y$, $x = c^{\mathfrak{A}}$. Then $\mathfrak{A} \models e < c$ for some $e \in E$. But $e^{\mathfrak{A}} \in Y$. QED(Lemma 11)

Another – very typical – application is:

Lemma 12 *Let W be an inner model of* ZFC. *Suppose that, in W, U is a normal measure on κ. Let $\tau > \kappa$ be regular in W and set $M = \langle H_\tau^W, U\rangle$. Assume that M is countable in* $\mathbf{V}$. *Then for any $\alpha \leq \kappa$ there is $\overline{M} = \langle \overline{H}, \overline{U}\rangle$ s.t. $\overline{U}$ is a normal measure in $\overline{M}$ and $\overline{M}$ iterates to M in exactly α many steps.* (*Hence $\overline{M}$ is iterable, since M is*).

Proof. The case $\alpha = 0$ is trivial, so assume $\alpha > 0$. Let δ be least s.t. $L_\delta(M)$ is admissible. Then $N = L_\delta(M)$ is countable. Let $\mathcal{L}$ be the $\in$-theory on N with:

Predicate: $\dot\in$

Constants: $\underline{x}$ $(x \in N)$, $\mathring{M}$

Axioms: The basic axioms; ZFC$^-$; $\mathring{M} = \langle \mathring{H}, \mathring{U}\rangle$ is a transitive ZFC$^-$ model; $\mathring{M}$ iterates to $\underline{M}$ in $\underline{\alpha}$ many steps.

It suffices to prove:

Claim $\mathcal{L}$ is consistent.

We first show that the claim implies the theorem. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Then $N \subset \text{wfc}(\mathfrak{A})$. Hence $M, \overline{M} \in \text{wfc}(\mathfrak{A})$, where $\overline{M} = \mathring{M}^{\mathfrak{A}}$. There is $\langle \overline{M}_i \mid i < \alpha\rangle$ which, in $\mathfrak{A}$, is an iteration from $\overline{M}$ to M. But then $\langle \overline{M}_i \mid i < \alpha\rangle \in \text{wfc}(\mathfrak{A})$ really is an iteration by absoluteness. QED

We now prove the claim.

Case 1 $\alpha < \kappa$.

Iterate $\langle W, U\rangle$ α many times, getting $\langle W_i, U_i\rangle$ $(i \leq \alpha)$ with iteration maps $\pi_{ij} : \langle W_i, U_i\rangle \prec \langle W_j, U_j\rangle$. Set $M_i = \pi_{0i}(M)$. Then $\langle M_i, U_i\rangle$ $(i \leq \alpha)$ is the iteration of $\langle M, U\rangle$ with maps $\pi'_{ij} = \pi_{ij} \upharpoonright M_i$. It suffices to show that $\mathcal{L}_\alpha = \pi_{0,\alpha}(\mathcal{L})$ is consistent. This is clear, however, since $\langle H_{\tau^+}, M\rangle$ models $\mathcal{L}_\alpha$ (with M interpreting the constant $\mathring{M}_\alpha = \pi_{0,\alpha}(\mathring{M})$). QED(Case 1)

Case 2 $\alpha = \kappa$.

This time we iterate $\langle W, U\rangle$ β many times where $\pi_{0\beta}(\kappa) = \beta$ and $\beta \leq \kappa^+$. $\langle H_{\tau^+}, M\rangle$ again models $\mathcal{L}_\beta$. QED(Lemma 12)

Barwise theory is useful in situations where one is given a transitive structure Q and wishes to find a transitive structure $\overline{Q}$ with similar properties inside an inner model. Another tool used in such situations is Schoenfield's lemma, which, however requires coding $\overline{Q}$ by a real. Unsurprisingly, Schoenfield's lemma can itself be derived from Barwise theory. We first note the well known fact that every Σ^1_2 condition on a real is equivalent to a $\Sigma_1(H_{\omega_1})$ condition, and conversely. Thus it suffices to show:

Lemma 13 *Let $H_{\omega_1} \models \varphi[a]$, $a \subset \omega$, where φ is Σ_1. Then*

$$H_{\omega_1}^{L[a]} \models \varphi[a].$$

Proof. Let $\varphi = \bigvee z\, \psi$, where ψ is Σ_0. Let $H_{\omega_1} \models \psi[z, a]$, where $\text{rn}(z) < \alpha$ and α is admissible in a. Let $\mathcal{L}$ be the language on $L_\alpha(a)$ with:

Predicate: $\dot{\in}$

Constants: $\overset{\circ}{z}$, $\underline{x}$ $(x \in L_\alpha(a))$

Axioms: Basic axioms, ZFC^-, $\psi(\overset{\circ}{z}, \underline{a})$.

Then $\mathcal{L}$ is consistent since $\langle H_{\omega_1}, z\rangle$ is a model. Applying Löwenheim-Skolem in $L(a)$, we find a countable $\overline{\alpha}$ and a map $\pi : L_{\overline{\alpha}}(a) \prec L_\alpha(a)$. Let w.l.o.g. $\pi(\overset{\circ}{z}) = \overset{\circ}{z}$ and let $\overline{\mathcal{L}}$ be defined over $L_{\overline{\alpha}}(a)$ like $\mathcal{L}$ over $L_\alpha(a)$. Then $\overline{\mathcal{L}}$ is consistent and has a solid model $\mathfrak{A}$ in $L(a)$. Let $z = \overset{\circ}{z}^{\mathfrak{A}}$. Then $z \in L(a)$ and $H_{\omega_1} \models \psi[z, a]$ in $L(a)$.

QED(Lemma 13)

Chapter 3

Subcomplete Forcing

3.1 Introduction

In §10 of [PF] Shelah defines the notion of *complete forcing*:

Definition Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is a *complete forcing* iff for sufficiently large θ we have: Let $\mathbb{B} \in H_\theta$. Let $\sigma : \overline{H} \prec H$, where $\overline{H}$ is countable and transitive. Let $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$. If $\overline{G}$ is $\overline{\mathbb{B}}$-generic over $\overline{H}$, then there is $b \in \mathbb{B}$ which forces that, that whenever $G \ni b$ is $\mathbb{B}$-generic, then $\sigma''\overline{G} \subset G$.

Note If $\overline{G}$, G, $\overline{H}$, H, σ are as above, then σ extends uniquely to a σ^* s.t. $\sigma^* : \overline{H}[\overline{G}] \prec H[G]$ and $\sigma^*(\overline{G}) = G$.

Proof. To see uniqueness, note that each $x \in \overline{H}[\overline{G}]$ has the form $x = t^{\overline{G}}$ where $t \in \overline{H}$ is a $\overline{\mathbb{B}}$-name. Thus $\sigma^*(x) = \sigma(t)^G$. To see existence, note that:

$$\overline{H}[\overline{G}] \models \varphi(t_1^{\overline{G}}, \ldots, t_n^{\overline{G}}) \longleftrightarrow \bigvee b \in \overline{G}\ b \Vdash_{\overline{\mathbb{B}}}^{\overline{H}} \varphi(t_1, \ldots, t_n) \longrightarrow$$
$$\longrightarrow \bigvee b \in G \Vdash_{\mathbb{B}}^{H} \varphi(\sigma(t_1), \ldots, \sigma(t_n)) \longrightarrow H[G] \models \varphi(\sigma(t_1)^G, \ldots, \sigma(t_n)^G).$$

Hence there is $\sigma^* : \overline{H}[\overline{G}] \prec H[G]$ defined by: $\sigma^*(t^{\overline{G}}) = \sigma(t)^G$. But then $\sigma^* \supset \sigma$ since

$$\sigma^*(x) = \sigma^*(\check{x}^{\overline{G}}) = \sigma(\check{x})^G = \sigma(x)$$

for $x \in \overline{H}$. Letting $\dot{\overline{G}}$ be the $\overline{\mathbb{B}}$-generic name and $\dot{G}$ the $\mathbb{B}$-generic name we then have:

$$\sigma^*(\overline{G}) = \sigma^*(\dot{\overline{G}}^{\overline{G}}) = \dot{G}^G = G.$$

QED

Lemma 1.1 *Let $\mathbb{B}$ be a complete forcing. Let G be $\mathbb{B}$-generic. Then $\mathbf{V}[G]$ has no new countable sets of ordinals.*

Proof. Let $\Vdash f : \check{\omega} \to On$.
Claim $f^G \in \mathbf{V}$.
Suppose not. Then $b \Vdash f \notin \check{\mathbf{V}}$ for some b. Let θ be big enough and let $\sigma : \overline{H} \prec H_\theta$ s.t. $\sigma(\overline{f}, \overline{b}, \overline{B}) = f, b, \mathbb{B}$, where $\overline{H}$ is countable and transitive. Let $\overline{G} \ni \overline{b}$ be $\overline{\mathbb{B}}$-generic over $\overline{H}$. Let G be $\mathbb{B}$-generic s.t. $\sigma''\overline{G} \subset G$. Let σ^* be the above mentioned extension

of σ. Then $\sigma^*(\overline{\overline{f}^G}) = f^G$. But clearly $\sigma^*(\overline{\overline{f}^G}) = \sigma''\overline{f}^G \in \mathbf{V}$, where $b = \sigma(\overline{b}) \in G$. Contradiction! QED(Lemma 1)

We note without proof that

Lemma 1.2 *If $\mathbb{B}$ is the result of a countable support iteration of complete forcings, then $\mathbb{B}$ is complete.*

Remark In fact, the notion of complete forcing reduces to that of an ω-closed set of conditions. ($\mathbb{P}$ is called ω-closed iff whenever $\langle p_i \mid i < \omega \rangle$ is a sequence with $p_i \leq p_j$ for all $j \leq i$, then there is q with $q \leq p_i$ for all i.) It is shown in [FA] that:

Lemma 1.3 *$\mathbb{B}$ is a complete forcing iff it is isomorphic to* $\mathrm{BA}(\mathbb{P})$ *for some ω-closed set of conditions $\mathbb{P}$.*

The properties of ω-closed forcing are well known and Lemmas 1.1, 1.2 follow easily from Lemma 1.3.

The knowledgable reader will recognize the complete forcings as being a subclass of Shelah's *proper forcings*. Proper forcings satisfy Lemma 1.2 but not Lemma 1.1. In fact, many proper forcings add new reals. However, a proper forcing can never change the cofinality of an uncountable regular cardinal to ω. Thus, the notion is useless in dealing e.g. with Namba forcing. What we want is a class of forcings which do not add new reals but do permit new sets of ordinals – even to the extent of changing cofinalities. We of course want these forcings to be iterable – i.e. some reasonable analogue of Lemma 1.2 should hold. The proof of Lemma 1.1 gives us a clue as to how such a class might be defined: The proof depends strongly on the fact that $\sigma''\overline{G} \subset G$ for a $\sigma \in \mathbf{V}$. Instead, we might require that, if $\overline{H}$, σ, θ, $\mathbb{B}$, $\overline{G}$ are as in the definition of "completed forcing", then there is $b \in \mathbb{B}$ which forces that, if $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma' \in \mathbf{V}[G]$ s.t. $\sigma' : \overline{H} \prec H_\theta$, $\sigma'(\overline{\mathbb{B}}) = \mathbb{B}$ and $\sigma'\,''\overline{G} \subseteq G$. We can even require b to force $\sigma'(\overline{s}) = \sigma(\overline{s})$ for an arbitrarily chosen $\overline{s} \in \overline{H}$. If we now try to carry out the proof of Lemma 1 with a $\sigma' : \overline{H} \prec H_\theta$ s.t. $\sigma'(\overline{f}, \overline{b}, \overline{\mathbb{B}}) = f, b, \mathbb{B}$, in place of σ, we can conclude only that $f^G = \sigma'\,''\overline{\overline{f}^G}$. Since we do not know that $\sigma' \in \mathbf{V}$, we cannot conclude that $f^G \in \mathbf{V}$. However, if we assume $\Vdash f : \omega \to \omega$, then $f^G = \sigma'\,''\overline{\overline{f}^G}$, where $\overline{\overline{f}^G} \in \mathbf{V}$ and $\overline{\overline{f}^G} \subset \omega^2$. Since $\sigma' \restriction \omega = \mathrm{id}$, we can then conclude that $f^G \in \mathbf{V}$.

Thus such forcings will add no reals, but may permit us to add new countable sets of ordinals.

In order to carry out this program we must address several difficulties, the first being this: Suppose that H_θ has definable Skolem functions. (This is certainly the case if $\mathbf{V} = L$.) We could then form $\sigma : \overline{H} \prec H_\theta$ s.t. $\sigma(\overline{b}, \overline{f}, \overline{\mathbb{B}}) = b, f, \mathbb{B}$ simply by transitivizing the Skolem closure of $\{b, f, \mathbb{B}\}$. But then σ is the *only possible* elementary map to H_θ with $\sigma(\overline{b}, \overline{f}, \overline{\mathbb{B}}) = b, f, \mathbb{B}$. Thus we perforce have: $\sigma' = \sigma$. In order to avoid this we must place a stronger condition on $\overline{H}$ which implies the possibility of many maps to the top. We shall define such a condition for the case that $\overline{H}$ is a ZFC^--model.

Definition Let N be transitive. N is *full* iff $\omega \in N$ and there is γ s.t. $L_\gamma(N)$ models ZFC^- and N is *regular* in $L_\gamma(N)$ – i.e. if $f : x \to N$, $x \in N$, $f \in L_\gamma(N)$, then $\mathrm{rng}(f) \in N$.

It follows that N itself is a ZFC^- model. In fact, regularity in $L_\gamma(N)$ is equivalent to saying that N models 2nd order ZFC^- in $L_\gamma(N)$.

If $\overline{N}$ is full and $\sigma : \overline{N} \prec N$, then there will, indeed, be many different maps $\sigma' : \overline{N} \prec N$. Since fullness is a property of ZFC^- models, however, we shall have to reformulate Shelah's definition so that we do not work directly with H_θ but rather with ZFC^- models containing H_θ. It also turns out that, in order to prove iterability, we must apparently impose a stronger similarity between σ' and σ than we have hitherto stated. In order to formulate this we define:

Definition Let $\mathbb{B}$ be a complete BA.

$$\delta(\mathbb{B}) = \text{ the smallest cardinality of a set which lies dense in } \mathbb{B} \setminus \{0\}.$$

Note If we were working with sets $\mathbb{P}$ of conditions rather than complete BA's, we would normally choose $\mathbb{P}$ to have cardinality $\delta(BA(\mathbb{P}))$. Hence the above definition would be superfluous and we would work with $\overline{\overline{\mathbb{P}}}$ instead.

Definition Let $N = L_\tau^A =_{\mathrm{Df}} \langle L_\tau[A], \in, A \cap L_\tau[A] \rangle$ be a ZFC^- model. Let $X \cup \{\delta\} \subset N$.

$$C_\delta^N(X) =_{\mathrm{Df}} \text{ the smallest } Y \prec N \text{ s.t. } X \cup \delta \subset Y.$$

We are now ready to define:

Definition Let $\mathbb{B}$ be a complete BA. $\mathbb{B}$ is a *subcomplete forcing* iff for sufficiently large cardinals θ we have: $\mathbb{B} \in H_\theta$ and for any ZFC^- model $N = L_\tau^A$ s.t. $\theta < \tau$ and $H_\theta \subset N$ we have: Let $\delta : \overline{N} \prec N$ where $\overline{N}$ is countable and full. Let $\sigma(\overline{\theta}, \overline{s}, \overline{\mathbb{B}}) = \theta, s, \mathbb{B}$ where $\overline{s} \in \overline{N}$. Let $\overline{G}$ be $\overline{\mathbb{B}}$-generic over $\overline{N}$. Then there is $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic over $\mathbf{V}$, there is $\sigma' \in \mathbf{V}[G]$ s.t.

(a) $\sigma' : \overline{N} \prec N$,
(b) $\sigma'(\overline{\theta}, \overline{s}, \overline{\mathbb{B}}) = \theta, s, \mathbb{B}$,
(c) $C_\delta^N(\mathrm{rng}(\sigma')) = C_\delta^N(\mathrm{rng}(\sigma))$ where $\delta = \delta(\mathbb{B})$,
(d) $\sigma' {}'' \overline{G} \subset G$.

(Hence σ' extends uniquely to a $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$ s.t. $\sigma^*(\overline{G}) = G$.)

Note We define $N[G]$ in such a way that A is still a predicate. Thus $N = L_\tau^A$ is $N[G]$-definable.

Note This is expressible in $\mathbf{V}$, since the last part can be expressed as:

$$\bigvee b \in \mathbb{B} \;\; b \Vdash \varphi(\check{\mathbb{B}}, \check{\overline{N}}, \check{N}, \mathring{\sigma}, \overline{G}, \mathring{G}),$$

$\mathring{G}$ being the generic name.

Note If we omitted (c) from the definition of subcompleteness, the resulting class of forcings would still satisfy Lemma 1.2 for countable support iterations of length $\leq \omega_2$. Since such forcings might change the cofinality of ω_2 to ω, we would thereafter have to use the revised countable support (RCS) iteration. (We will also have to make some further assumptions on the component forcings $\mathbb{B}_i$ of the iteration $\langle \mathbb{B}_i \mid i < \alpha \rangle$.) (c) appears to be needed to get past regular limits points λ of the iteration s.t. $\lambda > \delta(\mathbb{B}_i)$ for $i < \lambda$.

Definition θ *verifies* the subcompleteness of $\mathbb{B}$ iff θ is as in the definition of subcompleteness.

In the following discussion write 'ver$(\mathbb{B}, \theta)$' to mean 'θ verifies the subcompleteness of $\mathbb{B}$'. Now let $\mathbb{B} \in H_\theta$ and let $\theta' > \overline{\overline{H}}_\theta$ be a cardinal. A Löwenheim-Skolem argument that, in order to determine whether ver$(\mathbb{B}, \theta)$, we need only consider $N = L_\tau^A$ s.t. $N \in H_{\theta'}$. By the well known fact: $H_{\theta'}[G] = (H_{\theta'})^{\mathbf{V}[G]}$ for $\mathbb{B}$-generic G, where $\mathbb{B} \in H_{\theta'}$, we see that, in fact, the definition of ver$(\mathbb{B}, \theta)$ relativizes to $H_{\theta'}$ – i.e.

Lemma 2.1 *Let $\mathbb{B} \in H_\theta$. Let $\theta' > \overline{\overline{H}}_\theta$ be a cardinal. The statement* ver$(\mathbb{B}, \theta)$ *is absolute in $H_{\theta'}$.*

This holds in particular for $\theta' = (\overline{\overline{H}}_\theta)^+$. But then the elements of $H_{\theta'}$ can be coded by subsets of H_θ and we get:

Lemma 2.2 *Let $\theta > \omega_1$ be a cardinal.* $\{\mathbb{B} \mid \text{ver}(\mathbb{B}, \theta)\}$ *is uniformly* 2*nd order definable over H_θ.*

Hence:

Corollary 2.3 *Let W be an inner model s.t. $\mathfrak{P}(H_\theta) \subset W$. Then* ver$(\mathbb{B}, \theta)$ *is absolute in W.*

Finally, we note:

Lemma 2.4 *Let θ verify the subcompleteness of $\mathbb{B}$. Then $\mathbb{B}$ is subcomplete.*

(Thus "sufficiently large θ" can be replaced by "some θ" in the definition of 'subcomplete'.)

Proof of Lemma 2.4. It suffices to show:

Claim Let $\mathbb{B} \in H_\theta$. Let $\theta' > \overline{\overline{H}}_\theta$ be a cardinal. Then ver$(\mathbb{B}, \theta')$.

Proof. We can assume w.l.o.g. that θ is least with ver$(\mathbb{B}, \theta)$. Then by Lemma 2.1:

(1) $H_{\theta'} \models \theta$ is least s.t. ver$(\mathbb{B}, \theta)$.

Now let $N = L_\tau^A$ s.t. $\theta' < \tau$ and $H_{\theta'} \subset N$. Then $\theta < \tau$ and $H_\theta \subset N$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full. Let $\sigma(\overline{\mathbb{B}}, \overline{\theta}', \overline{s}) = \mathbb{B}, \theta', s$. By (1) there is $\overline{\theta}$ s.t. $\sigma(\overline{\theta}) = \theta$. By ver$(\mathbb{B}, \theta)$ there is then a $b \in \mathbb{B}$ with the desired property.

QED(Lemma 2.4)

When actually verifying the subcompleteness of a specific $\mathbb{B}$ we often find it convenient to employ an additional parameter. Thus we define:

Definition $\langle\theta, p\rangle$ verifies the subcompleteness of $\mathbb{B}$ (ver$(\mathbb{B}, \theta, p)$) iff $p, \mathbb{B} \in H_\theta$ and for any ZFC$^-$ model $N = L_\tau^A$ with $\theta < \tau$ and $H_\theta \subset N$ we have: Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full. Let $\sigma(\overline{p}, \overline{\theta}, \overline{s}, \overline{\mathbb{B}}) = p, \theta, s, \mathbb{B}$. Let $\overline{G}$ be $\overline{\mathbb{B}}$-generic over $\overline{N}$. Then the previous conclusion holds.

The natural analogues of Lemma 2.1 – Corollary 2.3 follow as before. But then we can repeat the proof of Lemma 2.4 to get:

Lemma 2.5 *Let $\langle\theta, p\rangle$ verify the subcompleteness of $\mathbb{B}$. Then $\mathbb{B}$ is subcomplete.*

This will often be tacitly used in verifications of subcompleteness.

3.2 Liftups

In order to better elucidate the concept of fullness, we make a digression on the topic of cofinal embeddings.

Definition Let $\overline{\mathfrak{A}}$, $\mathfrak{A}$ be models which satisfy the extensionality axiom. Let $\pi : \overline{\mathfrak{A}} \to \mathfrak{A}$ be a structure preserving map. We call π *cofinal* (in symbols: $\pi : \overline{\mathfrak{A}} \to \mathfrak{A}$ cofinally) iff for all $x \in \mathfrak{A}$ there is $u \in \overline{\mathfrak{A}}$ s.t. $x \in_{\mathfrak{A}} \pi(u)$.

Note In this definition we did not require $\overline{\mathfrak{A}}$, $\mathfrak{A}$ to be transitive or even well founded. Most of our applications will be to transitive models, but we must occasionally deal with ill founded structures. We shall, however, normally assume such structures to be *solid* in the sense of Chapter 1. (I.e. the well founded core of $\mathfrak{A}$ (wfc$(\mathfrak{A})$) is transitive and $\in^{\mathfrak{A}} \cap \text{wfc}(\mathfrak{A})^2 = \in \cap \text{wfc}(\mathfrak{A})^2$.)

Definition Let τ be a cardinal in $\mathfrak{A}$. $H_\tau^{\mathfrak{A}}$ = the set of x s.t. $\mathfrak{A} \models x \in H_\tau$.

Note Even if $\mathfrak{A}$ were a transitive ZFC$^-$ model, we would *not* necessarily have: $H_\tau^{\mathfrak{A}} \in \mathfrak{A}$.

Definition Let $\tau \in \overline{\mathfrak{A}}$ be a cardinal in $\overline{\mathfrak{A}}$. We call $\pi : \overline{\mathfrak{A}} \to \mathfrak{A}$ τ-*cofinal* iff for all $x \in \mathfrak{A}$ there is $u \in \overline{\mathfrak{A}}$ s.t. $\overline{\overline{u}} < \tau$ in $\overline{\mathfrak{A}}$ and $x \in_{\mathfrak{A}} \pi(u)$.

We shall generally work with elementary embeddings but must sometimes consider a finer degree of preservation:

Definition $\pi : \overline{\mathfrak{A}} \to \mathfrak{A}$ is Σ_n-*preserving* ($\pi : \overline{\mathfrak{A}} \to_{\Sigma_n} \mathfrak{A}$) iff for all Σ_n-formulae φ and all $x_1, \ldots, x_n \in \overline{\mathfrak{A}}$:

$$\overline{\mathfrak{A}} \models \varphi[x_1, \ldots, x_n] \longleftrightarrow \mathfrak{A} \models \varphi[\pi(x_1), \ldots, \pi(x_n)].$$

Definition Let $\overline{\mathfrak{A}}$ be a solid model of ZFC$^-$. Let $\tau \in \text{wfc}(\overline{\mathfrak{A}})$ be an uncountable cardinal in $\overline{\mathfrak{A}}$. Set $\overline{H} = H_\tau^{\overline{\mathfrak{A}}}$. (Hence $\overline{H} \subset \text{wfc}(\overline{\mathfrak{A}})$.) Let $\overline{\pi} : \overline{H} \to_{\Sigma_0} H$ cofinally, where H is transitive. Then by a *liftup* of $\langle\overline{\mathfrak{A}}, \overline{\pi}\rangle$ we mean a pair $\langle\mathfrak{A}, \pi\rangle$ s.t $\pi \supset \overline{\pi}$,

$H \subset \text{wfc}(\mathfrak{A})$, and $\pi : \overline{\mathfrak{A}} \to_{\Sigma_0} \mathfrak{A}$ τ-cofinally, where $\mathfrak{A}$ is solid.
(We also say: $\pi : \overline{\mathfrak{A}} \to \mathfrak{A}$ is a *liftup* of $\overline{\mathfrak{A}}$ by $\overline{\pi} : \overline{H} \to H$.)

Lemma 3.1 *Let $\overline{\mathfrak{A}}$, τ, $\overline{H}$, H, $\overline{\pi}$ be as in the above definition. The liftup $\langle \mathfrak{A}, \pi \rangle$ of $\langle \overline{\mathfrak{A}}, \overline{\pi} \rangle$ (if it exists) is determined up to isomorphism (i.e. if $\langle \mathfrak{A}', \pi' \rangle$ is another liftup, there is $\sigma : \mathfrak{A} \overset{\sim}{\leftrightarrow} \mathfrak{A}'$ with $\sigma\pi = \pi'$).*

Proof. Set $\Delta =$ the set of $f \in \overline{\mathfrak{A}}$ s.t. $\overline{\mathfrak{A}} \models (f \text{ is a function } \wedge \text{dom}(f) \in H_\tau)$. For each $f \in \Delta$ let $d(f) =$ that $u \in \overline{H}$ s.t. $u = \text{dom}(f)$ in $\overline{\mathfrak{A}}$. Set:

$$\Gamma = \{\langle f, x \rangle \mid f \in \Delta \wedge x \in \overline{\pi}(d(f))\}.$$

It is easily seen by τ-cofinality that each $a \in \mathfrak{A}$ has the form: $a = \pi(f)(x)$ in $\mathfrak{A}$, where $\langle f, x \rangle \in \Gamma$. The same holds for $\langle \mathfrak{A}', \pi' \rangle$ if $\langle \mathfrak{A}', \pi' \rangle$ is another liftup. But:

$$\begin{aligned} \pi(f)(x) \in \pi(g)(y) \text{ in } \mathfrak{A} &\longleftrightarrow \langle x, y \rangle \in \overline{\pi}(\{\langle z, w \rangle \mid f(z) \in g(w) \text{ in } \overline{\mathfrak{A}}\}) \\ &\longleftrightarrow \pi'(f)(x) \in \pi'(g)(y) \text{ in } \mathfrak{A}\,! \end{aligned}$$

Similarly:

$$\pi(f)(x) = \pi(g)(y) \text{ in } \mathfrak{A} \longleftrightarrow \pi'(f)(x) = \pi'(g)(y) \text{ in } \mathfrak{A}'.$$

Hence there is $\sigma : \mathfrak{A} \overset{\sim}{\leftrightarrow} \mathfrak{A}'$ defined by $\sigma(\pi(f)(x)_{\mathfrak{A}}) = \pi'(f)(x)_{\mathfrak{A}}$ for $\langle f, x \rangle \in \Gamma$. But for any $a \in \overline{\mathfrak{A}}$, we have: $\overline{\mathfrak{A}} \models a = k_a(0)$, where $k_a = \{\langle a, 0 \rangle\}$ in $\overline{\mathfrak{A}}$. Thus $\pi(a) = \pi(k_a)(0)$ in $\mathfrak{A}$, where $\langle k_a, 0 \rangle \in \Gamma$. Hence $\sigma(\pi(a)) = \pi'(k_a)(0) = \pi'(a)$. QED(Lemma 3.1)

Since the identity is the only isomorphism of a transitive structure onto a transitive structure, we have:

Corollary 3.2 *Let $\langle \mathfrak{A}, \pi \rangle$ be the liftup $\langle \overline{\mathfrak{A}}, \overline{\pi} \rangle$, where $\mathfrak{A}$, $\overline{\mathfrak{A}}$ are transitive. Then $\langle \mathfrak{A}, \pi \rangle$ is the unique liftup.*

Proof. Let $\langle \mathfrak{A}', \pi' \rangle$ be a liftup. Let $\sigma : \mathfrak{A} \overset{\sim}{\leftrightarrow} \mathfrak{A}'$ s.t. $\pi' = \sigma\pi$. Then $\mathfrak{A}'$ is well founded, hence transitive, by solidity. Hence $\sigma = \text{id}$ and $\pi' = \pi$, $\mathfrak{A}' = \mathfrak{A}$.QED(Corollary 3.2)

A transitive liftup does not always exist, even when $\overline{\mathfrak{A}}$ is transitive. However, a straightforward modification of the ultrapower construction does give us:

Lemma 3.3 *Let $\overline{\mathfrak{A}}$ be a solid model of* ZFC$^-$*. Let $\tau > \omega$, $\tau \in \text{wfc}(\overline{\mathfrak{A}})$ be a cardinal in $\overline{\mathfrak{A}}$ and set: $\overline{H} = H_\tau^{\overline{\mathfrak{A}}}$. Let $\overline{\pi} : \overline{H} \to_{\Sigma_0} H$ cofinally, where H is transitive. Then $\langle \overline{\mathfrak{A}}, \overline{\pi} \rangle$ has a liftup $\langle \mathfrak{A}, \pi \rangle$.*

Proof. Define Δ, Γ as above. Let $\mathfrak{A} = \langle |\mathfrak{A}|, \in_{\mathfrak{A}}, A_1^{\mathfrak{A}}, \ldots, A_n^{\mathfrak{A}} \rangle$. Define an equality model $\Gamma^* = \langle \Gamma, =^*, \in^*, A_1^*, \ldots, A_n^* \rangle$ by:

$$\begin{aligned} \langle f, x \rangle =^* \langle g, y \rangle &\longleftrightarrow \langle x, y \rangle \in \overline{\pi}(\{\langle z, w \rangle \mid f(z) \in g(w) \text{ in } \overline{\mathfrak{A}}\}) \\ \langle f, x \rangle \in^* \langle g, y \rangle &\longleftrightarrow \langle x, y \rangle \in \overline{\pi}(\{\langle z, w \rangle \mid f(z) \in g(w) \text{ in } \overline{\mathfrak{A}}\}) \\ \langle f, x \rangle \in A_i^* &\longleftrightarrow x \in \overline{\pi}(\{z \mid f(z) \in A_i \text{ in } \overline{\mathfrak{A}}\}). \end{aligned}$$

A straightforward modification of the usual proof gives us *Łos' Theorem* for Γ^*:

(1) $\Gamma^* \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle] \longleftrightarrow$
$$\longleftrightarrow \langle x_1, \ldots, x_n\rangle \in \overline{\pi}(\{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \varphi[f_1(z_1), \ldots, f_n(z_n)]\}).$$

This is proven by induction on φ. The case that φ is a primitive formula is immediate. We display the induction step for $\varphi = \varphi(v_1 \ldots, v_n) = \bigvee v_0\ \psi(v_0, \ldots, v_n)$.

($\rightarrow$) Let $\Gamma^* \models \varphi[\langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle]$. Then $\Gamma^* \models \psi[\langle f_0, x_0\rangle, \ldots, \langle f_n, x_n\rangle]$ for some $\langle f_0, x_0\rangle \in \Gamma$. Hence

$$\begin{array}{c}\langle x_0, \ldots, x_n\rangle \in \overline{\pi}(\{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \psi[f_0(z_0), \ldots, f_n(z_n)\}) \\ \bigcap \\ \overline{\pi}(d(f_0) \times \{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \varphi[f_1(z_1), \ldots, f_n(z_n)]\}) \\ \longrightarrow \langle x_1, \ldots, x_n\rangle \in \overline{\pi}(\{(\vec{z}\,) \mid \overline{\mathfrak{A}} \models \varphi[f_1(z_1), \ldots, f_n(z_n)]\}).\end{array}$$

($\leftarrow$) Set $u = \{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \varphi[f_1(z_1), \ldots, f_n(z_n)]\}$. Then $u \in \overline{H}$ and $\langle \vec{x}\,\rangle \in \pi(u)$. In $\overline{\mathfrak{A}}$ we have $\bigwedge \vec{z} \bigvee y(y, f_1(z_1), \ldots, f_n(z_n))$. Hence, by ZFC$^-$, there is $f_0 \in \overline{\mathfrak{A}}$ s.t.

$$\bigwedge \vec{z}\ \psi(f_0(\vec{z}), f_1(z_1), \ldots, f_n(z_n)) \quad \text{in } \overline{\mathfrak{A}}.$$

But then $\langle f_0, \langle \vec{z}\rangle\rangle \in \Gamma$ and

$$\langle\langle \vec{x}\,\rangle, x_1, \ldots, x_n\rangle \in \overline{\pi}(\{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \psi[z_0, \ldots, z_n]\}).$$

Hence $\Gamma^* \models \psi[\langle f_0, \langle \vec{x}\,\rangle\rangle, \langle f_1, x_1\rangle, \ldots, \langle f_n, x_n\rangle]$. QED(1)

Now let $\Gamma' = \langle |\Gamma'|, \in', A'_1, \ldots, A'_n\rangle$ be the result of factoring Γ^* by $=^*$, the elements being the $=^*$-equivalence classes x' of $x \in \Gamma$. Since Γ' satisfies extensionality, there is an isomorphism $\sigma : \Gamma' \overset{\sim}{\leftrightarrow} \mathfrak{A}$, where $\mathfrak{A}$ is solid. Set: $[f, x] = \sigma(\langle f, x\rangle')$, where $\langle f, x\rangle \in \Gamma$. Then $\mathfrak{A} \models \text{ZFC}^-$ by (1). We now define $\pi : \overline{\mathfrak{A}} \prec \mathfrak{A}$ by:

Definition For $a \in \overline{\mathfrak{A}}$ let $k = \{\langle a, 0\rangle\}$ in $\overline{\mathfrak{A}}$. Set: $\pi(a) =_{\text{Df}} [k, 0]$. Then:

(2) $\pi : \overline{\mathfrak{A}} \prec \mathfrak{A}$.

Proof.

$$\begin{aligned}\overline{\mathfrak{A}} \models \varphi[a_1, \ldots, a_n] &\longleftrightarrow \langle 0 - 0\rangle \in \{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \varphi[k_{a_1}(z_1), \ldots, k_{a_n}(z_n)]\} \\ &\longleftrightarrow \langle 0 - 0\rangle \in \pi(\{\langle \vec{z}\rangle \mid \overline{\mathfrak{A}} \models \varphi[k_{a_1}(z_1), \ldots, k_{a_n}(z_n)]\}) \\ &\longleftrightarrow \mathfrak{A} \models \varphi[\pi(a_1), \ldots, \pi(a_n)]\end{aligned}$$

by (1). QED(2)

Now set:

Definition $\Delta^0 =$ the set of functions $f \in \overline{H}$.

$$\Gamma^0 = \text{ the set of } \langle f, x\rangle \text{ s.t. } f \in \Delta^0 \text{ and } x \in \overline{\pi}(\text{dom}(f)).$$

Since $\overline{\pi} : \overline{H} \to H$ cofinally, H is the set of $\overline{\pi}(f)(x)$ s.t. $\langle f, x\rangle \in \Gamma^0$. Now set:

Definition $\tilde{H} = \{[f, x] \mid \langle f, x\rangle \in \Gamma^0\}$.

(3) $\tilde{H}$ is "$\mathfrak{A}$-transitive" – i.e if $a \in_{\mathfrak{A}} b \in \tilde{H}$, then $a \in \tilde{H}$.

Proof. Let $a = [f,x]$, $b = [g,y]$, where $\langle g,y\rangle \in \Gamma^0$ and $\langle f,x\rangle \in \Gamma$. Set: $u = \{z \in d(f) \mid f(z) \in \overline{H}\}$: Then $\langle f,x\rangle \in^* \langle g,y\rangle$ implies $\langle f,x\rangle =^* \langle f \upharpoonright u, x\rangle$, where $\langle f \upharpoonright u, x\rangle \in \Gamma^0$. QED(3)

But for $\langle f,x\rangle, \langle g,y\rangle \in \Gamma^0$ we have:

$$[f,x] \in [g,y] \text{ in } \mathfrak{A} \longleftrightarrow \langle x,y\rangle \in \overline{\pi}(\{\langle z,w\rangle \mid f(z) \in g(w)\}) \longleftrightarrow \overline{\pi}(f)(x) \in \overline{\pi}(g)(y).$$

Similarly: $[f,x] = [g,y] \leftrightarrow \overline{\pi}(f)(x) = \pi(g)(y)$. Hence there is an isomorphism $\sigma : \langle \tilde{H}, \in_{\mathfrak{A}}\rangle \overset{\sim}{\leftrightarrow} \langle H, \in\rangle$ defined by: $\sigma([f,x]) = \overline{\pi}(f)(x)$ for $\langle f,x\rangle \in \Gamma^0$. Hence $\langle \tilde{H}, \in_{\mathfrak{A}}\rangle$ is well founded. Since $\tilde{H}$ is $\mathfrak{A}$-transitive it follows that $\tilde{H} \subset \text{wfc}(\mathfrak{A})$; hence $\in_{\mathfrak{A}} \cap \tilde{H}^2 = \in \wedge H^2$ by solidity. Hence $\tilde{H}$ is transitive. Thus $\sigma = \text{id}$ and

(4) $\tilde{H} = H \subset \text{wfc}(\mathfrak{A})$ and $[f,x] = \overline{\pi}(f)(x)$ for $\langle f,x\rangle \in \Gamma^0$.

But then:

(5) $[f,x] = \pi(f)(x)$ in $\mathfrak{A}$ for all $\langle f,x\rangle \in \Gamma$.

Proof. $x \in \pi(d(f))$, where

$$d(f) = \{x \mid f(x) = f(x)\} = \{x \mid f(x) = (k_f(0))(\text{id} \upharpoonright d(f))(x) \text{ in } \overline{\mathfrak{A}}\}$$

where $k_f = \{\langle f, 0\rangle\}$ in $\overline{\mathfrak{A}}$. Hence

$$\langle x, 0, x\rangle \in \pi(\{\langle z,y,w\rangle \mid f(z) = k_f(y)(\text{id} \upharpoonright d(f))(z) \text{ in } \overline{\mathfrak{A}}\}.$$

Thus $[f,x] = [k_f, 0]([(\text{id} \upharpoonright d(f)), x])$ in $\mathfrak{A}$, where: $[k_f, 0] = \pi(f)$ and $[\text{id} \upharpoonright d(f), x] = \overline{\pi}(\text{id} \upharpoonright d(f))(x) = x$ by (4). QED(5)

(6) $\pi \upharpoonright \overline{H} = \overline{\pi}$, since for $a \in \overline{H}$ we have $\pi(a) = [k_a, 0] = \overline{\pi}(k_a)(0) = k_{\overline{\pi}(a)}(0) = \overline{\pi}(a)$ by (4).

Finally, since every $a \in \mathfrak{A}$ has the form $\pi(f)(x)$ for an $x \in H$, it follows that $a \in \pi(\text{rng}(f))$ in $\mathfrak{A}$, where $\overline{\overline{\text{rng}(f)}} < \tau$ in $\overline{\mathfrak{A}}$. Thus

(7) $\pi : \overline{\mathfrak{A}} \prec \mathfrak{A}$ τ-cofinally. QED(Lemma 3.3)

The above proof yields more than we have stated. For instance:

Lemma 3.4 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ confinally, where $\overline{N}$ is a* ZFC$^-$ *model and N is transitive. Then $\pi : \overline{N} \prec N$. (Hence N is a* ZFC$^-$ *model.)*

Proof. Repeat the above proof with $\tau = On \cap \overline{N}$ (hence $\overline{H} = \overline{N}$). All steps go through and we get $\mathfrak{A} = \tilde{H} = N$. QED(Lemma 3.4)

Lemma 3.5 *Let $\overline{\mathfrak{A}}$, $\mathfrak{A}$, $\overline{H}$, H, τ, π be as in Lemma* 3.3*. Set $\tilde{\tau} = On \cap H$. Then $\tilde{\tau} \in \text{wfc}(\mathfrak{A})$ and $H = H^{\mathfrak{A}}_{\tilde{\tau}}$.*

Proof. By the definition of wfc($\mathfrak{A}$) we have:

($*$) If $x \in \mathfrak{A}$ and $y \in \text{wfc}(\mathfrak{A})$ whenever $y \in_{\mathfrak{A}} x$, then $x \in \text{wfc}(\mathfrak{A})$.

We consider two cases:

Case 1 τ is regular in $\overline{\mathfrak{A}}$.

Claim $H = H^{\mathfrak{A}}_{\pi(\tau)}$ (hence $\pi(\tau) = \tilde{\tau} \in \text{wfc}(\mathfrak{A})$).

Proof. ($\subset$) is trivial. We prove ($\supset$).
Let $x \in H_{\pi(\tau)}$ in $\mathfrak{A}$. We claim that $x \in H$. Let $x \in \pi(u)$ in $\mathfrak{A}$, where $u \in \overline{\mathfrak{A}}$, $\overline{\overline{u}} < \tau$ in $\overline{\mathfrak{A}}$. Let $v = u \cap H_\tau$ in $\overline{\mathfrak{A}}$. Then $v \in \overline{H} = H^{\overline{\mathfrak{A}}}_\tau$ by regularity of τ. But then $x \in \pi(v) \in H$. Hence $x \in H$. QED(Case 1)

Case 2 Case 1 fails.

Let $\kappa = \text{cf}(\tau)$ in $\overline{\mathfrak{A}}$. Then $\kappa \in \overline{H}$. Let $f : \kappa \to \tau$ in $\overline{\mathfrak{A}}$ be normal and cofinal in τ. Then $f \in \text{wfc}(\overline{\mathfrak{A}})$ by ($*$). Let $\tilde{\kappa} = \sup \pi''\kappa$. Then $\tilde{\kappa} \leq \pi(\kappa) \in H$. Hence $\tilde{\kappa} \in H$. Let $g = \pi(f) \upharpoonright \tilde{\kappa}$ in $\mathfrak{A}$. It follows easily by ($*$) that $g \in \text{wfc}(\mathfrak{A})$. Thus $\tilde{\tau} = \sup g''\tilde{\kappa} \in \text{wfc}(\mathfrak{A})$.

Claim $H = H^{\mathfrak{A}}_{\tilde{\tau}}$

($\subset$) Let $x \in H$. Then $x \in \pi(u)$ where $u \in \overline{H}$. Hence $x \in \pi(u) \in H_{\tilde{\tau}}$. Hence $x \in H_{\tilde{\tau}}$.
($\supset$) Let $x \in H^{\mathfrak{A}}_{\tilde{\tau}}$. Then $x \in H^{\mathfrak{A}}_{\pi(\nu)}$ for a $\nu < \tau$ which is regular in $\overline{\mathfrak{A}}$, since $\tilde{\tau} = \sup \pi''\tau$ and $\tilde{\tau}$ is a limit cardinal in $\mathfrak{A}$. Let $x \in \pi(u)$ in $\mathfrak{A}$, where $u \in \overline{\mathfrak{A}}$, $\overline{\overline{u}} < \tau$ in $\overline{\mathfrak{A}}$. We can choose ν large enough that $\overline{\overline{u}} < \nu$ in $\overline{\mathfrak{A}}$. Let $v = u \cap H_\nu$ in $\overline{\mathfrak{A}}$. Then $v \in H_\nu \subset \overline{H}$ and $x \in \pi(v) \in H$. QED)(Lemma 3.5)

An immediate corollary of the proof is:

Corollary 3.6 *If τ is regular or $\text{cf}(\tau) = \omega$ in $\overline{\mathfrak{A}}$. Then $\tilde{\tau} = \pi(\tau)$ and $H = H^{\mathfrak{A}}_{\pi(\tau)}$.*

Note that if $\overline{N}$, N are transitive ZFC^- models, $\tau \in \overline{N}$ is a cardinal in $\overline{N}$ and $\pi : \overline{N} \prec N$ τ-cofinally, then π is κ cofinal for every $\kappa \geq \tau$ which is a cardinal in $\overline{N}$. Hence, by Corollary 3.6 we conclude:

Corollary 3.7 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ τ-cofinally, where $\overline{N}$, N are transitive, $\tau \in \overline{N}$ is a cardinal in $\overline{N}$, and $\overline{N} \models \text{ZFC}^-$. Let $\kappa \geq \tau$ be regular in $\overline{N}$ or $\text{cf}(\kappa) = \omega$ in $\overline{N}$. Then $\pi(\kappa) = \sup \pi''\kappa$ and $H^N_{\pi(\kappa)} = \bigcup_{u \in H^{\overline{N}}_\kappa} \pi(u)$.*

⋆ ⋆ ⋆ ⋆ ⋆

We are now ready to develop the concept of fullness further. We first generalize it as follows:

Definition Let N be a transitive ZFC^- model. N is *almost full* iff $\omega \in N$ and there is a solid $\mathfrak{A}$ s.t.

- $\mathfrak{A} \models \text{ZFC}^-$,
- $N \in \text{wfc}(\mathfrak{A})$,
- N is regular in $\mathfrak{A}$ – i.e. if $f : x \in N$, $x \in N$, and $f \in \mathfrak{A}$, then $\text{rng}(f) \in N$.

The last condition can be alternatively expressed by: $|N| = H^{\mathfrak{A}}_\tau$, where $\tau = On \cap N$.

Definition $\mathfrak{A}$ *verifies* the almost fullness of N iff the above holds.

Clearly every full structure is almost full. By Lemma 3.3 and 3.5 we then have:

Lemma 4.1 *Let $\overline{N}$ be almost full. Let $\overline{\pi} : \overline{N} \to_{\Sigma_0} N$ cofinally, where N is transitive. Then N is almost full. (In fact, if $\overline{\mathfrak{A}}$ verifies the almost fullness of $\overline{N}$ and $\langle \mathfrak{A}, \pi \rangle$ is a liftup of $\langle \overline{\mathfrak{A}}, \overline{\pi} \rangle$, then $\mathfrak{A}$ verifies the almost fullness of N.)*

Definition Let N be a transitive ZFC^- model. $\delta_N =$ the least δ s.t. $L_\delta(N)$ is admissible.

By Chapter 1 Corollary 21.1 we then have:

Lemma 4.2 *If $\mathfrak{A}$ verifies the almost fullness of N, then $L_{\delta_N}(N) \subset \mathrm{wfc}(\mathfrak{A})$.*

Combining this with Lemma 4.1 we get a conclusion that is rich in consequences:

Lemma 4.3 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ cofinally where $\overline{N}$ is almost full and N is transitive. Let φ be a Π_1 condition. Let $a_1, \ldots, a_n \in \overline{N}$. Then*

$$L_{\delta_{\overline{N}}}(\overline{N}) \models \varphi[\overline{N}, \vec{a}\,] \longrightarrow L_{\delta_N}(N) \models \varphi[N, \pi(\vec{a}\,)].$$

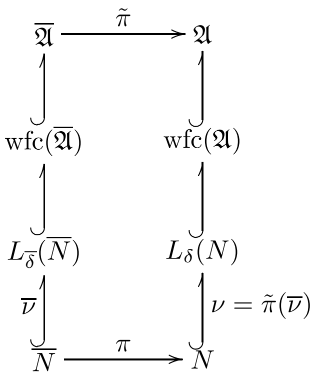
Proof. Let $\overline{\mathfrak{A}}$ verify the almost fullness of $\overline{N}$ and let $\langle \mathfrak{A}, \tilde{\pi} \rangle$ be a liftup of $\langle \overline{\mathfrak{A}}, \overline{\pi} \rangle$. We assume:

$$L_{\delta_N}(N) \models \psi[N, \pi(\vec{a}\,)],$$

where ψ is a Σ_1 condition, and prove:

Claim $L_{\delta_{\overline{N}}}(\overline{N}) \models \psi[\overline{N}, \vec{a}\,]$.

Set: $\nu =$ the least ordinal s.t. $L_\nu(N) \models \psi[N, \pi(\vec{a}\,)]$. Then $\nu < \delta_N$. Noting that $\mathfrak{A} \models \psi[N, \pi(\vec{a}\,)]$, we see that ν is $\mathfrak{A}$-definable, hence has a preimage $\overline{\nu}$ under $\tilde{\pi}$



Since $\nu \in \mathrm{wfc}(\mathfrak{A})$, we conclude that $\overline{\nu} \in \mathrm{wfc}(\overline{\mathfrak{A}})$. Hence $L_{\overline{\nu}}(\overline{N}) \models \psi[\overline{N}, \vec{a}\,]$. But $L_\eta(N)$ is not admissible for any $\eta \leq \nu$. Hence $L_\eta(\overline{N})$ is not admissible for any $\eta \leq \overline{\nu}$. Hence $\overline{\nu} < \delta_{\overline{N}}$ and the conclusion follows. QED(Lemma 4.3)

We now combine this with Barwise' theory. Recall that by a *theory* or *axiomatized language* on an admissible structure M we mean a pair $\langle \mathcal{L}_0, A \rangle$ where $\mathcal{L}_0$ is a language (i.e. a set of predicates and constants) in M-finitary predicate logic, and A is a set of axioms in $\mathcal{L}_0$.

We defined $\mathcal{L} = \langle \mathcal{L}_0, A\rangle$ to be $\Sigma_1(M)$ in parameters $p_1, \ldots, p_n \in M$ iff $\mathcal{L}_0$ is $\Delta_1(M)$ in $\vec{p}$ and A is $\Sigma_1(M)$ in $\vec{p}$.

By Chapter 2 Corollary 4 we get:

Lemma 4.4 *Let M be admissible. Let $\mathcal{L} = \langle \mathcal{L}_0, A\rangle$ be a theory on M which is $\Sigma_1(M)$ in parameters $p_1, \ldots, p_n \in M$. The statement: '$\mathcal{L}$ is consistent' is then $\Pi_1(M)$ in $\vec{p}$ (uniformly in the Σ_1 definition of A from $\vec{p}$).*

Hence

Lemma 4.5 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ cofinally, where $\overline{N}$ is almost full. Let $\overline{\mathcal{L}}$ be an infinitary theory on $L_{\delta_{\overline{N}}}(\overline{N})$ which is Σ_1 in parameters $\overline{N}$, $p_1, \ldots, p_n \in \overline{N}$. Let the theory $\mathcal{L}$ on $L_{\delta_N}(N)$ be Σ_1 in N, $\pi(\vec{p})$ by the same definition. If $\overline{\mathcal{L}}$ is consistent, so is $\mathcal{L}$.*

A typical application is:

Corollary 4.6 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ cofinally, where $\overline{N}$ is almost full. Let $\varphi(v_1, \ldots, v_n)$ be a first order (finite) formula in the $\overline{N}$-language with one additional predicate $\mathring{A}$. Let $\text{card}(\overline{N}) = \overline{\tau}$, $\text{card}(N) = \tau$. Let $x_1, \ldots, x_n \in \overline{N}$. If $\text{coll}(\omega, \overline{\tau})$ forces $\bigvee A\langle \overline{N}, A\rangle \models \varphi[\vec{x}]$. Then $\text{coll}(\omega, \tau)$ forces $\bigvee A\langle N, A\rangle \models \varphi[\pi(\vec{x})]$, ($\text{coll}(\omega, \tau)$ being the usual conditions for collapsing τ to ω).*

Proof. Let $\overline{\mathcal{L}}$ be the language on $L_{\delta_{\overline{N}}}(\overline{N})$ with the basic axioms. The additional constant $\mathring{a}$, and the additional axiom:

$$\langle \underline{N}, \mathring{a}\rangle \models \varphi[\underline{x}_1, \ldots, \underline{x}_n].$$

Let $\mathcal{L}$ have the same definition over $L_{\delta_N}(N)$ in the parameters $\pi(x_1), \ldots, \pi(x_n)$. By Barwise' completeness theorem, $\overline{\mathcal{L}}$ is consistent iff $\text{coll}(\omega, \overline{\tau})$ forces $\bigvee A\langle \overline{N}, A\rangle \models \varphi[\vec{x}]$. Similarly for $\mathcal{L}$, N, $\pi(\vec{x})$. The conclusion then follows by Lemma 4.5.

QED(Corollary 4.6)

The theory of liftups also reveals the import of condition (c) in the definition of "subcomplete". To this end we prove the *interpolation lemma*:

Lemma 5.1 *Let $\pi : \overline{N} \prec N$ where $\overline{N}$ is a transitive* ZFC$^-$ *model and N is transitive. Let τ be a cardinal in $\overline{N}$. Set: $\overline{H} = H_\tau^{\overline{N}}$ and $\tilde{H} = \bigcup\{\pi(u) \mid u \in \overline{N}$ and $\overline{\overline{u}} < \tau$ in $\overline{N}\}$. Then:*

(a) *The transitive liftup $\langle \tilde{N}, \tilde{\pi}\rangle$ of $\langle \overline{N}, \pi \restriction \overline{H}\rangle$ exists.*
(b) *There is $\sigma : \tilde{N} \prec N$ s.t. $\sigma\tilde{\pi} = \pi$ and $\sigma \restriction \tilde{H} = \text{id}$.*
(c) *σ is the unique $\sigma' : \tilde{N} \to_{\Sigma_0} N$ s.t. $\sigma'\tilde{\pi} = \pi$ and $\sigma' \restriction \tilde{\tau} = \text{id}$, where $\tilde{\tau} = On \cap \tilde{H}$.*

Proof. Let $\langle \mathfrak{A}, \tilde{\pi}\rangle$ be a liftup of $\langle \overline{N}, \pi \restriction \overline{H}\rangle$. Letting Γ be as in the proof of Lemma 3.3 we see that each $y \in \mathfrak{A}$ has the form $\tilde{\pi}(f)(x)$ in $\mathfrak{A}$ for some $\langle f, x\rangle \in \Gamma$.

Moreover:

$$\begin{aligned}\mathfrak{A} \models \varphi[\tilde{\pi}(f_1)(x_1), \ldots, \tilde{\pi}(f_n)(x_n)] \longleftrightarrow \\ \longleftrightarrow \langle x_1, \ldots, x_n \rangle \in \pi(\{\langle \vec{z} \rangle \mid \overline{N} \models \varphi[f_1(z_1), \ldots, f_n(z_n)]\}) \\ \longleftrightarrow N \models \varphi[\pi(f_1)(x_1), \ldots, \pi(f_n)(x_n)].\end{aligned}$$

Hence there is $\sigma : \mathfrak{A} \prec N$ defined by: $\sigma(\tilde{\pi}(f)(x)) = \pi(f)(x)$ for $\langle f, x \rangle \in \Gamma$. Thus $\mathfrak{A}$ is well founded, hence transitive by solidity. This proves (a), (b). We now prove (c). Let σ' be as in (c). Since $\pi : \overline{N} \prec \tilde{N}$ τ-cofinally, it follows that any $y \in \tilde{N}$ has the form $\pi(f)(\nu)$ for an $\langle f, \nu \rangle \in \Gamma$ s.t. $\mathrm{dom}(f) \subset \tau$. Hence $\sigma'(y) = \pi(f)(\nu) = \sigma(y)$.

QED(Lemma 5.1)

Just as in the proof of Lemma 3.4 we can repeat this using $\tau = On \cap N$, getting:

Lemma 5.2 *Let $\pi : \overline{N} \prec N$ where $\overline{N}$, N are transitive* ZFC$^-$ *models. Set: $\tilde{N} = \bigcup_{u \in \overline{N}} \pi(u)$. (Hence $\pi : \overline{N} \prec \tilde{N}$ cofinally.) Then $\tilde{N} \prec N$.*

We now utilize this to examine the meaning of (c) in the definition of "subcomplete".

Lemma 5.3 *Let $\sigma : \overline{N} \prec N$ where $\overline{N} = L^A_\alpha$ is a* ZFC$^-$ *model and N is transitive. Let $\sigma(\overline{\delta}) = \delta$, where δ is a cardinal in N. Set $C = C^N_\delta(\mathrm{rng}(\sigma))$, $\overline{H} = (H_{\delta^+})^{\overline{N}}$, $\tilde{H} = \bigcup_{u \in \overline{H}} \sigma(u)$. Let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \overline{N}, \sigma \restriction \overline{H} \rangle$ and let $k = \tilde{N} \prec N$ s.t. $k\tilde{\sigma} = \sigma$ and $k \restriction \tilde{H} = \mathrm{id}$. Then $C = \mathrm{rng}(k)$.*

Proof. ($\subset$) $\mathrm{rng}(\sigma) \subset \mathrm{rng}(k)$ and $\sigma \subset \mathrm{rng}(k)$.
($\supset$) Let $x \in \mathrm{rng}(k)$, $x = k(\tilde{x})$ where $\tilde{x} \in \tilde{\sigma}(u)$, $u \in \overline{N}$, $\overline{\overline{u}} < \overline{\delta}^+$ in $\overline{N}$. Let $f \in N$, $f : \overline{\delta} \xrightarrow{\text{onto}} u$. Then $x = k\tilde{\sigma}(f)(\nu) = \sigma(f)(\nu)$ for a $\nu < \delta$. Hence $x \in C$.

QED(Lemma 5.3)

Stating this differently, we can recover $\tilde{N}$, k from C by the definition: $k : \tilde{N} \stackrel{\sim}{\leftrightarrow} C$, where $\tilde{N}$ is transitive. We can then recover $\tilde{\sigma}$ from C by $\tilde{\sigma} = k^{-1} \cdot \sigma$. If we now have another $\sigma' : \overline{N} \prec N$ s.t. $\sigma'(\overline{\delta}) = \delta$ and $C = C^N_\delta(\mathrm{rng}(\sigma'))$, then $\langle \tilde{N}, \tilde{\sigma}' \rangle$ is the liftup of $\langle \overline{N}, \sigma' \restriction \overline{H} \rangle$, where $\tilde{\sigma}' = k^{-1}\sigma'$. Thus $\sigma = k\tilde{\sigma}$, $\sigma' = k\sigma'$ where $\tilde{\sigma}$, $\tilde{\sigma}'$ are determined entirely by $\sigma \restriction \overline{H}$, $\sigma' \restriction \overline{H}$, respectively. Hence

Corollary 5.4 *Let σ, σ' be as above. Let $\overline{\tau} \in \overline{N}$ be regular in $\overline{N}$ s.t. $\overline{\tau} > \overline{\delta}$ and $\sigma(\tau) = \sigma'(\tau)$. Then $\sup \sigma''\tau = \sup \sigma'{}''\tau$.*

Proof. Let $k(\tilde{\tau}) = \sigma(\tau) = \sigma'(\tau)$. Then $\tilde{\tau} = \sup \tilde{\sigma}''\tau = \sup \tilde{\sigma}'{}''\tau$, since $\tilde{\sigma}$, $\tilde{\sigma}'$ are τ-cofinal and τ is regular in $\overline{N}$. But then: $\sup \sigma''\tau = \sup \sigma'{}''\tau = \sup k''\tilde{\tau}$.

QED(Corollary 5.4)

A similar argument yields:

Corollary 5.5 *Let $\tau = On \cap \overline{N}$, where σ, σ' are as above. Then $\sup \sigma''\tau = \sup \sigma'{}''\tau = \sup k''\tilde{\tau}$, where $\tilde{\tau} = On \cap \tilde{N}$.*

Our original version of (c) was weaker, and can be stated as:

(c′) Let $\overline{s} = \langle \overline{s}_0, \overline{\lambda}_1, \ldots, \overline{\lambda}_n \rangle$ and $s = \langle s_0, \lambda_1, \ldots, \lambda_n \rangle$ where $\lambda_i > \delta$ is regular in N. Let $\overline{\lambda}_0 = On \cap \overline{N}$. Then $\sup \sigma'' \overline{\lambda}_i = \sup \sigma' {}'' \overline{\lambda}_i$ for $i = 0, \ldots, n$.

This is, of course, an immediate consequence of the above two corollaries. The weaker definition of 'subcomplete' should not be forgotten, since we might someday encounter a forcing which sastisfies the weaker version but not the stronger one. That has not happened to date, however, and in fact our original verifications of (c′) turned essentially on first verifying (c).

Before leaving the topic of τ-cofinal embeddings, we mention that these concepts can be applied to structures that are not ZFC^- models. For our purposes it will suffice to deal with the class of *smooth* models:

Definition Let N be a transitive model. N is *smooth* iff either $N \models \mathrm{ZFC}^-$ or else there is a sequence $\langle \langle N_i, \alpha_i \rangle \mid i < \lambda \rangle$ of limit length s.t. $N = \bigcup_i N_i$ and $N_j \models \mathrm{ZFC}^-$, N_j is transitive, and $N_i \in N_j$ s.t. α_i is regular in N_j and $N_i = H^{N_j}_{\alpha_i}$ for $i < j < \lambda$.

Then:

Lemma 5.6 *If $\overline{N}$ is smooth, N transitive, and $\pi : \overline{N} \to_{\Sigma_0} N$ cofinally, then N is smooth.*

Proof. If $\overline{N} \models \mathrm{ZFC}^-$, this is immediate from the foregoing. Otherwise there is a sequence $\langle \langle \overline{N}_i, \overline{\alpha}_i \rangle$ which verifies the smoothness of $\overline{N}$. Set $N_i = \pi(\overline{N}_i)$, $\alpha_i = \pi(\overline{\alpha}_i)$. Then $\langle \langle N_i, \alpha_i \rangle \mid i < \lambda \rangle$ verifies the smoothness of N. QED(Lemma 5.6)

Note It does *not* follow that $\pi : \overline{N} \prec N$.

The concepts "τ-cofinal" and "liftup" are defined as before, and it follows as before that if $\overline{N}$ is smooth, τ is a cardinal in $\overline{N}$ and $\overline{\pi} : H^{\overline{N}}_{\tau} \to_{\Sigma_0} H$ cofinally, then $\langle \overline{N}, \overline{\pi} \rangle$ has at most one transitive liftup.

Lemma 5.7 *Let $\pi : \overline{N} \to_{\Sigma_0} N$ τ-cofinally, where $\overline{N}$ is smooth. Let $\kappa \in \overline{N}$ be regular in $\overline{N}$, where $\kappa > \tau$. Let $\overline{H} = H^{\overline{N}}_{\kappa}$, $H = H^{N}_{\pi(\kappa)}$. Then $\pi \upharpoonright \overline{H} : \overline{H} \to_{\Sigma_0} H$ τ-cofinally.*

Proof. Exactly as in Case 1 of Lemma 3.5.

Lemma 5.8 *Let $\langle \langle \overline{N}_i, \overline{\alpha}_i \rangle \mid i < \lambda \rangle$ verify the smoothness of $\overline{N}$. Let $\tau \in \overline{N}$ be a cardinal. Let $\overline{\pi} : H^{\overline{N}}_{\tau} \to_{\Sigma_0} H$ cofinally. The transitive liftup of $\langle \overline{N}, \overline{\pi} \rangle$ exists iff for each i s.t. $\tau < \alpha_i$ the transitive liftup of $\langle \overline{N}_i, \overline{\pi} \rangle$ exists.*

Proof. ($\rightarrow$) Let $\langle N, \pi \rangle$ be the liftup of $\langle \overline{N}, \overline{\pi} \rangle$. Set: $\alpha_i = \pi(\overline{\alpha}_i)$, $N_i = \pi(\overline{N}_i)$. Then $\langle N_i, \pi \upharpoonright \overline{N}_i \rangle$ is the liftup of $\langle \overline{N}_i, \overline{\pi} \rangle$.
($\leftarrow$) Let $\langle N_i, \pi_i \rangle$ be the liftup of $\langle \overline{N}_i, \overline{\pi} \rangle$ for $\tau < \overline{\alpha}_i$. By Lemma 5.7 we have: $\pi_j \upharpoonright \overline{N}_i : \overline{N}_i \to N_i$ τ-cofinally. Hence $\pi_j \upharpoonright \overline{N}_i = \pi_i$ and we can set: $\pi = \bigcup_i \pi_i$. $\pi : \overline{N} \to_{\Sigma_0} N$ is then τ-cofinal and $\overline{\pi} = \pi \upharpoonright \overline{H}$. QED(Lemma 5.8)

Lemma 5.9 *Let $\overline{N}$ be smooth and $\pi : \overline{N} \to_{\Sigma_0} N$, where N is transitive. Let τ be a cardinal in $\overline{N}$. Set $\overline{H} = H^{\overline{N}}_{\tau}$. Then:*

(a) *The transitive liftup* $\langle \tilde{N}, \tilde{\pi}\rangle$ *of* $\langle \overline{N}, \pi \upharpoonright \overline{H}\rangle$ *exists.*
(b) *There is* $\sigma : \tilde{N} \to_{\Sigma_0} N$ *s.t.* $\sigma\tilde{\pi} = \pi$ *and* $\sigma \upharpoonright \tilde{H} = \mathrm{id}$*, where* $\tilde{H} = \bigcup_{u \in \overline{H}} \pi(u)$*.*
(c) σ *is the unique* $\sigma : \tilde{N} \to_{\Sigma_0} N$ *s.t.* $\sigma\tilde{\pi} = \pi$ *and* $\sigma \upharpoonright \tilde{\tau} = \mathrm{id}$*, where* $\tilde{\tau} = On \cap \tilde{H}$*.*

Proof.
Case 1 $\overline{N} \models \mathrm{ZFC}^-$.
Set: $N' = \bigcup_{u \in \overline{N}} \pi(u)$. Then $\pi : \overline{N} \to_{\Sigma_0} N'$ cofinally. Hence $\pi : \overline{N} \prec N' \subset N$ and we apply our previous lemmas.

Case 2 Case 1 fails.
Let $\langle\langle \overline{N}_i, \overline{\alpha}_i\rangle \mid i < \lambda\rangle$ verify the smothness of $\overline{N}$. Assume w.l.o.g. that $\tau \in \overline{N}_0$. (Hence $\overline{H} = H_\tau^{\overline{N}_i}$ for all $i < \lambda$.) (a) follows by Lemma 5.8. Moreover $\langle \overline{N}_i, \tilde{N}_i\rangle$ is the liftup of $\langle \overline{N}, \pi \upharpoonright \overline{H}\rangle$ by Lemma 5.7, where $\tilde{N}_i = \pi(\overline{N}_i)$. Let $\sigma_i : \tilde{N}_i \to_{\Sigma_0} \pi(\overline{N}_i)$ be defined by $\sigma_i\tilde{\pi}_i = \pi \upharpoonright \overline{N}_i$, $\sigma_i \upharpoonright \tilde{H} = \mathrm{id}$. Set $\sigma = \bigcup_i \sigma_i$. Then $\sigma : \tilde{N} \to_{\Sigma_0} N$ and $\sigma\tilde{\pi} = \pi$, $\sigma \upharpoonright \tilde{H} = \mathrm{id}$. This proves (b). But σ_i is unique s.t. $\sigma_i : \tilde{N}_i \to_{\Sigma_0} \pi(\overline{N}_i)$, $\sigma_i\tilde{\pi} = \pi \upharpoonright \overline{N}_i$ and $\sigma_i \upharpoonright \tilde{\tau} = \mathrm{id}$. Hence $\sigma \upharpoonright \tilde{N}_i = \sigma_i$ for $i < \lambda$ if σ is as in (c). This proves (c). QED(Lemma 5.9)

3.3 Examples

We are now ready to prove that some specific forcings are subcomplete. Since these forcings will be presented as sets of conditions rather than Boolean algebras, we set:

Definition Let $\mathbb{P}$ be a set of conditions.

$$\mathbf{V}^{\mathbb{P}} =_{\mathrm{Df}} \mathbf{V}^{\mathrm{BA}(\mathbb{P})}, \quad \delta(\mathbb{P}) =_{\mathrm{Df}} \delta(\mathrm{BA}(\mathbb{P}))$$

where $\mathrm{BA}(\mathbb{P})$ is the canonical Boolean algebra over $\mathbb{P}$ as defined in Chapter 0.

We may refer to the elements of $\mathbf{V}^{\mathbb{P}}$ as '$\mathbb{P}$-names'. We note:

Fact 1 Let $N = L_\tau^A$ be a ZFC^- model. Let $\mathrm{BA}(\mathbb{P}) \in H_\theta \subset N$. Let $\delta \subset C \prec N$, where $\mathrm{BA}(\mathbb{P}) \in C$ and $\delta = \delta(\mathbb{P})$. Then for each $p \in \mathbb{P}$ there is $q \in C \cap \mathbb{P}$ s.t. $[q] \subset [p]$. (Hence every set predense in $C \cap \mathbb{P}$ is predense in $\mathbb{P}$.)

Proof. Let $\mathbb{B} = \mathrm{BA}(\mathbb{P})$. By definition there are $f, \Delta \in H_\theta$ s.t. Δ is dense in $\mathbb{B}$ and $f : \delta \leftrightarrow \Delta$. Hence there are such $f, \Delta \in C$. But $\Delta \subset C$, since $\delta \subset C$. Let $p \in \mathbb{P}$. There is $b \in \Delta$ s.t. $b \subset [p]$. Hence there is $q \in C \cap \mathbb{P}$ s.t. $[q] \subset b$, since $C \prec N$. Hence $[q] \subset [p]$. QED(Fact 1)

Our first example is Prikry forcing.

Lemma 6.1 *Prikry forcing is subcomplete.*

Proof. Let U be a normal ultrafilter on a measurable cardinal κ. We define the Prikry forcing determined by U to be the set $\mathbb{P} = \mathbb{P}_U$ consisting of all pairs $\langle s, X\rangle$

s.t. $X \in U$ and $s \subset \kappa$ is finite. The extension relation $\leq_{\mathbb{P}}$ is defined by:

$$\langle s, X\rangle \leq \langle t, Y\rangle \quad \text{iff} \quad X \subset Y, s \supset t, \text{ and } \quad t = \text{lub}(t) \cap s.$$

$\mathbb{P}$ does not collapse cardinals or add new bounded subsets of κ If G is $\mathbb{P}$-generic, the $\mathbb{P}$-*sequence* added by G is

$$S = S_G = \bigcup\{s \mid \bigvee X \langle s, X\rangle \in G\}.$$

Then S is unbounded in κ and has order type ω. G is, in turn, definable from S by:

$$G = G_S = \{\langle s, X\rangle \in \mathbb{P} \mid s = S \cap \text{lub}(s) \wedge S \setminus s \subset X\}.$$

Definition We call $S \subset \kappa$ a $\mathbb{P}$-*sequence* (or *Prikry sequence*) iff $S = S_G$ for some $\mathbb{P}$-generic G.

The following characterization of Prikry sequences is well known:

Fact 2 S is a Prikry sequence iff $S \subset \kappa$ has order type ω and is almost contained in every $X \in U$ (i.e. $\bigvee \nu < \kappa \ S \setminus \nu \subset X$).

We now prove that $\mathbb{P}$ is subcomplete. To this end we let $\theta > 2^{2^\kappa}$ and let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $\tau > \theta$ and $H_\theta \subset N$. Furthermore we assume that $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full. We also suppose that

$$\sigma(\overline{\theta}, \overline{U}, \overline{\mathbb{P}}, \overline{s}) = \theta, U, \mathbb{P}, s.$$

Hence $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$, where $\mathbb{B} = \text{BA}(\mathbb{P})$ and $\overline{B} = \text{BA}(\overline{\mathbb{P}})$ in $\overline{N}$. We must show:

Main Claim There is $p \in \mathbb{P}$ s.t. whenever $\mathbb{G} \ni p$ is $\mathbb{P}$-generic. Then there is $\sigma' \in \mathbf{V}[G]$ s.t.

(a) $\sigma' : \overline{N} \prec N$,
(b) $\sigma'(\overline{\theta}, \overline{U}, \overline{\mathbb{P}}, \overline{s}) = \theta, U, \mathbb{P}, s$,
(c) $C_\delta^N(\text{rng}\,\sigma') = C_\delta^N(\text{rng}\,\sigma)$, where $\delta = \delta(\mathbb{P})$.
(d) $\sigma'\,''\overline{G} \subset G$.

Note that if we set: $S = S_G$ and $\overline{S} = S_{\overline{G}}$ in $\overline{N}[\overline{G}]$, then (d) becomes equivalent to:

(d′) $\sigma'\,''\overline{S} = S$.

Let $C = C_\delta^N(\text{rng}\,\sigma)$. Using Fact 1 we get:

(1) Let $X \in U$. Then there is $Y \in C \cap U$ s.t. $Y \subset X$.

Proof. Suppose not. Then for each $\nu < \kappa$ the set Δ_ν is dense in $\mathbb{P} \cap C$ where

$$\Delta_\nu = \{\langle s, Y\rangle \in \mathbb{P} \cap C \mid s \setminus \nu \not\subset X\}.$$

Hence Δ_ν is predense in $\mathbb{P}$ by Fact 1. Let G be $\mathbb{P}$-generic. Then $G \cap \Delta_\nu \neq \emptyset$ for $\nu < \kappa$. Hence S_G is not almost contained in X. Contradiction! by Fact 2. QED(1)

Hence:

(2) S is a $\mathbb{P}$-generic sequence iff S has order type ω and is almost contained in every $X \in C \cap U$.

(3) $\delta \geq \kappa$,

since otherwise $\overline{\overline{C}} < \kappa$ and $C \cap U$ would have a minimal element $Y = \bigcap(C \cap U)$.

Definition We define N_0, k_0, σ_0 by: $k_0 : N_0 \overset{\sim}{\leftrightarrow} C$, where N_0 is transitive $\sigma_0 = k_0^{-1} \circ \sigma$. We also set: $\Theta_0, \mathbb{P}_0, U_0, s_0 = \sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{U}, \overline{s})$.

By Chapter 3.2, however, we have an alternative characterization:

(4) Let $\sigma_0(\overline{\delta}) = \delta$, $\overline{\nu} = \overline{\delta}^{+\overline{N}}$, $\overline{H} = H_{\overline{\nu}}^{\overline{N}}$. Then $\langle N_0, \sigma_0 \rangle$ is the liftup of $\langle \overline{N}, \sigma \upharpoonright \overline{H} \rangle$. Moreover k_0 is defined by the condition:

$$k_0 : N_0 \prec N, \quad k_0 \sigma_0 = \sigma, \quad k_0 \upharpoonright \nu_0 = \mathrm{id},$$

where $\nu_0 = \sup \sigma'' \overline{\nu}$.

Since $\overline{\nu}$ is regular in $\overline{N}$, we conclude:

(5) σ_0 is a $\overline{\nu}$-cofinal map and $\sigma_0(\overline{\nu}) = \nu_0$.

Definition $\alpha_0 = \delta_{N_0}$ = the least α s.t. $L_\alpha(N_0)$ is admissible.

Our Main Claim will reduce to the assertion that a certain language $\mathcal{L}_0$ on $L_{\alpha_0}(N_0)$ is consistent. We define:

Definition $\mathcal{L}_0$ is the language on $L_{\alpha_0}(N_0)$ with:

Predicate: $\in$

Constants: $\mathring{S}$, $\mathring{\sigma}$, $\underline{x}$ $(x \in L_{\alpha_0}(N_0))$

Axioms:
- Basic axioms and ZFC$^-$
- $\mathring{S}$ is $\underline{\mathbb{P}}_0$-sequence over N_0
- $\mathring{\sigma} : \underline{\overline{N}} \prec \underline{N}_0$ $\underline{\overline{\kappa}}$-cofinally, where $\sigma(\overline{\kappa}) = \kappa$
- $\mathring{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{P}}}, \underline{\overline{U}}, \underline{\overline{s}}) = \underline{\theta}_0, \underline{\mathbb{P}}_0, \underline{U}_0, \underline{s}_0$
- $\mathring{\sigma}'' \underline{\overline{S}} = \mathring{S}$.

We first show that $\mathcal{L}_0$ is consistent. To this end we define:

Definition $\langle N_1, \sigma_1 \rangle$ = the liftup of $\langle \overline{N}, \sigma \upharpoonright H_{\overline{\kappa}}^{\overline{N}} \rangle$.
k_1 = the unique $k : N_1 \prec N_0$ s.t. $k\sigma_1 = \sigma_0$ and $k \upharpoonright \kappa_1 = \mathrm{id}$, where $\kappa_1 = \sup \sigma'' \overline{\kappa}$.
$\theta_1, \mathbb{P}_1, U_1, s_1 = \sigma_1(\overline{\theta}, \overline{\mathbb{P}}, \overline{U}, \overline{s})$, $S_1 = \sigma_1{}'' \overline{S}$.

Note that $\kappa_1 = \sigma_1(\overline{\kappa})$, since σ_1 is $\overline{\kappa}$-cofinal into N_1 and $\overline{\kappa}$ is regular in $\overline{N}$. Then:

(6) (a) S_1 is a $\mathbb{P}_1$-sequence over N_1,
(b) $\sigma_1 : \overline{N} \prec N_1$ $\overline{\kappa}$-cofinally,
(c) $\sigma_1(\overline{\theta}, \overline{\mathbb{P}}, \overline{U}, \overline{s}) = \theta_1, \mathbb{P}_1, U_1, s_1$,
(d) $\sigma_1{}'' \overline{S} = S_1$.

Proof. (b)–(d) are immediate. (a) follows by:

Claim Let $X \in U_1$. Then S_1 is almost contained in X.

Proof. Let $X \in \sigma_1(w)$, where $\overline{\overline{w}} < \overline{\kappa}$ in $\overline{N}$. Then $\overline{Y} = \bigcap(\overline{U} \cap w)$ is almost contained in every $z \in \overline{U} \cap w$ and $\overline{Y} \in \overline{U}$. Hence $Y = \sigma_1(Y)$ is almost contained in every $Y \in U_1 \cap \sigma_1(w)$. In particular, Y is almost contained in X. But S_1 is almost contained in Y. QED(6)

Now let:

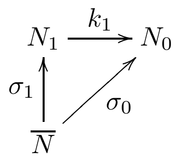
$$\alpha_1 = \delta_{N_1} = \text{ the least } \alpha \text{ s.t. } L_\alpha(N_1) \text{ is admissible.}$$

Let $\mathcal{L}_1$ be the language $L_{\alpha_1}(N_1)$ which is defined as $\mathcal{L}_0$ was defined on $L_{\alpha_0}(N_0)$, substituting θ_1, $\mathbb{P}_1$, U_1, s_1, κ_1 for θ_0, $\mathbb{P}_0$, U_0, s_0, κ_0. Then

(7) $\mathcal{L}_1$ is consistent.

Proof. $\langle H_\kappa, S_1, \sigma_1 \rangle$ models $\mathcal{L}_1$ by (6). QED(7)

Note, however, that:



where all maps are cofinal and all models are almost full.

Then $\mathcal{L}_0$ is $\Sigma_1(L_{\alpha_0}[N_0])$ in N_0 and the parameters:

$$\kappa,\ \mathbb{P}_0,\ \overline{\kappa},\ \overline{N},\ \overline{\theta},\ \overline{\mathbb{P}},\ \overline{U},\ \overline{s},\ \theta_0,\ \mathbb{P}_0,\ U_0,\ s_0.$$

But $\mathcal{L}_1$ is $\Sigma_1(L_{\alpha_1}[N_1])$ in N_1 and the k_1-preimages of these parameters by the same Σ_1-formula. Since N_1 is almost full and $k_1 : N_1 \prec N_0$ cofinally, we conclude by Lemma 4.5:

(8) $\mathcal{L}_0$ is consistent.

From this we now derive the Main Claim: Work in a generic extension $\mathbf{V}[F]$ of $\mathbf{V}$ in which $L_{\alpha_0}[N]$ is countable. Then $\mathcal{L}_0$ has a solid model

$$\mathfrak{A} = \langle |\mathfrak{A}|, \mathring{S}^{\mathfrak{A}}, \mathring{\sigma}^{\mathfrak{A}} \rangle.$$

Set $S = \mathring{S}^{\mathfrak{A}}$, $\sigma' = k_0 \circ \mathring{\sigma}^{\mathfrak{A}}$. Then

(8) (a) $\sigma' : \overline{N} \prec N$,
(b) S is $\mathbb{P}$-generic over $\mathbf{V}$,
(c) $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{U}, \overline{s}) = \theta, \mathbb{P}, U, s$,
(d) $C^N_\delta(\operatorname{rng} \sigma') = C$,
(e) $S = \sigma'{}''\overline{S}$.

Proof. (a), (c) are immediate. To see (d) note that $N_0 = C^{N_0}_\delta(\operatorname{rng} \mathring{\sigma}^{\mathfrak{A}})$, since $\mathring{\sigma}^{\mathfrak{A}}$ is $\overline{\kappa}$-cofinal and $\delta \geq \kappa = \mathring{\sigma}^{\mathfrak{A}}(\overline{\kappa})$. Hence:

$$C = k_0{}''N_0 = C^N_\delta(\operatorname{rng} k_0 \circ \mathring{\sigma}^{\mathfrak{A}}).$$

Since $k_0 \restriction (\kappa+1) = \mathrm{id}$ we have $U \cap N_0 = U \cap C$. Hence (b) follows by (2). (e) follows by $\sigma' \restriction \overline{\kappa} = \overset{\circ}{\sigma}^{\mathfrak{A}} \restriction \overline{\kappa}$. QED(9)

We have almost proven the Main Claim, the only problem being that σ' is not necessarily an element of $\mathbf{V}[S]$. We now show:

(10) There is $\sigma' \in \mathbf{V}[S]$ satisfying (9).

Proof. Work in $\mathbf{V}[S]$. Let μ be regular in $\mathbf{V}[S]$ s.t. $N \in H_\mu$. Set:
$M = \langle H_\mu, N, S, \theta, \mathbb{P}, U, s, \sigma\rangle$. We define a language $\mathcal{L}_2$ on the admissible structure M as follows:

Definition $\mathcal{L}_2$ is the language on M with

Predicate: $\in$
Constants: $\overset{\circ}{\sigma}$, $\underline{x}$, $(x \in M)$
Axioms:
- ZFC$^-$ and basic axioms
- $\overset{\circ}{\sigma} : \underline{\overline{N}} \prec \underline{N}$
- $\overset{\circ}{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{P}}}, \underline{\overline{U}}, \underline{\overline{s}}) = \underline{\theta}, \underline{\mathbb{P}}, \underline{U}, \underline{s}$
- $C^{\underline{N}}_{\underline{\delta}}(\mathrm{rng}\,\overset{\circ}{\sigma}) = \underline{C}$
- $\underline{S} = \overset{\circ}{\sigma}''\underline{\overline{S}}$

$\mathcal{L}_2$ is clearly consistent, since $\langle M, \sigma'\rangle$ is a model of $\mathcal{L}_2$ in $\mathbf{V}[F]$, where σ' is defined as above.

Now let $\pi : \tilde{M} \prec M$, where $\tilde{M}$ is countable and transitive. Let $\tilde{\mathcal{L}}_2$ be the language on $\tilde{M}$ with the same Σ_1 definition, replacing all parameters by their preimages under π. Then $\tilde{\mathcal{L}}_2$ is consistent. Since $\tilde{M} \in H^{\mathbf{V}[S]}_{\omega_1} = H^{\mathbf{V}}_{\omega_1}$, it follows that $\tilde{\mathcal{L}}_2$ has a solid model $\tilde{\mathfrak{A}}$ in $\mathbf{V}$. Let $\tilde{\sigma} = \overset{\circ}{\sigma}^{\tilde{\mathfrak{A}}}$ and set: $\sigma' = \pi \circ \tilde{\sigma}$. The verification of (9) is then straightforward. QED(10)

But, since S is a Prikry sequence, there must be $p \in G_S$ which forces the existence of such a σ'. This proves the Main Claim. QED(Lemma 6.1)

Lemma 6.2 *Assume* CH. *Then Namba forcing is subcomplete.*

Proof. We first define Namba forcing. The set $\omega_2^{<\omega}$ of monotone finite sequences in ω_2 is a tree ordered by inclusion. The set $\mathbb{N}$ of *Namba conditions* is the collection of all subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ s.t. T is downward closed in $\omega_2^{<\omega}$ and for each $s \in T$ the set $\{t \mid s \leq_T t\}$ has cardinality ω_2. The extension relation $\leq_{\mathbb{N}}$ is defined by:

$$T \leq T' \longleftrightarrow_{\mathrm{Df}} T \subset T'.$$

If G is $\mathbb{N}$-generic, then $S = \bigcup\bigcap G$ is a cofinal map of ω into $\omega_2^{\mathbf{V}}$. We rite $S = S_G$ and call any such S a *Namba sequence*. G is then recoverable from S by:

$$G = G_S = \{T \in \mathbb{N} \mid \bigwedge n < \omega \;\; S \restriction n \in T\}.$$

It is known that, if CH holds, then Namba forcing adds no reals.
We shall also make use of the following fact, which is proven in the Appendix to [DSF]:

Fact Let S be a Namba sequence. Let $S' \in \mathbf{V}[S]$ be a cofinal ω-sequence in $\omega_2^{\mathbf{V}}$. Then S' is a Namba sequence and $\mathbf{V}[S'] = \mathbf{V}[S]$.

Note that $\delta(\mathbb{N}) \geq \omega_2$, since otherwise ω_2 would not be collapsed.

We now turn to the proof. Let $\theta > 2^{2^{\omega_2}}$. Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $\tau > \theta$ and $H_\theta \subset N$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full. Let $\sigma(\overline{\theta}, \overline{\mathbb{N}}, \overline{s}) = \theta, \mathbb{N}, s$. Let $\overline{G}$ be $\overline{\mathbb{N}}$-generic over $\overline{N}$. It suffices to show:

Main Claim There is $p \in \mathbb{N}$ s.t. whenever $G \ni p$ is $\mathbb{N}$-generic, then there is $\sigma' \in \mathbf{V}[G]$ with:

(a) $\sigma' : \overline{N} \prec N$,
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{N}}, \overline{s}) = \theta, \mathbb{N}, s$,
(c) $C_\delta^N(\operatorname{rng} \sigma') = C_\delta^N(\operatorname{rng} \sigma)$ where $\delta = \delta(\mathbb{N})$,
(d) $\sigma' {}'' \overline{G} \subset G$.

Note We shall actually prove a stronger form of (c):

$$C_{\omega_2}^N(\operatorname{rng} \sigma') = C_{\omega_2}^N(\operatorname{rng} \sigma).$$

Note (d) can equivalently be replaced by:

$$\sigma' {}'' \overline{S} = S, \text{ where } \overline{S} = S_{\overline{G}}, \quad S = S_G.$$

Definition Set $C = C_{\omega_2}^N(\operatorname{rng} \sigma)$. Define k_0 by:

$$k_0 : N_0 \overset{\sim}{\leftrightarrow} C, \text{ where } N_0 \text{ is transitive, } \sigma_0 = k_0^{-1} \circ \sigma, \quad \theta_0, \mathbb{N}_0, s_0 = \sigma_0(\overline{\theta \mathbb{N}}, \overline{s}).$$

Just as before we get:

(1) $\langle N_0, \sigma_0 \rangle$ is the liftup of $\langle \overline{N}, \sigma \upharpoonright H_{\omega_3}^{\overline{N}} \rangle$,
k_0 is the unique $k : N_0 \prec N$ s.t. $k_0 \sigma_0 = \sigma$ and $k_0 \upharpoonright \omega_3^{N_0} = \text{id}$,
(where $\omega_3^{N_0} = \sup \sigma_0 {}'' \omega_3^{\overline{N}}$).

Now let α_0 be the least α s.t. $L_\alpha(N_0)$ is admissible. We define a language $\mathcal{L}_0$ on $L_{\alpha_0}(N_0)$ as follows:

Definition $\mathcal{L}_0$ is the language on $L_{\alpha_0}(N_0)$ with:

Predicate: $\in$
Constants: $\mathring{\sigma}$, $\underline{x}$ $(x \in L_{\alpha_0}(N_0))$
Axioms:
- Basic axioms and ZFC$^-$
- $\mathring{\sigma} : \underline{\overline{N}} \prec \underline{N}_0$ $\underline{\omega_2^{\overline{N}}}$-cofinally
- $\mathring{\sigma}(\overline{\theta}, \overline{N}, \overline{s}) = \theta_0, \mathbb{N}_0, s_0$.

(2) $\mathcal{L}_0$ is consistent.

Proof. Let $\langle N_1, \sigma_1 \rangle$ be the liftup of $\langle \overline{N}, \sigma \upharpoonright H_{\omega_2}^{\overline{N}} \rangle$. Define $k_1 : N_1 \prec N_0$ by:

$$k_1 \sigma_1 = \sigma_0, \quad k_1 \upharpoonright \gamma_1 = \text{id}, \text{ where } \gamma_1 = \sup \sigma' \omega_2^{\overline{N}} = \sigma_1(\omega_2^{\overline{N}}).$$

Let $\mathcal{L}_1$ be the corresponding language on $L_{\alpha_1}(N_1)$, where $\alpha_1 = \delta_{N_1}$. Just as before it suffices to show that $\mathcal{L}_1$ is consistent. This clear, however, since $\langle H_{\omega_2}, \sigma_1 \rangle$ is a model. QED(2)

Now let S' be a Namba sequence. Work in $\mathbf{V}[S']$. Let μ be a regular cardinal in $\mathbf{V}[S']$ with $N \in H_\mu$. Set:

$$M = \langle H_\mu, N, \sigma, \mathbb{N}, s\rangle.$$

Let $\pi : \tilde{M} \prec M$, where $\tilde{M}$ is transitive and countable. Then $\tilde{M} \in H_{\omega_1} \subset \mathbf{V}$ in $V[S']$. Let

$$\pi(\tilde{\mathbb{N}}, \tilde{\sigma}, \tilde{N}, \tilde{\mathcal{L}}, \tilde{k}, \tilde{C}) = \mathbb{N}, \sigma, N, \mathcal{L}_0, k_0, C_0.$$

Let $\mathfrak{A} \in \mathbf{V}$ be a solid model of $\tilde{\mathcal{L}}$. Set $\tilde{\sigma} = \tilde{k} \circ \overset{\circ}{\sigma}{}^{\mathfrak{A}}$; $\sigma' = \pi \circ \tilde{\sigma}$. It follows easily that:

(3) (a) $\sigma' : \overline{N} \prec N$
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{N}}, \overline{s}) = \theta, \mathbb{N}, s$
(c) $C^N_{\omega_1}(\operatorname{rng}\sigma') = C$

Now let $\overline{S} = S_{\overline{G}}$ and set: $S = \sigma' {}''\overline{S}$. Then $S \in \mathbf{V}[S']$ is a cofinal ω-sequence in $\omega_2^{\mathbf{V}}$; hence:

(4) S is a Namba sequence and $\mathbf{V}[S] = \mathbf{V}[S']$. (Hence $\sigma' \in \mathbf{V}[S]$.)

But we know:

(5) $S = \sigma' {}''\overline{S}$.

Let $G = G_S$. There is then a $p \in G$ which forces the existence of a $\sigma' \in \mathbf{V}[S]$ satisfying (3), (5). This proves the Main Claim. QED(Lemma 6.2)

Now let $\kappa > \omega_1$ be a regular cardinal. Let $A \subset \kappa$ be a stationary set of ω-cofinal ordinals. Our final example is the forcing $\mathbb{P}_A$ which is designed to shoot a cofinal normal sequence of order type ω_1 through A:

Definition $\mathbb{P}_A$ is the set of normal functions $p : \nu + 1 \to A$, where $\nu < \omega_1$. The extension relation is defined by:

$$p \le q \quad \text{in} \quad \mathbb{P}_A \longleftrightarrow_{\text{Df}} q \subset p.$$

Clearly, if G is $\mathbb{P}_A$-generic, then $\bigcup G : \omega_1 \to A$ is normal and cofinal in κ. $\mathbb{P}_A$ adds no new countable subsets of the ground model. If, however, $\{\lambda < \kappa \mid \operatorname{cf}(\lambda) = \omega \wedge \lambda \notin A\}$ is stationary, then $\mathbb{P}_A$ will not be a complete forcing.

Lemma 6.3 *$\mathbb{P}_A$ is subcomplete.*

Proof. Clearly $\delta(\mathbb{P}_A) \ge \kappa$, since otherwise κ would remain regular. Now let $\theta > 2^{2^\kappa}$. Let $N = L^B_\tau$ be a ZFC$^-$ model s.t. $\tau > \theta$ and $H_\theta \subset N$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full. Let $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{A}, \overline{\kappa}, \overline{s}) = \theta, \mathbb{P}_A, A, \kappa, s$. Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}$. It suffices to show:

Main Claim There is $p \in \mathbb{P}$ s.t. whenever $G \ni p$ is $\mathbb{P}_A$-generic, there is $\sigma' \in \mathbf{V}[G]$ s.t.

(a) $\sigma' : \overline{N} \prec N$,

(b) $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{\kappa}, \overline{A}, \overline{s}) = \theta, \mathbb{P}, \kappa, A, s$,
(c) $C^N_\kappa(\operatorname{rng}\sigma') = C^N_\kappa(\operatorname{rng}\sigma)$,
(d) $\sigma'\,''\overline{G} \subset G$.

(1) Let $\sigma' \in \mathbf{V}$ satisfy (a), (b), (c) and

(e) $\sup \sigma'\,''\overline{\kappa} \in A$.

Then the Main Claim holds.

Proof. Let $\overline{F} = \bigcup \overline{G}$. Then $\overline{F}$ is a cofinal normal map of $\omega_1^{\overline{N}}$ into $\overline{A}$, where $\sigma(\overline{A}) = A$. Define $p \in \mathbb{P}_A$ by:

$$p(\xi) = \begin{cases} \sigma'\overline{F}(\xi) & \text{for } \xi < \omega_1^{\overline{N}}, \\ \sup \sigma'\,''\overline{\kappa} & \text{for } \xi = \omega_1^{\overline{N}}. \end{cases}$$

Clearly $p \leq \sigma'(q)$ for $q \in \overline{G}$. Hence if $G \ni p$ is generic, then $\sigma'\,''\overline{G} \subset G$. QED(1)

We must produce a σ' satisfying (a), (b), (c) and (e). For $\xi < \kappa$ set: $C_\xi = C^N_\xi(\operatorname{rng}\sigma)$. Set:

$$D = \{\tau < \kappa \mid \tau = \kappa \cap C_\tau\}.$$

Then D is club in κ. Hence then is $\kappa_0 \in D \cap A$. Set:

Definition $k_0 : N_0 \overset{\sim}{\leftrightarrow} C_{\kappa_0}$, where N_0 is transitive; $\sigma_0 = k_0^{-1} \circ \sigma$; $\theta_0, \mathbb{P}_0, s_0, A_0 = \sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}, \overline{A})$. (Hence $\kappa_0 = \sigma_0(\overline{\kappa})$.)

We again let $\alpha_0 = \delta_{N_0}$ be least s.t. $L_{\alpha_0}(N_0)$ is admissible and define:

Definition $\mathcal{L}_0$ is the language on $L_{\alpha_0}(N_0)$ with:

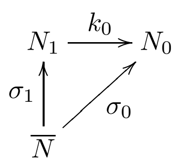
Predicate: $\in$
Constants: $\overset{\circ}{\sigma}$, $\underline{x}$ $(x \in L_{\alpha_0}(N_0))$
Axioms:
- Basic axioms and ZFC$^-$
- $\overset{\circ}{\sigma} : \underline{\overline{N}} \prec \underline{N_0}$ $\underline{\overline{\kappa}}$-cofinally
- $\overset{\circ}{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{P}}}, \underline{\overline{s}}, \underline{\overline{\kappa}}, \underline{\overline{A}}) = \underline{\theta}_0, \underline{\mathbb{P}}_0, \underline{s}_0, \underline{\kappa}_0, \underline{A}_0$.

(2) $\mathcal{L}_0$ is consistent.

Proof. Let $\langle N_1, \sigma_1 \rangle$ be the liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\overline{\kappa}} \rangle$. Note that $\sigma \restriction H^{\overline{N}}_{\overline{\kappa}} = \sigma_0 \restriction H^{\overline{N}}_{\overline{\kappa}}$. Hence there is $k_1 : N_1 \prec N_0$ defined by:

$$k_1\sigma_1 = \sigma_0, \quad k_1 \restriction \kappa_1 = \mathrm{id},$$

where $\kappa_1 = \sigma_1(\overline{\kappa}) = \sup \sigma''\overline{\kappa}$. Let $\mathcal{L}_1$ be the corresponding language on $L_{\alpha_1}(N_1)$, where $\alpha_1 = \delta_{N_1}$. By the usual argument it suffices to show that $\mathcal{L}_1$ is consistent: Since $N_0 = C^{N_0}_{\kappa_0}(\operatorname{rng}\sigma_0)$, we can conclude that $\sigma_0 : \overline{N} \prec N_0$ is $\overline{\kappa}^{+\overline{N}}$-cofinal. Hence:



where all maps are cofinal and all structures are almost full. $\mathcal{L}_1$ is trivially consistent, however, since $\langle H_\kappa, \sigma_1 \rangle$ models $\mathcal{L}_1$. QED(2)

Now let $M = \langle H_\kappa, N_0, \kappa_0, A_0, \sigma_0 \rangle$. Let $\pi : \tilde{M} \prec M$ s.t. $\tilde{M}$ is countable and transitive. Let $\pi(\tilde{\mathcal{L}}) = \mathcal{L}_0$. Then $\tilde{\mathcal{L}}$ is a consistent language on $L_{\tilde{\alpha}}(\tilde{N}) = \pi^{-1}(L_{\alpha_0}(N_0))$. Hence $\tilde{\mathcal{L}}$ has a solid model $\mathfrak{A}$. Set:

$$\sigma' = k_0 \circ \pi \circ \overset{\circ}{\sigma}^{\mathfrak{A}}.$$

Then σ' satisfies (a), (b), (c), (e) of (1). QED(Lemma 6.3)

Chapter 4

Iterating subcomplete forcing

The *two step iteration theorem* for subcomplete forcing says that if $\mathbb{A}$ is subcomplete and

$$\Vdash_{\mathbb{A}} \overset{\circ}{\mathbb{B}} \quad \text{is subcomplete,}$$

then $\mathbb{A} * \overset{\circ}{\mathbb{B}}$ is subcomplete. Equivalently:

Theorem 1 *Let $\mathbb{A} \subseteq \mathbb{B}$ where $\mathbb{A}$ is subcomplete and*

$$\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\overset{\circ}{G} \quad \textit{is subcomplete.}$$

Then $\mathbb{B}$ is subcomplete.

(**Note** The definitions of $\mathbb{A} * \overset{\circ}{\mathbb{B}}$, $\check{\mathbb{B}}/\overset{\circ}{G}$ and other Boolean conventions employed here can be found in Chapter 0. $\overset{\circ}{G}$ is the canonical generic name – i.e. $\Vdash_{\mathbb{A}} \overset{\circ}{G}$ is $\check{\mathbb{A}}$-generic over $\check{\mathbf{V}}$, and $[[\check{a} \in \overset{\circ}{G}]] = a$ for $a \in \mathbb{A}$.)

Proof of Theorem 1. Let θ be big enough that θ verifies the subcompleteness of $\mathbb{A}$ and:

$$\Vdash_{\mathbb{A}} \check{\theta} \text{ verifies the subcompleteness of } \check{\mathbb{B}}/\overset{\circ}{G}.$$

Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$ and $\tau > \theta$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and sound. Let:

$$\sigma(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, s$$

where $s \in N$. Let $\overline{G}$ be $\overline{\mathbb{B}}$-generic over $\overline{N}$. We must find $b \in \mathbb{B} \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}$-generic, there is $\sigma' \in \mathbf{V}[G]$ satisfying (a)–(d) in the definition of subcompleteness.

Let $\overline{G}_0 = \overline{G} \cap \overline{\mathbb{A}}$. Then $\overline{G}_0$ is $\overline{\mathbb{A}}$-generic over $\overline{N}$. Since θ verifies the subcompleteness of $\mathbb{A}$, there exist $a \in \overline{\mathbb{A}} \setminus \{0\}$, $\overset{\circ}{\sigma}_0 \in \mathbf{V}^{\mathbb{A}}$ s.t. whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \overset{\circ}{\sigma}{}_0^{G_0}$, then (a)–(d) hold with $\overline{\mathbb{A}}$, $\overline{G}_0$, $\mathbb{A}$, G_0, σ_0 in place of $\overline{\mathbb{B}}$, $\overline{G}$, $\mathbb{B}$, G, σ'. Let $\overline{\mathbb{B}}^* = \overline{\mathbb{B}}/\overline{G}_0$. Let $G_0 \ni a$ be $\mathbb{A}$-generic. Set: $\mathbb{B}^* = \mathbb{B}/G_0$. Clearly, σ_0 extends to σ_0^* s.t

$$\sigma_0^* : \overline{N}[\overline{G}_0] \prec N[G_0] \quad \text{and} \quad \sigma_0^*(\overline{G}_0) = G_0$$

In other words, $\sigma_0^* : \overline{N}^* \prec N^*$ where: $\overline{N} = L_{\overline{\tau}}^{\overline{A}}$, $\overline{N}^* = L_{\overline{\tau}}^{\overline{A},\overline{G}_0}$, $N = L_\tau^A$, $N^* = L_\tau^{A,G_0}$. Note that $H_\theta^{\mathbf{V}[G_0]} = H_\theta^{\mathbf{V}}[G_0] \subset N^*$. Moreover $\overline{G}^*$ is $\overline{\mathbb{B}}^*$-generic over $\overline{N}^*$ where:

$$\overline{G}^* = \overline{G}/\overline{G}_0 = \{b/\overline{G}_0 \mid b \in \overline{G}\}.$$

Clearly

$$\sigma_0^*(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{\mathbb{B}}^*, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, \mathbb{B}^*, s.$$

Since θ verifies the subcompleteness of $\mathbb{B}^*$ in $\mathbf{V}[G_0]$, we conclude that there is $b^* \in \mathbb{B}^*$ s.t. whenever $G^* \ni b^*$ is $\mathbb{B}^*$-generic over $\mathbf{V}[G_0]$, then there is $\sigma^* \in \mathbf{V}[G_0][G^*]$ with:

(a*) $\sigma^* : \overline{N}^* \prec N^*$,
(b*) $\sigma^*(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{\mathbb{B}}^*, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, \mathbb{B}^*, s$
(c*) $C_{\delta^*}^{N^*}(\text{rng}(\sigma^*)) = C_{\delta^*}^{N^*}(\text{rng}(\sigma_0^*))$, where $\delta^* = \delta(\mathbb{B}^*)$.
(d*) $\sigma^* {}''\overline{G}^* \subset G^*$.

Note that $\overline{G} = \overline{G}_0 * \overline{G}^* =_{\text{Df}} \{b \in \overline{\mathbb{B}} \mid b/\overline{G}_0 \in \overline{G}^*\}$. Set $G = G_0 * G^*$. Set $\sigma' = \sigma^* \upharpoonright \overline{N}$. Then $\sigma' \in \mathbf{V}[G] = \mathbf{V}[\overline{G}_0][G^*]$. We show:

Claim σ' satisfies:

(a) $\sigma' : \overline{N} \prec N$
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, s$
(c) $C_\delta^N(\text{rng}(\sigma')) = C_\delta^N(\text{rng}(\sigma))$, where $\delta = \delta(\mathbb{B})$.
(d) $\sigma' {}''\overline{G} \subset G$.

We note first that the claim proves the theorem, since G is $\mathbb{B}$-generic and there must, therefore, be a $b \in G$ which forces the existence of such a σ'. We now prove the claim. (a), (b), (d) are immediate. We prove (c). Note that $\delta \geq \delta^*$, we have:

$$C_\delta^{N^*}(\text{rng}(\sigma^*)) = C_\delta^{N^*}(\text{rng}(\sigma_0^*)).$$

Since $C_\delta^N(\text{rng}(\sigma_0)) = C_\delta^N(\text{rng}(\sigma))$, it suffices to show:

(1) $C_\delta^N(\text{rng}(\sigma')) = N \cap C_\delta^{N^*}(\text{rng}(\sigma^*))$

(2) $C_\delta^N(\text{rng}(\sigma_0)) = N \cap C_\delta^{N^*}(\text{rng}(\sigma_0^*))$.

We proof (1), the proof of (2) being virtually identical. ($\subset$) is trivial. We prove ($\supset$).
Let $x \in N \cap C_\delta^{N^*}(\text{rng}(\sigma^*))$. Then x is $N[G_0]$-definable in ξ, $\sigma^*(z)$, G_0, where $\xi < \delta$, $z \in \overline{N}$. But, letting $t \in \overline{N}^{\mathbb{A}}$ s.t. $\langle z, \overline{G}_0 \rangle = t^{\overline{G}_0}$, we have:

$$\langle \sigma^*(z), G_0 \rangle = \sigma^*(\langle z, \overline{G}_0 \rangle) = \sigma^*(t^{\overline{G}_0}) = \sigma'(t)^{G_0}.$$

Hence:

$$x = \text{ that } x \text{ s.t. } N[G_0] \models \varphi[x, \xi, \sigma'(t)^{G_0}].$$

But since $\sigma'(\overline{\mathbb{B}}) = \mathbb{B}$, we have: $\sigma'(\overline{\delta}) = \delta$, where $\overline{\delta} = \delta(\overline{\mathbb{B}})$. Since $\overline{\delta} \geq \delta(\overline{\mathbb{A}})$, there is $f \in \overline{N}$ mapping $\overline{\delta}$ onto a dense subset of $\overline{A}$. Hence $\sigma'(f)$ maps δ onto a dense subset of $\mathbb{A}$. Hence there is $\nu < \delta$ s.t. $\sigma'(f)(\nu) \in G_0$ and $\sigma'(f)(\nu)$ forces $\varphi(\check{x}, \check{\xi}, \sigma'(t))$. Thus: x = that x s.t. $\sigma'(f)(\nu) \Vdash^N_{\mathbb{A}} \varphi(\check{x}, \check{\xi}, \sigma'(t))$. Hence $x \in C^N_\delta(\text{rng}(\sigma'))$. QED(Theorem 1)

The proof of Theorem 1 shows more than we have stated. We can omit the assumption that $\mathbb{A}$ is subcomplete and omit the map σ, assuming, however, that $\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\mathring{G}_0$ is subcomplete, where $\mathring{G}_0$ is the canonical $\mathbb{A}$-generic name. Let θ be big enough that

$$\Vdash_{\mathbb{A}} \check{\theta} \quad \text{verifies the subcompleteness of } \check{\mathbb{B}}/\mathring{G}_0.$$

Let N be as before and let $\overline{N}$ be countable and full. Suppose that $a \in \mathbb{A} \setminus \{0\}$ and $\mathring{\sigma} \in \mathbf{V}^{\mathbb{A}}$ are given s.t. whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \mathring{\sigma}_0^{G_0}$, then

- $\sigma_0 : \overline{N} \prec N$
- $\sigma_0(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, s$
- $\sigma_0{}''\overline{G}_0 \subset G_0$.

Our proof then yields a $b^* \in \mathbb{B}^* = (\mathbb{B}/G_0) \setminus \{0\}$ s.t. if $G \supset G_0$ is $\mathbb{B}$-generic and $b^* \in G^* = G/G_0 = \{c/G_0 \mid c \in G\}$, then there is $\sigma' \in \mathbf{V}[G]$ s.t. (a), (b), (d) and:

(c′) $C^N_\delta(\text{rng}(\sigma')) = C^N_\delta(\text{rng}(\sigma_0))$

hold, where $\delta = \delta(\mathbb{B})$. We can improve on this still further. Suppose that $t \in \mathbf{V}^{\mathbb{A}}$ s.t. and $a \Vdash t \in \check{\overline{N}}$. This means that $t^{G_0} \in \overline{N}$ whenever $G_0 \ni a$ is $\mathbb{A}$-generic. We can then select our b^* so as to force:

(e*) $\sigma^*(t^{G_0}) = \sigma_0(t^{G_0})$.

in addition to (a*)–(d*). It then follows that:

(e′) $\sigma'(t^{G_0}) = \sigma_0(t^{G_0})$.

Since, whenever $G_0 \ni a$ is $\mathbb{A}$-generic, we can find a $b^* \in \mathbb{B}/G_0$ forcing (a), (b), (d), (c′), (e′), we conclude that there is $b^0 \in \mathbf{V}^{\mathbb{A}}$ s.t. a forces $b^* = \mathring{b}^{G_0}$ to have these properties. We may assume w.l.o.g. that $\Vdash_{\mathbb{A}} \mathring{b} \in \check{\mathbb{B}}/\mathring{G}_0$ and $[[\mathring{b} \neq 0]]_{\mathbb{A}} = a$. By Chapter 0, Fact 4 there is then a unique $b \in \mathbb{B}$ s.t. $\Vdash_{\mathbb{A}} \check{b}/\mathring{G}_0 = \mathring{b}$. Letting $h = h_{\mathbb{A},\mathbb{B}}$ be defined as in Chapter 0 by $h(c) = \bigcap\{a \in \mathbb{A} \mid c \subset a\}$ for $a \in \mathbb{B}$, we conclude by Chapter 0, Fact 3 that:

$$h(b) = [[\check{b}/\mathring{G}_0 \neq 0]] = [[\mathring{b} \neq 0]] = a.$$

Clearly, if $G \ni b$ is $\mathbb{B}$-generic, then $G_0 \ni a$ is $\mathbb{A}$-generic, where $G_0 = G \cap \mathbb{A}$. Thus $b/G_0 = \mathring{b}^{G_0} = b^*$ has the above properties and (a), (b), (d), (c′),(e′) hold.

Putting all of this together, we get a very useful technical lemma:

Lemma 1.1 *Let $\mathbb{A} \subseteq \mathbb{B}$ and let: $\Vdash_{\mathbb{A}} \check{\mathbb{B}}/\mathring{G}$ is subscomplete. Let θ be big enough that $\mathbb{B} \in H_\theta$ and: $\Vdash_{\mathbb{A}} \check{\theta}$ verifies the subcompleteness of $\check{\mathbb{B}}/\mathring{G}$. Let $N = L^A_\tau$ be a* ZFC$^-$

model s.t. $H_\theta \subset N$ and $\theta < \tau$. Let $\overline{N}$ be countable and full. Let $\overline{\mathbb{A}} \subseteq \overline{\mathbb{B}}$ in $\overline{N}$, where $\overline{G}$ is $\overline{\mathbb{B}}$-generic over $\overline{N}$. Set: $\overline{G}_0 = \overline{G} \cap \overline{\mathbb{A}}$. Suppose that $a \in \mathbb{A} \setminus \{0\}$, $\overset{\circ}{\sigma}_0 \in \mathbf{V}^{\mathbb{A}}$ s.t. whenever $G_0 \ni a$ is $\mathbb{A}$-generic and $\sigma_0 = \overset{\circ}{\sigma}{}_0^{G_0}$, then:

(i) $\sigma_0 : \overline{N} \prec N$
(ii) $\sigma_0(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, s$
(iii) $\sigma_0{}''\overline{G}_0 \subset G_0$
(iv) $t^{G_0} \in \overline{N}$.

Let $h = h_{\mathbb{A},\mathbb{B}}$. Then there are $b \in \mathbb{B} \setminus \{0\}$, $\overset{\circ}{\sigma} \in \mathbf{V}^{\mathbb{B}}$ s.t. $a = h(b)$ and whenever $G \ni b$ is $\mathbb{B}$-generic, $\sigma = \overset{\circ}{\sigma}{}^G$, and $G_0 = G \cap \mathbb{A}$, then

(a) $\sigma : \overline{N} \prec N$
(b) $\sigma(\overline{\theta}, \overline{\mathbb{A}}, \overline{\mathbb{B}}, \overline{s}) = \theta, \mathbb{A}, \mathbb{B}, s$
(c) $C^N_\delta(\text{rng}(\sigma)) = C^N_\delta(\text{rng}(\sigma_0)) \quad (\delta = \delta(\mathbb{B}))$
(d) $\sigma{}''\overline{G} \subset G$
(e) $\sigma(t^{G_0}) = \sigma_0(t^{G_0})$.

⋆ ⋆ ⋆ ⋆ ⋆

We now prove a theorem about iterations of length ω.

Theorem 2 *Let $\langle \mathbb{B}_i \mid i < \omega \rangle$ be s.t. $\mathbb{B}_0 = 2$, $\mathbb{B}_i \subseteq \mathbb{B}_{i+1}$ and $\Vdash_{\mathbb{B}_i} (\check{\mathbb{B}}_{i+1}/\overset{\circ}{G}$ is subcomplete) for $i < \omega$. Let $\mathbb{B}_\omega$ be the inverse limit of $\langle \mathbb{B}_i \mid i < \omega \rangle$. Then $\mathbb{B}_\omega$ is subcomplete.*

Proof. Let θ be big enough that $\Vdash_{\mathbb{B}_i} \check{\theta}$ verifies the subcompleteness of $\check{\mathbb{B}}_{i+1}/\overset{\circ}{G}$ for $i < \omega$. Let $N = L^A_\tau$ s.t. $H_\theta \subset N$, $\theta < \tau$, and N is a ZFC$^-$ model. Let $\sigma : \overline{N} \prec N$ s.t. $\sigma(\overline{\theta}, \langle \overline{\mathbb{B}}_i \mid i \leq \omega \rangle, \overline{s}) = \theta, \langle \mathbb{B}_i \mid i \leq \omega \rangle, s$, and $\overline{N}$ is countable and full. Let $\overline{G} = \overline{G}_\omega$ be $\overline{\mathbb{B}}_\omega$-generic over $\overline{N}$ and set $\overline{G}_i = \overline{G} \cap \overline{\mathbb{B}}_i$. Then $\overline{G}_i$ is $\overline{\mathbb{B}}_i$-generic over $\overline{N}$. We claim that there is $b \in \mathbb{B}_\omega \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}_\omega$-generic, there is $\sigma' \in \mathbf{V}[G]$ s.t.

(a) $\sigma' : \overline{N} \prec N$
(b) $\sigma'(\overline{\theta}, \langle \overline{\mathbb{B}}_i \mid i < \omega \rangle, \overline{s}) = \theta, \langle \mathbb{B}_i \mid i < \omega \rangle, s$
(c) $C^N_\delta(\text{rng}(\sigma')) = C^N_\delta(\text{rng}(\sigma))$, where $\delta = \delta(\mathbb{B}_\omega)$.
(d) $\sigma'{}''\overline{G} \subset G$.

We first construct a sequence b_i, $\overset{\circ}{\sigma}_i$ $(i < \omega)$ s.t. $b_i \in \mathbb{B}_i$, $h_i(b_{i+1}) = b_i$ (where $h_i = h_{\mathbb{B}_i,\mathbb{B}_{i+1}}$ and whenever $G_i \ni b_i$ is $\mathbb{B}_i$-generic, then, letting $\sigma_i = \overset{\circ}{\sigma}{}_i^{G_i}$, we have:

(a′) $\sigma_i : \overline{N} \prec N$
(b′) $\sigma_i(\overline{\theta}, \langle \overline{\mathbb{B}}_i \mid i \leq \omega \rangle, \overline{s}) = \theta, \langle \mathbb{B}_i \mid i \leq \omega \rangle, s$
(c′) $C^N_\delta(\text{rng}(\sigma_i)) = C^N_\delta(\text{rng}(\sigma))$
(d′) $\sigma_i''\overline{G} \subset G_i$.

Now let $\langle x_i \mid i < \omega\rangle$ enumerate $\overline{N}$. Set: $u_i =$ the $\overline{N}$-least u s.t. $\sigma(x_i) \in \sigma_i(u)$ and $\overline{\overline{u}} \leq \overline{\delta} =_{\mathrm{Df}} \delta(\overline{\mathbb{B}})$ in $\overline{N}$. (This exists, since $\mathrm{rng}(\sigma) \subset C^N_\delta(\mathrm{rng}(\sigma_i)) = \bigcup\{\sigma_i(u) \mid \overline{\overline{u}} \leq \overline{\delta}$ in $\overline{N}\}$ by Chapter 3, Lemma 5.5.) σ_i will satisfy the additional requirements:

(e′) $\sigma_0 = \sigma$
(f′) $\sigma_i(x_h) = \sigma_h(x_h)$ for $h < i$, where $\sigma_h =_{\mathrm{Df}} \overset{\circ}{\sigma}_h(G_i \cap \mathbb{B}_h)$.

(**Note** Then $\sigma_h = \overset{\circ}{\sigma}{}^{G_i}_h$, since we assume: $\mathbf{V}^{\mathbb{B}_h} \subseteq \mathbf{V}^{\mathbb{B}_i}$ (i.e. the identity is the natural injection of $\mathbf{V}^{\mathbb{B}_h}$ into $\mathbf{V}^{\mathbb{B}_i}$). Thus $t^{G_i \cap \mathbb{B}_h} = t^{G_i}$ for $t \in \mathbf{V}^{\mathbb{B}_h}$, $h < i$.)

(g′) $\sigma_i(u_h) = \sigma_h(u_h)$ for $h < i$.

Note that $u_i = \overset{\circ}{u}{}^{G_i}_i$ for a $\overset{\circ}{u}_i \in \mathbf{V}^{\mathbb{B}_i}$. We set: $b_0 = 1$, $\overset{\circ}{\sigma}_0 = \check{\sigma}$. Given b_i, $\overset{\circ}{\sigma}_i$, Lemma 1.1 then gives us b_{i+1}, $\overset{\circ}{\sigma}_{i+1}$. (Take $\sigma_{i+1}(t^{G_i}) = \sigma_i(t^{G_i})$ where $\Vdash_{\mathbb{B}_i} t = \langle \check{x}_0, \ldots, \check{x}_i, \overset{\circ}{u}_0, \ldots, \overset{\circ}{u}_i\rangle$.) Since $h_i(b_{i+1}) = b_i$, the sequence $\vec{b} = \langle b_i \mid i < \omega\rangle$ is a thread in $\langle \mathbb{B}_i \mid i < \omega\rangle$. Hence $b = \bigcap_i b_i \neq 0$ in $\mathbb{B}_\omega$, since $\mathbb{B}_\omega$ is the inverse limit. Now let $G \ni b$ be $\mathbb{B}_\omega$-generic. Set $G_i = G \cap \mathbb{B}_i$, $\sigma_i = \overset{\circ}{\sigma}{}^G_i = \overset{\circ}{\sigma}{}^{G_i}_i$. Then (a′)–(g′) hold for $i < \omega$. By (f′) we can define $\sigma' : \overline{N} \prec N$ by: $\sigma'(x) = \sigma_i(x)$ for i s.t. $\sigma_i(x) = \sigma_j(x)$ for $i \leq j$. (a), (b) are then trivial. We prove:

(c) $C^N_\delta(\mathrm{rng}(\sigma')) = C^N_\delta(\mathrm{rng}(\sigma))$.

Proof. Set $C_i = C^N_\delta(\mathrm{rng}(\sigma_i))$ for $i < \omega$. (Hence $C_0 = C^N_\delta(\mathrm{rng}(\sigma))$.)
($\subset$) It suffices to show $\mathrm{rng}(\sigma') \subset C_0$. But $\sigma'(x_i) = \sigma_i(x_i) \in C_i = C_0$.
($\supset$) We show $\mathrm{rng}(\sigma) \subset C^N_\delta(\mathrm{rng}(\sigma'))$.
$\sigma(x_i) \in \sigma_i(u_i) = \sigma'(u_i) \subset \bigcup\{\sigma'(u) \mid \overline{\overline{u}} \leq \overline{\delta}$ in $\overline{N}\} = C^N_\delta(\mathrm{rng}(\sigma'))$. QED(c)

Finally we show:

(d) $\sigma'{}''\overline{G} \subset G$.

Proof. We first note that $\sigma'{}''\overline{G}_i \subset G$ for $i < \omega$, since if $a \in \overline{G}_i$, then $\sigma'(a) = \sigma_j(a) \in G_j \subset G$ for some $j \geq i$. Now let $\overline{a} \in G$. Since $\overline{\mathbb{B}}_\omega$ is the inverse limit of $\langle \overline{\mathbb{B}}_i \mid i < \omega\rangle$, we may assume w.l.o.g. that $\overline{a} = \bigcap_{i<\omega} \overline{a}_i$ where $\langle \overline{a} \mid i < \omega\rangle$ is a thread in $\langle \overline{\mathbb{B}}_i \mid i < \omega\rangle$. Let $\sigma'(\langle \overline{a}_i \mid i < \omega\rangle) = \langle a_i \mid i < \omega\rangle$. Then $\langle a_i \mid i < \omega\rangle$ is a thread in $\langle \mathbb{B}_i \mid i < \omega\rangle$ and $\sigma'(\overline{a}) = \sigma'(\bigcap_i \overline{a}_i) = \bigcap_i a_i \in G$ by the completeness of G wrt. $\mathbf{V}$, since $a_i \in G$ for $i < \omega$. QED(Theorem 2)

Note Theorem 2 can be generalized to countable support iterations of length $< \omega_2$. At ω_2 it can fail, however, since in a countable support iteration we are required to take a direct limit at ω_2. If some earlier stage changed the cofinality of ω_2 to ω (e.g. if $\mathbb{B}_1$ were Namba forcing), then the direct limit would not be subcomplete. Hence for longer iterations we must employ *revised countable support* iterations, which we discuss in the next section.

Revised countable support iterations

Definition By an *iteration* we mean a sequence $\langle \mathbb{B}_i \mid i < \alpha \rangle$ s.t.

- $\mathbb{B}_0 = 2$
- $\mathbb{B}_i \subseteq \mathbb{B}_j$ for $i \leq j < \alpha$
- If $\lambda < \alpha$ is a limit ordinal, then $\bigcup\limits_{i<\lambda} \mathbb{B}_i$ generates $\mathbb{B}_\lambda$.

In dealing with an iteration we shall employ obvious notational simplifications, writing e.g. $\Vdash_i$ for $\Vdash_{\mathbb{B}_i}$, $[[\varphi]]_i$ for $[[\varphi]]_{\mathbb{B}_i}$ etc. We also write: $h_i(b) = h_{\mathbb{B}_i}(b) =_{\mathrm{Df}} \bigcap\{a \in \mathbb{B}_i \mid b \subset a\}$ in $\mathbb{B}_i$, for $b \in \bigcup\limits_{j<\alpha} \mathbb{B}_j$. Recall that:

- $h_i(b) \neq 0 \leftrightarrow b \neq 0$
- $h_i(\bigcup\limits_{j \in I} b_j) = \bigcup\limits_{i \in I} h_i(b_j)$
- $a \cap h_i(b) = h_i(a \cap b)$ for $a \in \mathbb{B}_i$
- $h_i(b) = [[\check{b}/\mathring{G} \neq 0]]_i$.

Our definition of "iteration" permits great leeway in defining $\mathbb{B}_\lambda$ at limit λ. In practice people usually employ one of a number of standard limiting procedures, such as *finite support* (FS), *countable support* (CS) or *revised countable support* (RCS) iterations. RCS iterations are particulary suited to subcomplete forcing. The definition of RCS iteration is given in Chapter 0. For present purposes all we need to know is:

Fact Let $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ be an RSC iteration. Then:

(a) If $\lambda < \alpha$ and $\langle \xi_i \mid i < \omega \rangle$ is monotone and cofinal in λ, then:

(i) If $\langle b_i \mid i < \omega \rangle$ is a thread through $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$, then $\bigcap\limits_{i<\omega} b_i \neq \emptyset$ in $\mathbb{B}_i$.

(ii) The set of such $\bigcap\limits_i b_i$ is dense in $\mathbb{B}_\lambda$.

(b) If $\lambda < \alpha$ and $\Vdash_i \mathrm{cf}(\check{\lambda}) > \omega$ for $i < \lambda$, then $\bigcup\limits_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$.

(c) If $i < \lambda$ and G is $\mathbb{B}_i$-generic, then the iteration $\langle \mathbb{B}_{i+j}/G \mid j < \alpha - i \rangle$ satisfies (a), (b) in $\mathbf{V}[G]$.

(**Note** By a "thread" through $\langle \mathbb{B}_i \mid i < \omega \rangle$ we mean a sequence $\langle b_i \mid i < \omega \rangle$ wrt. $b_0 \neq 0$, $b_i \in \mathbb{B}_i$, and $h_i(b_j) = b_i$ for $i \leq j < \omega$.)

Theorem 3 *Let* $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ *be an* RCS-*iteration s.t. for all* $i + 1 < \alpha$*:*

(a) $\mathbb{B}_i \neq \mathbb{B}_{i+1}$

(b) $\Vdash_i$ ($\check{\mathbb{B}}_{i+1}/\mathring{G}$ *is subcomplete*)

(c) $\Vdash_{i+1}$ ($\delta(\check{\mathbb{B}}_i)$ *has cardinality* $\leq \omega_1$).

Then every $\mathbb{B}_i$ *is subcomplete.*

Proof. Set: $\delta_i = \delta(\mathbb{B}_i)$. Then

(1) $\delta_i \leq \delta_j$ for $i \leq j < \alpha$,

since if X is dense in $\mathbb{B}_j$, then $\{h_i(a) \mid a \in X\}$ is dense in $\mathbb{B}_i$.

(2) $\overline{\overline{\nu}} \leq \delta_\nu$ for $\nu < \alpha$.

Proof. Suppose not. Let ν be the least counterexample. Then $\nu > 0$ is a cardinal. If $\nu < \omega$, then $\delta_\nu < \omega$ and hence $\mathbb{B}_\nu$ is atomic with δ_ν the number of atoms. Let $\nu = n+1$. Then $\delta_n < \delta_\nu < n+1$ by (a). Hence $\delta_n < n$. Contradiction! Hence $\nu \geq \omega$ is a cardinal. If ν is a limit cardinal, then $\delta_\nu \geq \sup_{i<\nu} \delta_\nu \geq \nu$. Contradiction! Thus ν is a successor cardinal. Let $X \subset \mathbb{B}_\nu$ be dense in $\mathbb{B}_\nu$ with $\overline{\overline{X}} = \delta_\nu < \nu$. Then $X \subset \mathbb{B}_\eta$ for an $\eta < \nu$ by the regularity of ν. Hence $\mathbb{B}_\eta = \mathbb{B}_\nu$, contradicting (a).

QED(2)

By induction on i we prove:

Claim Let G be $\mathbb{B}_h$-generic, $h \leq i$. Then $\mathbb{B}_i/G$ is subcomplete in $\mathbf{V}[G]$. (Hence $\mathbb{B}_i \simeq \mathbb{B}_i/\{1\}$ is subcomplete in $\mathbf{V}$, taking $h = 0$.) The case $h = i$ is trivial, since then $\mathbb{B}_i/G \simeq 2$. Hence $i = 0$ is trivial. Now let $i = j+1$. Then $\mathbb{B}_j/G \subset \mathbb{B}_i/G$. Let $\tilde{G}$ be $\mathbb{B}_j/G$-generic over $\mathbf{V}[G]$. Then $G' = G * \tilde{G} = \{b \in \mathbb{B}_j \mid b/G \in \tilde{G}\}$ is $\mathbb{B}_j$-generic over $\mathbf{V}$. But then $(\mathbb{B}_i/G)/\tilde{G} \simeq \mathbb{B}_i/G'$ is subcomplete in $\mathbf{V}[G'] = \mathbf{V}[G][\tilde{G}]$ by (b). Hence we have shown: $\Vdash_{\mathbb{B}_j/G} ((\check{\mathbb{B}_i}/G)/\mathring{G}$ is subcomplete).But $\mathbb{B}_j/G$ is subcomplete in $\mathbf{V}[G]$ by the induction hypothesis, so it follows by the two step theorem that $\mathbb{B}_i/G$ is subcomplete in $\mathbf{V}[G]$. There remains the case that $i = \lambda$ is a limit ordinal. By our induction hypothesis $\mathbb{B}_j/G_h$ is subcomplete in $\mathbf{V}[G_h]$ for $h \leq j < \lambda$. But then $\langle \mathbb{B}_{h+i}/G_h \mid i < \lambda - h\rangle$ satisfies the same induction hypothesis, since if $i \leq k < \lambda - h$ and $\hat{G}$ is $\mathbb{B}_{h+i}/G_h$-generic over $\mathbf{V}[G_h]$, then $G = G_h * \tilde{G}$ is $\mathbb{B}_{h+i}$-generic over $\mathbf{V}$ and $(\mathbb{B}_{h+k}/G_h)/\tilde{G} \simeq \mathbb{B}_{h+k}/G$ is subcomplete in $\mathbf{V}$.

Case 1 $\mathrm{cf}(\lambda) \leq \delta_i$ for an $i < \lambda$.
Then $\mathrm{cf}(\lambda) \leq \omega_1$ in $\mathbf{V}[G_j]$ for $i < j < \lambda$ whenever G_j is $\mathbb{B}_j$-generic. It suffices to prove the claim for such j, since if $h < j$ and G_h is $\mathbb{B}_h$-generic, we can then use the two step theorem to show – exactly as in the successor case – that $\mathbb{B}_\lambda/G_h$ is subcomplete in $\mathbf{V}[G_h]$. Hence it will suffice to prove:

Claim Assume $\mathrm{cf}(\lambda) \leq \omega_1$ in $\mathbf{V}$. Then $\mathbb{B}_\lambda$ is subcomplete,

since the same proof can then be carried out in $\mathbf{V}[G_j]$ to show that $\mathbb{B}_\lambda/G_j$ is subcomplete. Fix $f : \omega_1 \to \lambda$ s.t. $\sup f''\omega_1 = \lambda$. Let $\theta > \lambda$ be a cardinal s.t. $\overline{\overline{\mathbb{B}}} < \theta$ and θ is big enough that:

$$\Vdash_i (\check{\theta} \text{ witnesses the subcompleteness of } \check{\mathbb{B}}_j/\mathring{G})$$

for $i \leq j < \lambda$. Let $N = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset N$, $\theta < \tau$. Let $\sigma : \overline{N} \prec N$ s.t. $\overline{N}$ is countable and full. Suppose also that: $\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{f}, \overline{s}) = \theta, \mathbb{B}, \lambda, f, s$.

Claim There is $b \in \mathbb{B}_\lambda \setminus \{0\}$ s.t. whenever $G \ni b$ is $\mathbb{B}_\lambda$-generic, there is $\sigma' \in \mathbf{V}[G]$ s.t.

(a) $\sigma' : \overline{N} \prec N$
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{f}, \overline{s}) = \theta, \mathbb{B}, \lambda, f, s$
(c) $C^N_\delta(\text{rng}(\sigma')) = C^N_\delta(\text{rng}(\sigma))$, where $\sigma = \sup\{\delta_i \mid i < \lambda\}$.
(d) $\sigma'{}''\overline{G} \subset G$.

Set: $\tilde{\lambda} = \sup \sigma''\overline{\lambda}$. It is easily verified that there is a sequence $\langle \nu_i \mid i < \omega \rangle$ in $\omega_1^{\overline{N}}$ s.t., setting $\overline{\xi}_i = \overline{f}(\nu_i)$, we have: $\overline{\xi}_0 = 0$, and $\langle \overline{\xi} \mid i < \omega \rangle$ is monotone and cofinal in $\overline{\lambda}$. (We can assume w.l.o.g. that $\overline{f}(0) = 0$.) Set $\xi_i = f(\nu_i)$. Then $\xi_i = \sigma(\overline{\xi}_i)$ and $\langle \xi_i \mid i < \omega \rangle$ is monotone and cofinal in $\tilde{\lambda}$. Moreover:

(3) $\sigma'(\overline{\xi}_i) = \xi_i$ whenever $\sigma' : \overline{N} \prec N$ s.t. $\sigma'(\overline{f}) = f$.

We now closely imitate the proof of Theorem 2, constructing a sequence b_i, $\overset{\circ}{\sigma}_i$ $(i < \omega)$ s.t. $b_i \in \mathbb{B}_{\xi_i}$, $h_{\xi_i}(b_{i+1}) = b_i$, and whenever $G_i \ni b_i$ is $\mathbb{B}_{\xi_i}$-generic, then, letting $\sigma_i = \overset{\circ}{\sigma}{}_i^{G_i}$, we have:

(a′) $\sigma_i : \overline{N} \prec N$
(b′) $\sigma_i(\overline{\theta}, \overline{\mathbb{B}}, \overline{f}, \overline{s}) = \theta, \mathbb{B}, f, s$
(c′) $C^N_{\delta_i}(\text{rng}(\sigma_i)) = C^N_\delta(\text{rng}(\sigma))$
(d′) $\sigma_i{}''\overline{G}_i \subset G_i$
(e′) $\sigma_0 = \sigma$
(f′) $\sigma_i(x_h) = \sigma_h(x_h)$ $(h \le i)$ where $\sigma_h = \overset{\circ}{\sigma}{}_h^{G_i}$
($\langle x_\ell \mid \ell < \omega \rangle$ being an arbitrarily chosen enumeration of $\overline{N}$.)
(g′) $\sigma_i(u_h) = \sigma_h(u_h)$ $(h \le i)$, where $u_i =$ the $\overline{N}$-least u s.t. $\sigma(x_i) \in \sigma_i(u)$ and $\overline{\overline{u}} < \overline{\delta} =_{\text{Df}} \sigma^{-1}(\delta)$ in $\overline{N}$.

The construction is exactly as before using that $\sigma_i(\overline{\mathbb{B}}_{\overline{\xi}_j}) = \mathbb{B}_{\xi_j}$ for all j and that $\Vdash_{\mathbb{B}_{\xi_j}} (\check{\mathbb{B}}_{\xi_{j+1}}/\overset{\circ}{G}$ is subcomplete). As before set: $\sigma'(x) = \sigma_i(x)$, where i is big enough that $\sigma_i(x) = \sigma_j(x)$ for $i \le j$. The verification of (a)–(c) is exactly as before. To verify (d), we first note that, as before,

(2) $\sigma'{}''G_i \subset G$ for $i < \omega$.

We then consider two cases: If $\text{cf}(\lambda) = \omega$ in N, then $\text{cf}(\overline{\lambda}) = \omega$ in $\overline{N}$ and $\tilde{\lambda} = \lambda$. $\overline{\mathbb{B}}_{\overline{\lambda}}$ is then the inverse limit of $\langle \overline{\mathbb{B}}_{\overline{\xi}_i} \mid i < \omega \rangle$ and $\mathbb{B}_\lambda$ is the inverse limit of $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$. We then proceed exactly as before. If $\text{cf}(\lambda) = \omega_1$, $\overline{\mathbb{B}}_\lambda$ is the direct limit – i.e. $\bigcup_{i<\omega} \overline{\mathbb{B}}_i$ is dense in $\overline{\mathbb{B}}_\lambda$. The conclusion then follows by (2). QED(Case 1)

Case 2 Case 1 fails.
Then λ is regular and $\delta_i < \lambda$ for all $i < \lambda$. Hence $\lambda = \sup_{i<\lambda} \delta_i$. Let N, $\overline{N}$, θ, σ, $\overline{G}$ be as before with $\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{s}, \overline{\lambda}) = \theta, \mathbb{B}, s, \lambda$. (However, there is now nothing corresponding to the function f.) As before set: $\tilde{\lambda} = \sup \sigma''\lambda$. It suffices to show:

Claim There is $c \in \mathbb{B}_\lambda$ s.t. whenever $G \ni c$ is $\mathbb{B}_\lambda$-generic, there is $\sigma' \in \mathbf{V}[G]$ with:

(a) $\sigma' : \overline{N} \prec N$
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{s}) = \theta, \mathbb{B}, \lambda, s$

(c) $C^N_\lambda(\text{rng}(\sigma')) = C^N_\lambda(\text{rng}(\sigma))$
(d) $\sigma' {}''\overline{G} \subset G$.

Choose a sequence $\langle \overline{\xi}_i \mid i < \omega \rangle$ which is monotone and cofinal in $\overline{\lambda}$ with $\overline{\xi}_0 = 0$. Set: $\xi_i = \sigma(\overline{\xi}_i)$. As before, our strategy is to construct c_i, $\overset{\circ}{\sigma}_i$ $(i < \omega)$ s.t. $c_0 = 1$, $\overset{\circ}{\sigma}_0 = \check{\sigma}$, $\langle c_i \mid i < \omega \rangle$ is a thread in $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$, and c_i forces $\overset{\circ}{\sigma}_i : \check{\overline{N}} \prec \check{N}$. The intention is, again, that if $c = \bigcap_i c_i \in G$ and G is $\mathbb{B}_\lambda$-generic, then we can define the embedding $\sigma' \in \mathbf{V}[G]$ from the sequence $\sigma_i = \overset{\circ}{\sigma}{}^G_i$ $(i < \omega)$. However, since we no longer have the function f available in defining $\langle \xi_i \mid i < \omega \rangle$, we shall *not* be able to enforce: $\sigma_i(\overline{\xi}_j) = \xi_j$ for $j < \omega$. Nonetheless we can enforce: $\sup \sigma_i {}''\overline{\lambda} = \tilde{\lambda}$, and shall have to make do with that. We inductively construct $c_i \in \mathbb{B}_{\xi_i}$, $\overset{\circ}{\sigma}_i \in \mathbf{V}^{\mathbb{B}_{\xi_i}}$ with the properties:

(I) (a) $c_0 = 1$
  (b) $h_{\xi_h}(c_i) = c_h$ for $i = h + 1$.

(II) Let $G \ni c_i$ be $\mathbb{B}_{\xi_i}$-generic. Set: $G_\eta = G \cap \mathbb{B}_\eta$ $(\eta \leq \xi_i)$, $\overline{G}_\eta = \overline{G} \cap \overline{\mathbb{B}}_\eta$ $(\eta \leq \overline{\xi}_i)$, $\sigma_h = \overset{\circ}{\sigma}{}^G_h = \overset{\circ}{\sigma}{}^{G_{\xi_h}}_h$ for $h \leq i$. Then:
  (a) $\sigma_i : \overline{N} \prec N$
  (b) $\sigma_i(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{s}) = \theta, \mathbb{B}, \lambda, s$
  (c) $C^N_\lambda(\text{rng}(\sigma_i)) = C^N_\lambda(\text{rng}(\sigma))$
  (d) Let $\sigma_i(\overline{\xi}_m) \leq \xi_i < \sigma_i(\overline{\xi}_{m+1})$. Then $\sigma_i {}''\overline{G}_{\overline{\xi}_m} \subset G$.
  (e) $\sigma_i(x_h) = \sigma_h(x_h)$ for $h < i$, $\langle x_\ell \mid \ell < \omega \rangle$ being a fixed enumeration of $\overline{N}$.
  (f) $\sigma_i(u_h) = \sigma_h(u_h)$ for $h < i$, where $u_h =$ the $\overline{N}$-least u s.t. $\sigma(x_h) \in \sigma_h(u)$.
  (g) $\sigma_i = \sigma_h$ if $\sigma_h(\overline{\xi}_m) \leq \xi_h < \xi_i < \sigma_h(\overline{\xi}_{m+1})$

(I), (II) are easily seen to imply the claim. Set $c = \bigcap_i c_i$. Then $c \neq 0$, since c is a thread in $\langle \mathbb{B}_{\xi_i} \mid i < \omega \rangle$. Let $G \ni c$ be $\mathbb{B}_\lambda$-generic. Define $\sigma_i = \overset{\circ}{\sigma}{}^G_i$ $(i < \omega)$ and define $\sigma'(x) = \sigma_j(x)$ where $\sigma_j(x) = \sigma_k(x)$ for all $k \geq j$. (a)–(c) follow exactly as before. We prove (d). Since $\overline{\mathbb{B}}_{\overline{\lambda}}$ is the direct limit of $\langle \overline{\mathbb{B}}_{\overline{\xi}_i} \mid i < \omega \rangle$, it suffices to show:

(d') $\sigma' {}''\overline{G} \subset G$ for $i < \omega$, where $\overline{G}_\eta = \overline{G} \cap \overline{B}_\eta$.

Proof. Let $a \in \overline{G}_{\overline{\xi}_i}$. We first note that for $j \geq i$ sufficiently large we have: $\sigma_j(\overline{\xi}_m) \leq \xi_j < \sigma_j(\overline{\xi}_{m+1})$ for an $m \geq i$, since otherwise $\xi_j < \sigma_j(\overline{\xi}_i)$ for arbitrarily large j. But $\sigma_j(\overline{\xi}_i) = \sigma'(\overline{\xi}_i)$ for sufficient large j. Hence $\sigma'(\overline{\xi}_i) \geq \sup_j \xi_j = \lambda$. Contradiction! If we also pick j large enough that $\sigma_j(a) = \sigma'(a)$, then $\sigma'(a) = \sigma_j(a) \in G$, since $a \in G_{\xi_m}$. QED(d)

It remains only to construct c_i, $\overset{\circ}{\sigma}_i$ and verify (I), (II). This will be somewhat trickier than the construction in Theorem 2. We shall also have to add further induction hypotheses to (I), (II). Before defining c_i we define a $b_i \in \mathbb{B}_{\xi_i}$ s.t.

(III) (a) $b_0 = 1$, $\overset{\circ}{\sigma}_0 = \check{\sigma}$

(b) $h_{\xi_j}(b_i) = c_j$ if $i = j + 1$
(c) (II)(a)–(g) hold whenever $b_i \in G$.

$\overset{\circ}{\sigma}_i$ will be defined simultaneously with b_i, before defining c_i. Our next induction hypothesis states an important porperty of b_i:

Definition Let $\nu \leq \xi_i < \mu < \tilde{\lambda}$ s.t. $\xi_h < \nu$ for $h < i$,

$$a^{j\nu\mu} =_{\mathrm{Df}} b_i \cap [[\overset{\circ}{\sigma}_i(\check{\overline{\xi}}_j) = \check{\nu} \wedge \overset{\circ}{\sigma}_i(\check{\overline{\xi}}_{j+1}) = \check{\mu}]]_{\xi_i}.$$

It follows easily that:

(4) $a^{j\nu\mu} \cap a^{j'\nu'\mu'} = 0 \quad$ if $\quad \langle j, \nu, \mu \rangle \neq \langle j', \nu', \mu' \rangle$

Proof. Suppose $a^{j\nu\mu} \cap a^{j'\nu'\mu'} \in G$ where G is $\mathbb{B}_{\xi_i}$-generic. Then $j = j'$, since if e.g. $j < j'$, then $\mu \leq \sigma_i(\overline{\xi}_{j+1}) \leq \sigma_i(\overline{\xi}_{j'}) = \nu' \leq \xi_i$ Contradiction! But then $\nu = \sigma_i(\xi_j) = \nu'$, $\mu = \sigma_i(\overline{\xi}_{j+1}) = \mu'$. Contradiction! QED(1)

Our final induction hypothesis reads:

(IV) $a^{j\nu\mu} \cap [[\overset{\circ}{\sigma}_i(\check{x}) = \check{y}]]_{\xi_i} \in \mathbb{B}_\nu$ if $\sup\limits_{h<i} \xi_h < \nu \leq \xi_i < \mu$.

Hence $a^{j\nu\mu} = a^{j\nu\mu} \cap [[\overset{\circ}{\sigma}_i(\check{0}) = \check{0}]] \in \mathbb{B}_\nu$.

Definition $A = A_i =$ the set of all $a^{j\nu\mu} \neq 0$ s.t. $\sup\limits_{h<i} \xi_h < \nu \leq \xi_i < \mu$.

By (IV) we see that for each $a = a^{j\nu\mu} \in A$ there is $\overset{\circ}{\sigma}_a \in \mathbf{V}^{\mathbb{B}_\nu}$ s.t.

(5) $\overset{\circ}{\sigma}{}_a^{G_\nu} = \sigma_i^G$ for $\mathbb{B}_{\xi_i}$-generic $G \ni a$.

But:

(6) If $G \ni a$ is $\mathbb{B}_\nu$-generic, then G extends to a $\mathbb{B}_{\xi_i}$-generic G' s.t. $G = G' \cap \mathbb{B}_\nu$. Hence: $\overset{\circ}{\sigma}{}_a^G = \overset{\circ}{\sigma}{}_a^{G'} = \overset{\circ}{\sigma}{}_i^{G'}$.

Thus we have:

(7) Let $G \ni a$ be $\mathbb{B}_\nu$-generic, where $a = a^{j\nu\mu} \in A_i$. Then (II) holds with $\sigma_a = \overset{\circ}{\sigma}{}_a^G$ in place of σ_i, $\sigma_h = \overset{\circ}{\sigma}{}_h^G = \overset{\circ}{\sigma}{}_h^{G_{\xi_h}}$ for $h < i$, where $G_\eta =_{\mathrm{Df}} G \cap \mathbb{B}_\eta$ $(\eta \leq \nu)$.

Note Since $a \in G$, (d) then reduces to: $\sigma_a{}''\overline{G}_{\overline{\xi}_j} \subset G$.

Note We then have: $\sigma_a(x_h) = \sigma_h(x_h)$, $\sigma_a(u_h) = \sigma_h(u_h)$ for $h < i$.

Whenever $\nu < \mu < \lambda$ and G is $\mathbb{B}_\nu$-generic, we know that $\mathbb{B}_\mu/G$ is subcomplete in $\mathbf{V}[G]$. Then, using (7), Lemma 1.1, and repeating the construction of b_{i+1}, $\overset{\circ}{\sigma}_{i+1}$ from b_i, $\overset{\circ}{\sigma}_i$ in the proof of Theorem 2, we get:

(8) Let $a \in A_i$, $a = a^{j\nu\mu}$. There are $\tilde{a} \in \mathbb{B}_\mu$, $\overset{\circ}{\sigma}{}'_a \in \mathbf{V}^{\mathbb{B}_\mu}$ s.t. $h_\nu(\tilde{a}) = a$ and whenever $G \ni \tilde{a}$ is $\mathbb{B}_\mu$-generic, $\sigma_a = \overset{\circ}{\sigma}{}^G$, and $\sigma'_a = \overset{\circ}{\sigma}{}'^G_a$, then we have:

- (a) $\sigma'_a : \overline{N} \prec N$
- (b) $\tau'_a(\overline{\theta}, \overline{\mathbb{B}}, \overline{\lambda}, \overline{s}) = \theta, \mathbb{B}, \lambda, s$
- (c) $C^N_\lambda(\text{rng}(\sigma'_a)) = C^N_\lambda(\text{rng}(\sigma_a))$
- (d) $\sigma'_a{}''\overline{G}_{\overline{\xi}_{j+1}} \subset G$
- (e) Let r be least s.t. $\mu \le \xi_r$. Then $\sigma'_a(x_h) = \sigma_a(x_h)$ for $h < r$.
- (f) Let r be as above. Then $\sigma'_a(u^a_h) = \sigma_a(u^a_h)$ for $h < r$, where $u^a_h =$ the $\overline{N}$-least $u \in \overline{N}$ s.t. $\overline{\overline{u}} \le \overline{\lambda}$ in $\overline{N}$ and $\sigma(x_h) \in \sigma_a(u_h)$.
- (g) $\sigma'_a(\overline{\xi}_\ell) = \sigma_a(\overline{\xi}_\ell)$ for $\ell \le j+1$.

For each $a \in A_i$ fix such a pair $\tilde{a}$, $\overset{\circ}{\sigma}{}'_a$, which can be regarded as an instruction to be used later in forming b_r, where r is least s.t. $\mu \le \xi_r$. If G is $\mathbb{B}_{\xi_r}$-generic and $a \cap b_r \in G$, we shall want: $\tilde{a} \in G$ and $\sigma_r = \overset{\circ}{\sigma}{}'^G_a$ (where $\sigma_r = \overset{\circ}{\sigma}{}^G_r$). In particular, we want: $a \cap b_r = \tilde{a}$. But we shall also require: $h_{\xi_i}(b_r) = c_i$. Hence we need: $a \cap c_i = h_\xi(a \cap b_r) = h_\xi(\tilde{a})$. This is why b_i must be "shrunk" to c_i. Accordingly we define c_i as follows:

Definition Let b_i be given. Set $\overline{b} = b_i \setminus \bigcup A_i$. Then:

$$c_i =_{\text{Df}} \overline{b} \cup \bigcup_{a \in A_i} h_{\xi_i}(\tilde{a}).$$

We are working by induction on i. We assume (I)–(IV) to hold below i and (III), (IV) to hold at i. We must now verify (I), (II) at i. (II) is immediate by (III)(c), since $c_i \subset b_i$. (I)(b) holds, since $h_{\xi_j} h_{\xi_i}(\tilde{a}) = h_{\xi_j}(\tilde{a}) = h_{\xi_j} h_\nu(\tilde{a}) = h_{\xi_j}(a)$ for $i = j+1$ and $a = a^{\ell\mu\nu} \in A_i$. Hence:

$$h_{\xi_j}(c_i) = h_{\xi_j}(\overline{b}) \cup \bigcup_{a \in A} h_{\xi_j}(\tilde{a}) = h_{\xi_j}(\overline{b} \cup \bigcup_{a \in A} h_{\xi_i}(\tilde{a})) = h_{\xi_j}(b_i) = c_j.$$

For (I)(a) note that $A_0 = \{a\}$ where $a = a^{0,0,\xi_1} = 1$, since $\sigma_0 = \sigma$ by (III)(a). Hence $c_0 = h_0(\tilde{a}) = 1$. This completes the verification of (I)–(IV) at i, given (III), (IV) at i and (I)–(IV) below i. Now let (I)–(IV) hold below i. We must define b_i, $\overset{\circ}{\sigma}_i$ and verify (III), (IV) at i. For $i = 0$ set: $b_0 = 1$, $\overset{\circ}{\sigma}_0 = \check{\sigma}$. The verifications are trivial. Now let $i = j+1$. Note that A_ℓ, $\langle \tilde{a} \mid a \in A_\ell \rangle$, $\langle \sigma'_a \mid a \in A_\ell \rangle$ have been defined for $\ell \le j$. Set:

Definition $\hat{A}_j =$ the set of $a = a^{h\nu\mu} \in \bigcup_{\ell \le j} A_\ell$ s.t. $\xi_j < \mu$.

(9) Let $a, a' \in \hat{A}_j$, $a = a^{h\nu\mu}$, $a' = a^{h'\nu'\mu'}$. Then $a \cap a' = 0$ if $\langle h, \nu, \mu \rangle \ne \langle h', \nu', \mu' \rangle$.

Proof. Suppose not. Let $a \in A_\ell$, $a' \in A_{\ell'}$. Then $\ell \ne \ell'$ by (4). Let e.g. $\ell < \ell'$ Let $a \cap a' \in G$, where G is $\mathbb{B}_j$-generic. Set $\sigma_\ell = \overset{\circ}{\sigma}{}^G_\ell$ for $\ell \le j$. Then:

$$\sigma_\ell(\overline{\xi}_h) = \nu \le \xi_\ell < \nu' \le \xi_{\ell'} \le \xi_j < \mu = \sigma_\ell(\overline{\xi}_{h+1}).$$

Hence $\sigma_{\ell'} = \sigma_\ell$ by (II)(g). Hence $h < h'$, since $\sigma_\ell(\overline{\xi}_{h'}) = \nu' > \sigma_\ell(\overline{\xi}_h)$. Hence $\sigma_\ell(\overline{\xi}_{h+1}) \leq \nu' < \mu$. Contradiction! QED(9)

We now define:

Definition $b_i = \bigcup\{h_{\xi_i}(\tilde{a}) \mid a \in \hat{A}_j\}$ for $i = j+1$.

To define σ_i we set: $\tilde{A} =$ the set of $a^{i\nu\mu} \in \hat{A}_j$ s.t. $\mu \leq \xi_i$. $\mathring{\sigma}_i \in \mathbf{V}^{\mathbb{B}_i}$ is then a name s.t. $[[\mathring{\sigma}_i = \mathring{\sigma}'_a]] = \tilde{a}$ if $a \in \tilde{A}$, $[[\mathring{\sigma}_i = \mathring{\sigma}_j]] \cap b_i = b_i \setminus \bigcup \tilde{A}$.

Then:

(10) (III)(c) holds at i.

Proof. Let $G \ni b_i$ be $\mathbb{B}_{\xi_i}$-generic.

Case 1 $\tilde{a} \in G$ for an $a \in \tilde{A}$.

Let $a = a^{h\nu\mu} \in A_\ell$, $\mu \leq \xi_i$ (hence $\xi_j < \mu \leq \xi_i$). Thus $\sigma_i = \sigma'_a$. (II)(a)–(d) hold by (8)(a)–(d). Note that the r in (8)(e), (f) is $r = i$. But, if $a \in A_\ell$, $\ell \leq j$, then $\sigma_\ell = \sigma_a$. Hence $\sigma_\ell(\overline{\xi}_h) = \nu \leq \xi_\ell \leq \xi_{\ell'} < \sigma_\ell(\overline{\xi}_{h+1}) = \mu$ for $\ell \leq \ell' \leq j$. Hence: $\sigma_a = \sigma_{\ell'}$ for $\ell \leq \ell' \leq j$. But then (II)(e), (f) hold by (8)(e), (f). Finally (II)(g) holds vacuously, since $\xi_j < \mu = \sigma_i(\overline{\xi}_{h+1}) \leq \xi_i$, hence $\xi_j < \sigma_i(\overline{\xi}_m)$ where $\sigma_i(\overline{\xi}_m) \leq \xi_i < \sigma_i(\overline{\xi}_{m+1})$.

Case 2 Case 1 fails.

Then $\sigma_i = \sigma_j$. (II)(a)–(g) then follow trivially. QED(10)

(III)(a) holds vacuously at $i = j+1$. We prove:

(11) (III)(b) holds at i.

Proof. Clearly $h_{\xi_j}(b_i) = \bigcup_{a \in \hat{A}_j} h_{\xi_j}(\tilde{a})$. Hence we need:

Claim $c_j = \bigcup_{a \in \hat{A}_j} h_{\xi_j}(\tilde{a})$.

For $j = 0$ this is trivial, so let $j = \ell + 1$. Recall that $c_j = \overline{b} \cup \bigcup_{a \in A_j} h_{\xi_j}(\tilde{a})$, where $\overline{b} = b_j \setminus \bigcup_{a \in A_j} a$, so it suffices to show:

Claim $\overline{b} = \bigcup_{a \in A'} h_{\xi_j}(\tilde{a})$ where $A' = \hat{A}_j \setminus A_j$.

($\supset$) Let $a' \in A'$. Then $a' \in \hat{A}_\ell$. Hence $h_{\xi_j}(\tilde{a}') \subset \bigcup_{a \in \hat{A}_\ell} h_j(\tilde{a}) = b_j$. But for all $a \in A_j$ we have $a \cap a' = 0$ by (10). Hence $h_{\xi_j}(\tilde{a}) \cap h_{\xi_j}(\tilde{a}') = a \cap h_{\xi_j}(\tilde{a}') = h_{\xi_j}(a \cap \tilde{a}') = 0$, since $a \cap \tilde{a}' \subset a \cap a' = 0$. Hence $h_{\xi_j}(\tilde{a}') \subset \overline{b}$.

($\subset$) Suppose not. Then there is $a \in \hat{A}_j \setminus A'$ s.t. $\overline{b} \cap h_{\xi_j}(\tilde{a}) \neq 0$. But then $a \in A_j$ and $h_{\xi_j}(\tilde{a}) = a$. Hence $a \cap \overline{b} = 0$ by the definition of $\overline{b}$. QED(11)

It remains only to show:

(12) (IV) holds at i.

Proof. Let $a = a^{h,\nu,\mu} \in A_i$. Then $\xi_j < \nu \leq \xi_i$. $a \cap [[\overset{\circ}{\sigma}_i(\check{x}) = \check{y}]] = b_i \cap d$, where $d = [[\overset{\circ}{\sigma}_i(\check{\overline{\xi}}_h) = \check{\nu} \wedge \overset{\circ}{\sigma}_i(\overline{\xi}_{h+1}) = \check{\mu} \wedge \overset{\circ}{\sigma}_i(\check{x}) = \check{y}]]_{\mathbb{B}_{\xi_i}} = [[\varphi(\overset{\circ}{\sigma}_i)]]$, where the formula $\varphi(v)$ is Σ_0 in the parameters $\check{\overline{\xi}}_h$, $\check{\overline{\xi}}_{h+1}$, $\check{\nu}$, $\check{\mu}$, $\check{x}$, $\check{y}$, all of which lie in $\mathbf{V}^2$. Recall that we are assuming $\mathbf{V}^{\mathbb{B}_\eta} \subseteq \mathbf{V}^{\mathbb{B}_\tau}$ for $\eta \leq \tau$ (i.e. $\mathbb{B}_\eta$ is completely contained in $\mathbb{B}_\tau$ and the identity is the natural embedding of $\mathbf{V}^{\mathbb{B}_\eta}$ in $\mathbf{V}^{\mathbb{B}_\tau}$). As mentioned in Chapter 0, this has the consequence that if ψ is a Σ_0 formula and $t_1, \ldots, t_m \in \mathbf{V}^{\mathbb{B}_\eta}$, then:

$$a \Vdash_{\mathbb{B}_\tau} \psi(\vec{t}) \longleftrightarrow a \Vdash_{\mathbb{B}_\eta} \psi(\vec{t}) \quad \text{ for } \ a \in \mathbb{B}_\eta,$$

or in other words: $[[\psi(\vec{t})]]_{\mathbb{B}_\tau} = [[\psi(\vec{t})]]_{\mathbb{B}_\eta} \in \mathbb{B}_\eta$.
We shall make strong use of this. We know: $b_i = \bigcup_{e \in \hat{A}_j} h_{\xi_i}(\tilde{e})$. Hence it suffices to assign to each $e \in \hat{A}_j$ an $e^* \in \mathbb{B}_\nu$ s.t.

$$h_{\xi_i}(\tilde{e}) \cap d = e^*,$$

since then we have:

$$b_i \cap d = \bigcup_{e \in \hat{A}_j} e^* \in \mathbb{B}_\nu.$$

For $h_{\xi_i}(\tilde{e}) \cap d = 0$ we, of course, set $e^* = 0$. Now let $h_{\xi_i}(\tilde{e}) \cap d \neq 0$. Let $e = a^{\overline{h},\overline{\nu},\overline{\mu}} \in \hat{A}_j$. Let $G \ni h_{\xi_i}(\tilde{e}) \cap d$ be $\mathbb{B}_{\xi_i}$-generic. Set: $\sigma_i = \overset{\circ}{\sigma}^G$, $\overset{\circ}{\sigma}_j = \overset{\circ}{\sigma}_j^G$.

Case 1 $\overline{\mu} \leq \xi_i$.
Then $\tilde{e} = h_{\xi_i}(\tilde{e}) \in \mathbb{B}_{\overline{\mu}} \wedge G$. Hence $\sigma_i = \sigma'_e =_{\text{Df}} \overset{\circ}{\sigma}{}'^G_e$. Hence $[[\varphi(\overset{\circ}{\sigma}{}'_e)]] \in G$. Conversely, if $\tilde{e} \cap [[\varphi(\overset{\circ}{\sigma}{}'_e)]] \in G$, then $\sigma_i = \sigma'_e$ and hence $\tilde{e} \cap d \in G$. Since this holds for all G, we conclude:

$$\tilde{e} \cap d = \tilde{e} \cap [[\varphi(\overset{\circ}{\sigma}{}'_e)]] \in \mathbb{B}_{\overline{\mu}}.$$

However, $\overline{\mu} \leq \nu$, since otherwise we would have $\sigma_i(\overline{\xi}_\ell) = \sigma_j(\overline{\xi}_\ell)$ for $\ell \leq \overline{h} + 1$ and $\sigma_i(\overline{\xi}_h) = \nu < \overline{\mu} = \sigma_i(\overline{\xi}_{\overline{h}+1})$. Hence $h \leq \overline{h}$ and $\sigma_i(\overline{\xi}_h) \leq \sigma_j(\overline{\xi}_{\overline{h}}) = \overline{\nu} \leq \xi_j < \nu$. Contradiction! QED(Case 1)

Case 2 $\overline{\mu} > \xi_i$.
We show that this cannot occur. Clearly, if $G \ni h_{\xi_i}(\tilde{e}) \cap d$ is $\mathbb{B}_{\xi_i}$-generic, then $\sigma_i = \sigma_j = \sigma'_e$ by the definition of σ'_e. But then $\tilde{e} \cap d = 0$, since if $G \ni \tilde{e} \cap d$ were $\mathbb{B}_{\overline{\mu}}$-generic, then $\sigma_i(\overline{\xi}_{\overline{h}}) = \overline{\nu} \leq \xi_j < \nu \leq \xi_i < \overline{\mu} = \sigma_i(\overline{\xi}_{\overline{h}+1})$. Hence $\nu = \sigma_i(\overline{\xi}_h)$ is impossible. Contradiction!
Since $d \in \mathbb{B}_{\xi_i}$, we conclude: $h_{\xi_i}(\tilde{e}) \cap d = h_{\xi_i}(\tilde{e} \cap d) = 0$. Contradiction. QED(12)
This completes the proof of Theorem 3.

The above theorem can be adapted to iterations which allow more freedom in the formation of limit algebras.

Definition An iteration $\mathbb{B} = \langle \mathbb{B}_i \mid i < \alpha \rangle$ is *nicely subcomplete* iff the following hold:

(a) For all $i+1<\alpha$:

  (i) $\Vdash_i \check{\mathbb{B}}_{i+1}/\overset{\circ}{G}$ is subcomplete,
  (ii) $\Vdash_{i+1} \operatorname{card}(\delta(\check{\mathbb{B}}_i)) \leq \omega_1$.

(b) If $\lambda<\alpha$ and $\langle \xi_n \mid n<\omega\rangle$ is monotone and cofinal in λ, then

  (i) $\bigcap_n b_n \neq 0$ in $\mathbb{B}_\lambda$ whenever $b=\langle b_n \mid n<\omega\rangle$ is a thread in $\langle \mathbb{B}_{\xi_n} \mid n<\omega\rangle$,
  (ii) $\mathbb{B}_\lambda$ is subcomplete if $\mathbb{B}_i$ is subcomplete for $i<\lambda$.

(c) If $\lambda<\alpha$ and $\Vdash_i \operatorname{cf}(\check{\lambda})>\omega$ for all $i<\lambda$, then $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$.

(d) If $i<\alpha$ and G is $\mathbb{B}_i$-generic, then (a)–(c) hold for $\langle \mathbb{B}_{i+j}/G \mid j<\alpha-i\rangle$ in $\mathbf{V}[G]$.

(This allows greater freedom in forming limit algebras at points which acquire cofinality ω, but requires us to take direct limits at other points.)

Theorem 4 *Let $\mathbb{B}=\langle \mathbb{B}_i \mid i<\alpha\rangle$ be nicely subcomplete. Then every $\mathbb{B}_i$ is subcomplete.*

Proof. (sketch) By induction on i we again prove:

Claim Let $h\leq i$. Let G be $\mathbb{B}_h$-generic. Then $\mathbb{B}_i/G$ is subcomplete in $\mathbf{V}[G]$.
The cases $h=i$, $i=j+1$ are again trivial, so assume that $i=\lambda$ is a limit ordinal. We again have the two cases:

Case 1 $\operatorname{cf}(\lambda)\leq \overline{\overline{\delta(\mathbb{B}_h)}}$ for an $h<\lambda$.
Case 2 Case 1 fails.
In Case 1 it again suffices to prove the claim for sufficiently large $h<\lambda$, so we assume $\operatorname{cf}(\lambda)\leq\omega_1$ in $\mathbf{V}[G]$ whenever G is $\mathbb{B}_h$-generic. But then we can assume $\operatorname{cf}(\lambda)\leq\omega_1$ in $\mathbf{V}$, since the same proof can be carried out in $\mathbf{V}[G]$ for $\langle \mathbb{B}_{h+j}/G \mid j<\alpha-h\rangle$. This splits into two subcases:

Case 1.1 $\operatorname{cf}(\lambda)=\omega$.
Then $\mathbb{B}_\lambda$ is subcomplete by (b)(ii).

Case 1.2 $\operatorname{cf}(\lambda)=\omega_1$.
We then literally repeat the argument in the proof of Theorem 3, using that $\mathbb{B}_\lambda$ is the direct limit of $\langle \mathbb{B}_i \mid i<\lambda\rangle$.
(**Note** If we instead assumed $\operatorname{cf}(\lambda)=\omega$, the proof in Theorem 3 would no longer work, since the set of $\bigcap_n b$ s.t. $b=\langle b_n \mid n<\omega\rangle$ is a thread in $\langle \mathbb{B}_{\xi_n} \mid n<\omega\rangle$ may not be dense in $\mathbb{B}_\lambda$.) QED(Case 1)

In Case 2 we literally repeat the proof in Theorem 3, using that if $\tilde{\lambda}=\sup\sigma{}''\overline{\lambda}$, then by (b)(i), if $c=\bigcap_n c_n$ is a thread in $\langle \mathbb{B}_{\xi_n} \mid n<\omega\rangle$ ($\xi_n=\sigma(\overline{\xi}_n)$, where $\langle \overline{\xi}_n \mid n<\omega\rangle$ is monotone and cofinal in $\overline{\lambda}$), then $c\in\mathbb{B}_{\tilde{\lambda}}$. Just as before we utilize the fact that we can ensure that $\sup\sigma_n{}''\overline{\lambda}=\lambda$, even though we cannot fix the values of $\sigma_n(\overline{\xi}_i)$ ($i<\omega$). QED(Theorem 4)

Forcing Axioms

We say that a complete BA $\mathbb{B}$ *satifies Martin's Axiom* iff whenever $\langle \Delta_i \mid i < \omega_1 \rangle$ is a sequence of dense sets in $\mathbb{B}$, there is a filter G on $\mathbb{B}$ s.t. $G \cap \Delta_i \neq \emptyset$ for $i < \omega$. The original Martin's Axiom said that this holds for all $\mathbb{B}$ satisfying the countable chain condition. This axiom is consistent relative to ZFC. It was later discovered that very strong versions of Martin's Axiom can be proven consistent relative to a supercompact cardinal. The best known of these are the *proper forcing axiom* (PFA), which posits Martin's Axiom for proper forcings and *Martin's Maximum* (MM) which is equivalent to Martin's Axiom for semiproper forcings. Both of these strengthen the original Martin's Axiom, hence imply the negation of CH. Here we shall consider the *subcomplete forcing axiom* (SCFA), which says that Martin's Axiom holds for subcomplete forcings. This, it turns out, is compatible with CH, hence cannot be a strengthening of the original Martin's Axiom (though it is, of course, a strengthening of Martin's Axiom for complete forcings). Nonetheless it turns out that SCFA has some of the more striking consequences of MM. A fuller account of this can be found in [FA].

We recall from Chapter 3.1 that the notion of subcompleteness is "locally based" in the sense that, if θ, θ' are cardinals with $\overline{\overline{H_\theta}} < \theta'$, then we need only consider $N = L_\tau^A$ of size less than θ', in order to determine whether θ verifies the subcompleteness of a given $\mathbb{B}$. In other words, $\mathfrak{P}(H_\theta)$ contains all the information needed to determine this. As a consequence we get Chapter 3, Corollary 2.3, which says that, if W is an inner model and $\mathfrak{P}(H_\theta) \subset W$, then the question, whether θ verifies the subcompleteness of $\mathbb{B}$, is absolute in W.

Using this we prove:

Theorem 5 Let κ be supercompact. There is a subcomplete $\mathbb{B} \subset \mathbf{V}_\kappa$ s.t. whenever G is $\mathbb{B}$-generic, then:

(a) $\kappa = \omega_2$,
(b) CH holds,
(c) SCFA holds.

Proof. Let f be a Laver function (i.e. for each x and each cardinal β there is a supercompact embedding $\pi : \mathbf{V} \to W$ s.t. $x = \pi(f)(\kappa)$ and $W^\beta \subset W$). We define and RCS iteration $\langle \mathbb{B}_i \mid i \leq \kappa \rangle$ by:

- $\mathbb{B}_0 = 2$.
- If $\Vdash_i f(i)$ is a subcomplete forcing, then $\Vdash_i \check{\mathbb{B}}_{i+1}/\overset{\circ}{G} \simeq f(i) * \mathrm{coll}(w_1, \overline{\overline{f(i)}})$.
- If $\not\Vdash_i f(i)$ is a subcomplete forcing, then $\Vdash_i \check{\mathbb{B}}_{i+1}/\overset{\circ}{G} \simeq \mathrm{coll}(w_1, w_2)$.

Let G be $\mathbb{B} = \mathbb{B}_\kappa$-generic. Then CH holds in $\mathbf{V}[G]$, since there will be a stage $\mathbb{B}_{i+1}$ which makes CH true by collapsing. But then it remains true at later stages, since no reals are added. We now show that SCFA holds in $\mathbf{V}[G]$. Let $\mathbb{A} \in \mathbf{V}[G]$ be subcomplete. Let $\mathbb{A} = \overset{\circ}{\mathbb{A}}^G$. Let $U \in \mathbf{V}$, $U \subset \mathbf{V}^{\mathbb{B}_\kappa}$ s.t.

(1) $[[x \in \overset{\circ}{\mathbb{A}}]] \subset \bigcup_{z \in U} [[x = z]]$ for $x \in \mathbf{V}^{\mathbb{B}_\kappa}$.

We may also assume w.l.o.g. that $\overset{\circ}{\mathbb{A}}$ is forced to be subcomplete and in fact:

(2) $\Vdash_\kappa \check{\theta}$ verifies the subcompleteness of $\overset{\circ}{\mathbb{A}}$.

Let $\beta = \overline{\overline{\mathbf{V}}}_\beta$ where $\overset{\circ}{\mathbb{A}}, U, \theta \in \mathbf{V}_\beta$. Let $\pi : \mathbf{V} \to W$ be a supercompact embedding s.t. $\overset{\circ}{\mathbb{A}} = \pi(f)(\kappa)$ and $W^\beta \subset W$. (Hence $\mathbf{V}_{\beta+1} \subset W$.) Then:

(3) θ verifies the subcompleteness of $\mathbb{A}$ in $W[G]$,

since this depends only on $\mathfrak{P}(H_\theta) \subset W$. Now let: $\langle \mathbb{B}'_i \mid i \le \kappa' \rangle = \pi(\langle \mathbb{B}_i \mid i \le \kappa \rangle)$. Then $\mathbb{B}_\kappa = \mathbb{B}'_\kappa$ and G is $\mathbb{B}'_\kappa$-generic over W. Hence we can form $G' \supset G$ which is $\mathbb{B}'_\kappa$-generic over W. Since $\mathbb{B}'_{\kappa+1} \simeq \mathbb{A} * \mathrm{coll}(\omega_1, \overline{\overline{\mathbb{A}}})$, there is $A \in W[G']$ which is $\mathbb{A}$-generic over $W[G]$. Now let π^* be the unique $\pi^* \supset \pi$ s.t.

(4) $\pi^* : \mathbf{V}[G] \prec W[G'] \wedge \pi^*(G) = G'$.

Then

(5) $\pi^*(\mathbb{A}) = \mathbb{A}'$, where $\mathbb{A}' = \pi(\overset{\circ}{\mathbb{A}})^{\mathbb{B}'_{\kappa'}}$.

But

(6) $\pi^* \upharpoonright \mathbb{A} \in W[G']$, since $\pi \upharpoonright U \in W$ and $\pi^* \upharpoonright \mathbb{A}$ is definable as that $\tilde{\pi}$ s.t. $\tilde{\pi}(t^G) = \pi(t)^{G'}$ whenever $t \in U$ and $t^G \in \mathbb{A}$.

Let A' be the filter on $\mathbb{A}'$ generated by $\pi^{*\,\prime\prime}\mathbb{A}$. Let $\langle \Delta_i \mid i < \omega_1 \rangle$ be a sequence of dense sets in $\mathbb{A}$ in $\mathbf{V}[G]$. Let $\langle \Delta'_i \mid i < \omega_1 \rangle = \pi^*(\langle \Delta_i \mid i < \omega_1 \rangle)$. Obviously $A' \cap \Delta'_i \neq \emptyset$ for $i < \omega_1$. Since $\pi : \mathbf{V}[G] \prec W[G]$, we conclude that there is a filter $\tilde{A}$ on $\mathbb{A}$ in $\mathbf{V}[G]$ s.t. $\tilde{A} \cap \Delta_i \neq \emptyset$ for all $i < \omega_2$. QED(Theorem 5)

In [FA] we show that subcomplete forcings are $\Diamond$-preserving – i.e. if $\Diamond$ holds in $\mathbf{V}$, it continues to hold in $\mathbf{V}[G]$. It follows easily from this that $\Diamond$ holds in the model $\mathbf{V}[G]$ just constructed. If we do a prior application of Silver forcing to make GCH true, then the ultimate model will also satisfy GCH. Hence we have, in fact, shown the consistency of

$$\mathrm{SCFA} + \Diamond + \mathrm{GCH}$$

relative to a supercompact cardinal.

SCFA has two of the more striking consequences of MM: Friedman's principle and the singular cardinal hypothesis at singular strong limit cardinals. Friedman's principle at a regular cardinal $\tau > \omega_1$ says that if $A \subset \tau$ is any stationary set of ω-cofinal ordinals, then there is a normal function $f : \omega_1 \to A$ (i.e. f is monotone and continuous at limits). It is easily seen that Friedman's principle at β^+ implies the negation of $\Box_\beta$.

Lemma 6 *Assume* SCFA. *Let $\kappa > \omega_1$ be regular. Then Friedman's principle holds at κ.*

Proof. Let $\mathbb{P}_A$ be as in the final example of Chapter 3.3, where $A \subset \kappa$ is a stationary set of ω-cofinal ordinals. Let

$$\Delta_i = \text{ the set of } p \in \mathbb{P}_A \text{ s.t. } i+1 \subset \text{dom}(p) \text{ for } i < \omega_1.$$

Then Δ_i is dense in $\mathbb{P}_A$. By SCFA there is a set G of mutually compatible conditions s.t. $G \cap \Delta_i = \emptyset$ for $i < \omega_1$. But then the function $f = \bigcup G$ has the desired property. QED(Lemma 6)

By essentially the same proof we get.

Lemma 7.1 *Assume* SCFA. *Let $\tau > \omega_1$ be regular. Let $A_i \subset \tau$ be a stationary set of ω-cofinal points for $i < \omega_1$. Let $\langle D_i \mid i < \omega_1 \rangle$ be a partition of ω_1 into disjoint stationary sets. Then there is a normal function $f : \omega_1 \to \tau$ s.t. $f(j) \in A_i$ for $j \in D_i$.*

Proof. We need only to show that the appropriate forcing $\mathbb{P}$ is subcomplete The proof is exactly like Chapter 3.3, Lemma 6.3. QED(Lemma 7.1)

The singular cardinal hypothesis for strong limit cardinals then follows by a well known argument of Solovay:

Corollary 7.2 *Assume* SCFA. *Let τ be as above. Then $\tau^{\omega_1} = \tau$.*

Proof. Let $\langle A_\xi \mid \xi < \tau \rangle$ partition $\{\lambda < \tau \mid \text{cf}(\lambda) = \omega\}$ into disjoint stationary sets. For each $a \in [\tau]^{\omega_1}$ let $\langle \xi_i \mid i < \omega_1 \rangle$ enumerate a. Let $f : \omega_1 \to \bigcup_{i<\omega_1} A_{\xi_i}$ be normal s.t. $f(j) \in A_{\xi_i}$ if $j \in D_i$, where $\langle D_i \mid i < \omega_1 \rangle$ partitions ω_1 into stationary sets. Let $\lambda = \sup f''\omega_1$. Then

$$a = B_\lambda =_{\text{Df}} \{\xi \mid A_\xi \cap \lambda \text{ is stationary in } \lambda\}.$$

Hence $[\tau]^{\omega_1} \subset \{B_\lambda \mid \lambda < \tau\}$. QED(Corollary 7.2)

Corollary 7.3 *Assume* SCFA. *If $\text{cf}(\beta) \leq \omega_1 < \beta$ and $2^{\underset{\smile}{\beta}} \leq \beta^+$, then $2^\beta = \beta^+$.*

Proof. $2^\beta = (2^{\underset{\smile}{\beta}})^{\text{cf}(\beta)} \leq (\beta^+)^{\omega_1} = \beta^+$.

Using Silver's Theorem we conclude:

Corollary 7.4 *Assume* SCFA. *If β is a singular strong limit cardinal, then $2^\beta = \beta^+$.*

Chapter 5

$\mathcal{L}$-Forcing

In the following, assume CH. Let $\beta > \omega_1$ be a cardinal and assume: $2^{\underset{\smile}{\beta}} = \beta$ (i.e. $2^\alpha \leq \beta$ for $\alpha < \beta$). Let $M = L^A_\beta =_{\mathrm{Df}} \langle L_\beta[A], \in, A \cap L_\beta[A] \rangle$ s.t. $L_\beta[A] = H_\beta$ and $A \subset H_\beta$. Suppose we have forcing conditions which do not collapse ω_1, but do add a map collapsing β onto ω_1. The existence of such a map is equivalent to the existence of a commutative "tower" $\langle M_i \mid i < \omega_1 \rangle$, $\langle \pi_{ij} \mid i \leq j < \omega_1 \rangle$ s.t. each M_i is countable and transitive, $\pi_{ij} : M_i \to M_j$ for $i \leq j < \omega_1$, and the tower converges to M (i.e. there are $\langle \pi_i \mid i < \omega \rangle$ s.t. $M_i \langle \pi_i \mid i < \omega \rangle$ is the direct limit of $\langle M_i \mid i < \omega_1 \rangle$, $\langle \pi_{ij} \mid i \leq j < \omega_1 \rangle$).

In $\mathcal{L}$-forcing we attempt to collapse β onto ω_1 by conditions which directly describe such a tower (or at least a commutative directed system converging to M). The "$\mathcal{L}$" in "$\mathcal{L}$-forcing" refers to an infinitary language on a structure of the form:

$$N = \langle H_{\beta^+}, \in, M, \ldots \rangle$$

in the ground model $\mathbf{V}$. $\mathcal{L}$ then determines a set of conditions $\mathbb{P}_{\mathcal{L}}$. $\mathcal{L}$-forcing has been used to add new reals with interesting properties. In these notes, however, we shall concentrate wholly on a form of $\mathcal{L}$-forcing which does *not* add new reals. This means, of course, that H_{ω_1} is absolute. Hence all countable initial segments of our "tower" will lie in $\mathbf{V}$.

The theory of $\mathcal{L}$-forcing is developed in [LF]. In that paper, however, we dealt only with forcings which literally added a tower converging to M in the aforementioned sense. In later applications we found it better to replace the tower by other sorts of convergence systems. We therefore adopt a more general approach here. The proofs in [LF] can be readily adapted to this approach.

Recall that we are working in first order set theory, so we cannot literally quantify over arbitrary classes. Instead we work with "virtual classes", which are expressions of the form $\{x \mid \varphi(x)\}$ where $\varphi = \varphi(x)$ is a formula of ZF. Normally we suppose x to be the only variable occurring free in φ. We define:

Definition An *approximation system* is a pair $\langle \Gamma, \Pi \rangle$ of virtual classes s.t. (I)–(VII) below are provable in ZFC^-.

(I) Γ is a class of pairs $\langle M, C \rangle$ s.t.

(a) $M = L_\tau^{A_1,\ldots,A_n}$ for some $A_1,\ldots,A_n,\tau$.

(b) $C \subset M$.

(**Definition** For $u \in \Gamma$ set: $u = \langle M_u, C_u\rangle$.)

(II) Π is a class of triples $\langle \pi, u, v\rangle$ s.t. $u, v \in \Gamma$, $\pi : M_u \prec M_v$, $C_u = \pi^{-1}{}''C_v$.

(**Definition** $\pi : u \lhd v \leftrightarrow_{\text{Df}} \langle \pi, u, v\rangle \in \Pi$, $\quad u \lhd v \leftrightarrow_{\text{Df}} \bigvee \pi\ \pi : u \lhd v$)

(III) There is at most one π s.t. $\pi : u \lhd v$.

(**Definition** $\pi_{uv} \simeq_{\text{Df}}$ that π s.t. $\pi : u \lhd v$.)

(IV) (a) $u \lhd u \wedge \pi_{uu} = \text{id}$ for $u \in \Gamma$.

(b) $u \lhd v \lhd w \to (u \lhd w \wedge \pi_{uw} = \pi_{vw} \circ \pi_{uv})$.

(c) If $u, v \lhd w$ and $\text{rng}(\pi_{uw}) \subset \text{rng}(\pi_{vw})$, then $u \lhd v$ and $\pi_{uv} = \pi_{vw}^{-1} \circ \pi_{uw}$.

We say that a set $X \subset \Gamma$ is $\lhd$-*directed* iff for all $u, v \in X$ there is w s.t. $u, v \lhd w$. In this case we can form a direct limit v, $\langle \pi_u \mid u \in X\rangle$ of $\langle u \mid u \in X\rangle$, $\langle \pi_{uu'} \mid u \lhd u' \wedge u, u' \in X\rangle$. Then $v = \langle \mathfrak{A}, C\rangle$, where $\mathfrak{A}$ is a (possibly ill founded) ZFC$^-$ model. If $\mathfrak{A}$ is well founded, we can take it as transitive. Clearly the transitivized direct limit of X, if it exists, is uniquely determined by X.

(V) Let $X \subset \Gamma$ be $\lhd$-directed. Let $v, \langle \pi_u \mid u \in X\rangle$ be the transitivized direct limit. Then $v \in \Gamma$ and $\pi_u = \pi_{uv}$ for $u \in X$. Moreover, if $u \lhd w$ for all $u \in X$, then $v \lhd w$. (Hence π_{vw} is uniquely determined by: $\pi_{vw}\pi_u = \pi_{uw}$ for $u \in X$.)

If $t = \{x \mid \varphi(x)\}$ is a virtual class and W is any set or class, we can form t^W (the interpretation of t in W) by relativizing all quantifiers in φ to W.

(VI) If M is an admissible set, then $\Gamma \cap M = \Gamma^M$ and $\Pi \cap M = \Pi^M$.

If $\mathfrak{A} = \langle |\mathfrak{A}|, \in_{\mathfrak{A}}\rangle$ is any binary structure we can form the relativization $t^{\mathfrak{A}}$ by relativizing quantifiers to $|\mathfrak{A}|$ and simultaneously replacing $\in$ by $\in_{\mathfrak{A}}$ in φ.

(VIII) If $\mathfrak{A}$ is a solid model of ZFC$^-$ and $A = \text{wfc}(\mathfrak{A})$, then $\Gamma \cap A = \Gamma^{\mathfrak{A}} \cap A$, and $\Pi \cap A = \Pi^{\mathfrak{A}} \cap A$.

Hence:

(1) If $\mathbf{V}[G]$ is a generic extension of $\mathbf{V}$, then $\Gamma^{\mathbf{V}[G]} \cap \mathbf{V} = \Gamma^{\mathbf{V}}$, $\Pi^{\mathbf{V}[G]} \cap \mathbf{V} = \Pi^{\mathbf{V}}$.

Proof. Let $x \in \mathbf{V}$. Then $x \in M \in \mathbf{V}$, where M is admissible. Hence $x \in \Gamma^{\mathbf{V}[G]} \leftrightarrow x \in \Gamma^M \leftrightarrow x \in \Gamma^{\mathbf{V}}$, applying (VI) first in $\mathbf{V}[G]$, then in $\mathbf{V}$. QED(1)

Remark In practice (I)–(VII) will follow readily from the definitions given for Γ, Π, so we shall not bother to verify them in detail. In all cases Γ and Π will also be provably primitive recursive in ZFC$^-$, so the absoluteness properties (VI), (VII) will follow by Chapter 1.3. However, it will also be easy to verify these properties directly without going through the theory of pr functions.

⋆ ⋆ ⋆ ⋆ ⋆

A simple example of an approximation system is:

Γ is the set of all $\langle M, C\rangle$ s.t.

- $M = L_\tau^A$ for some A, τ.
- M models ZFC$^-$ and ω_1 exists and CH.
- C maps ω_1^M onto M.

Π is then the set of all $\langle \pi, u, v\rangle$ s.t. $u, v \in \Gamma$, $\pi : M_u \prec M_v$ and $\pi \circ C_u \subset C_v$. (Note that in this example we have $\Gamma, \Pi \subset H_{\omega_2}$.) The absoluteness properties are straightforward, since if $M, N, \pi \in A$ and A is admissible, then $\pi : M \prec N$ is uniformly expressible by a Σ_1 formula in any solid $\mathfrak{A}$ extending A.

Now let an approximation system $\langle \Gamma, \Pi\rangle$ be given. Let $M = L_\beta^A$ be as described at the outset with $\beta > \omega_1$ and $H_\beta = L_\beta[A]$. Our aim in $\mathcal{L}$-forcing is to generically add $C \subset M$ in the extension $\mathbf{V}[C]$ s.t. $\langle M, C\rangle \in \Gamma^{\mathbf{V}[C]}$ and $\langle M, C\rangle$ is the limit of a directed $X \subset \Gamma \cap H_{\omega_1}$. At the same time we want to add no reals, so that $\Gamma \cap H_{\omega_1}$ remains absolute. Since we are assuming CH it follows easily that card$(M) = \omega_1$ in $\mathbf{V}[G]$. (In the above example we would accomplish this explicitly, since C would map ω_1 onto M.)

$\mathcal{L}$ is a language on $N = \langle H_{\beta^+}, \in, M, <, \ldots\rangle$, where $<$ is a well ordering of H_{β^+}. (**Note** N remains a ZFC$^-$ model, hence admissible, no matter which predicates and constants we adjoin to it.)

The only nonlogical predicate of $\mathcal{L}$ is $\in$. In addition to the constants $\underline{x}$ $(x \in N)$ there will be one further constant $\overset{\circ}{C}$. We always suppose $\mathcal{L}$ to contain the following *core axioms*:

- ZFC$^-$ (here the usual finite axioms are meant, so we could write them as a single M-finite conjunction).
- $\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z\in x} v = \underline{z})$ for $x \in N$.
- $H_{\omega_1} = \underline{H_{\omega_1}}$ (or equivalently $\mathfrak{P}(\omega) = \underline{\mathfrak{P}(\omega)}$).
- $\langle \underline{M}, \overset{\circ}{C}\rangle \in \Gamma$.
- For all countable $X \subset \underline{M}$ there is $u \in \Gamma \cap H_{\omega_1}$ s.t. $X \subset \text{rng}(\pi_{u,\langle \underline{M}, \overset{\circ}{C}\rangle})$.

  ($\mathcal{L}$ might, of course, contain further axioms as well.)

Definition Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. $\Gamma^{\mathfrak{A}}$, $\Pi^{\mathfrak{A}}$, $\lhd^{\mathfrak{A}}$, $\pi_{uv}^{\mathfrak{A}}$ $(u \lhd^{\mathfrak{A}} v)$ are defined in the obvious way. Set:

$$\tilde{\Gamma} = \tilde{\Gamma}^{\mathfrak{A}} =_{\text{Df}} \{e \in \Gamma \cap H_{\omega_1} \mid e \lhd \langle M, \overset{\circ}{C}{}^{\mathfrak{A}}\rangle \text{ in } \mathfrak{A}\}.$$

For $e \in \tilde{\Gamma}$ set: $\pi_e^{\mathfrak{A}} =_{\text{Df}} \pi^{\mathfrak{A}}_{e,\langle M, \overset{\circ}{C}{}^{\mathfrak{A}}\rangle}$.

Lemma 1.1 *Let $\mathfrak{A}$ be as above. Then $\tilde{\Gamma}$ is a $\lhd$-directed system with limit $\langle M, \overset{\circ}{C}{}^{\mathfrak{A}}\rangle$, $\langle \pi_e^{\mathfrak{A}} \mid e \in \tilde{\Gamma}\rangle$.*

Proof. $M = \bigcup_{e\in\tilde{\Gamma}} \text{rng}(\pi_e^{\mathfrak{A}})$ is trivial. We show that $\tilde{\Gamma}$ is directed. Let $e_0, e_1 \in \tilde{\Gamma}$. Let $u \in \tilde{\Gamma}$ s.t. $\text{rng}(\pi_{e_0}^{\mathfrak{A}}) \cup \text{rng}(\pi_{e_1}^{\mathfrak{A}}) \subset \text{rng}(\pi_u^{\mathfrak{A}})$. Then $e_0, e_1 \lhd u$ and $\pi_{e_h u} = (\pi_u^{\mathfrak{A}})^{-1} \circ \pi_{e_h}^{\mathfrak{A}}$.

QED(Lemma 1.1)

Lemma 1.2 *Let $\mathfrak{A}$ be as above. Let $A \in \mathfrak{A}$ s.t. $A \subset M$. There is $e \in \tilde{\Gamma}$ s.t.* $\text{rng}(\pi_e^{\mathfrak{A}}) \prec \langle M, A\rangle$.

Proof. In $\mathfrak{A}$ construct $\langle e_i \mid i < \omega\rangle$, $\langle X_i \mid i < \omega\rangle$ s.t. $\bigcup_{h<i} \text{rng}(\pi_{e_h}^{\mathfrak{A}}) \subset X_i \subset \text{rng}(\pi_{e_i}^{\mathfrak{A}})$ and $X_i \prec \langle M, A\rangle$. It follows easily that $e_h \lhd e_i$ for $h \leq i < \omega$ and $\{e_h \mid h < \omega\}$ has a direct limit e, $\langle \pi_{e_i e} \mid i < \omega\rangle$. But then $e \lhd \langle M, \mathring{C}^{\mathfrak{A}}\rangle$ and $\text{rng}(\pi_e^{\mathfrak{A}}) = \bigcup_{i<\omega} \text{rng}(\pi_{e_i}^{\mathfrak{A}}) = \bigcup_{i<\omega} X_i \prec \langle M, A\rangle$. QED(Lemma 1.2)

Corollary 1.3 *Let $\mathfrak{A}$ be as above. Let $U \subset \mathfrak{P}(M)$ s.t. $U \in \mathfrak{A}$ is countable in $\mathfrak{A}$. There is $e \in \tilde{\Gamma}$ s.t. $\text{rng}(\pi_e^{\mathfrak{A}}) \prec \langle M, A\rangle$ for all $A \in U$.*

Proof. Let $\langle A_i \mid i < \omega\rangle \in \mathfrak{A}$ enumerate U and apply Lemma 1.2 to $A = \{\langle x, i\rangle \mid x \in A_i\}$. QED(Corollary 1.3)

Corollary 1.4 *Let $\mathfrak{A}$ be as above. Let U, V be countable in $\mathfrak{A}$ s.t. $U \subset M$, $V \subset \mathfrak{P}(M)$. There is $e \in \tilde{\Gamma}$ s.t. $U \subset \text{rng}(\pi_e^{\mathfrak{A}}) \prec \langle M, A\rangle$ for $A \in V$.*

Proof. Apply Corollary 1.3 to $U \cup V$. QED(Corollary 1.4)

If $\mathcal{L}$ is consistent, we can define a set $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ of conditions as follows:

Definition Let $\tilde{\mathbb{P}}$ be the set of $p = \langle p_0, p_1\rangle$ s.t. $p_0 \in \Gamma \cap H_{\omega_1}$ and

$$p_1 \subset \mathfrak{P}(M) \times \mathfrak{P}(M_{p_0}) \quad \text{is countable.}$$

For $p \in \tilde{\mathbb{P}}$ let φ_p be the conjunction of the $\mathcal{L}$ statements:

- $\underline{p}_0 \lhd \langle \underline{M}, \mathring{C}\rangle$
- If $\pi = \pi_{\underline{p_0}, \langle \underline{M}, \mathring{C}\rangle}$, then $\pi : \langle \underline{M}_{p_0}, \overline{a}\rangle \prec \langle \underline{M}, a\rangle$ for all $\langle a, \overline{a}\rangle \in \underline{p_1}$.

Set $\mathcal{L}(p) = \mathcal{L} + \varphi_p$. We set: $\mathbb{P} = \{p \in \tilde{\mathbb{P}} \mid \text{con}(\mathcal{L}(p))\}$, where $\text{con}(\mathcal{L}(p))$ is the statement that "$\mathcal{L}(p)$ is consistent". The extension relation on $\mathbb{P}$ is then defined by:

Definition Let $p, q \in \mathbb{P}$

$$p \leq q \longleftrightarrow_{\text{Df}} (q_0 \lhd p_0 \wedge \text{rng}(q_1) \subset \text{rng}(p_1) \wedge \pi_{q_0 p_0} : \langle M_{q_0}, \overline{a}\rangle \prec \langle M_{p_0}, a'\rangle$$
$$\text{whenever } \langle a, \overline{a}\rangle \in q_1,\ \langle a, a'\rangle \in p_1).$$

Lemma 2.1 $\leq$ *is a partial ordering.*

Proof. Transitivity is immediate. Now let $p \leq q$, $q \leq p$. We claim that $p = q$, $p_0 = q_0$ is immediate. But if $\langle a, \overline{a}\rangle \in q_1$, $\langle a, a'\rangle \in p_1$, then $\overline{a} = a'$, since $\pi_{q_0 p_0} = \text{id}$. Hence $q_1 = p_1$. QED(Lemma 2.1)

Definition Let $p \in \mathbb{P}$. $M_p = M_{p_0}$, $C_p = C_{p_0}$, $F^p = p_1$, $R^p = \text{rng}(p_1)$, $D^p = \text{dom}(p_1)$.

Lemma 2.2 *Let $p \in \mathbb{P}$. Then*

(a) $(F^p)^{-1}$ *is a function.*
(b) *If* R^p *is closed under set difference, then* $F^p : D^p \leftrightarrow R^p$.
(c) $F^p \upharpoonright M_p$ *injects* M_p *into* M.

Proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Let $\pi = \pi_p^{\mathfrak{A}} =_{\text{Df}} (\pi_{p_0, \langle M, \mathring{C}^{\mathfrak{A}} \rangle})^{\mathfrak{A}}$.
(a) Let $\langle a, \overline{a} \rangle, \langle a, \overline{a}' \rangle \in F^p$. Then $\overline{a} = \overline{a}' = \pi^{-1}{}''a$.
(b) Let $\langle a, \overline{a} \rangle, \langle b, \overline{b} \rangle \in F^p$. It suffices to show:

Claim: $\overline{a} \subset \overline{b} \to a \subset b$.
Set $c = a \setminus b, \overline{c} = \overline{a} \setminus \overline{b} = \emptyset$. Then $(F^p)^{-1}(c) = \pi^{-1}{}''c = \pi^{-1}{}''a \setminus \pi^{-1}{}''b = \overline{b} \setminus \overline{a} = \emptyset$.
Hence $c = \emptyset$, since $\pi : \langle M_{p_0}, \overline{c} \rangle \prec \langle M, c \rangle$. QED(b)
(c) Let $x \in M_{p_0}$, $\langle x, \overline{x} \rangle \in F^p$. Then $\pi(\overline{x}) = x \in M$ since $\pi : \langle M_p, \overline{x} \rangle \prec \langle M, x \rangle$.
QED(Lemma 2.2)

We define:

Definition $\pi^p = F^p \upharpoonright M_p$.

Note By the proof of (c) we have: $\mathcal{L}(p) \vdash \underline{\pi^p} \subset \pi_{\underline{p}, \langle \underline{M}, \mathring{C} \rangle}$.

We now prove the main lemma on extendability of conditions.

Lemma 3.1 $\mathbb{P} \neq \emptyset$. *Moreover, if* $p, q \in \mathbb{P}$ *and* $\mathcal{L}(p) \cup \mathcal{L}(q)$ *is consistent, there is* r *s.t.* $r \leq p, q$. *Moreover, if* $X \subset \mathfrak{P}(M)$ *is any countable set, we may choose* r *s.t.* $X \subset R^r$.

Proof. To see $\mathbb{P} \neq \emptyset$ let $\mathfrak{A}$ be any solid model of $\mathcal{L}$. Let $e \lhd \langle M, \mathring{C}^{\mathfrak{A}} \rangle$ in $\mathfrak{A}$ where $e \in \Gamma \cap H_{\omega_1}$. Then $\mathfrak{A} \models \mathcal{L}(p)$ where $p = \langle e, \emptyset \rangle$. Hence $p \in \mathbb{P}$.
Now let $\mathfrak{A} \models \mathcal{L}(p) \cup \mathcal{L}(q)$. Let $X \subset \mathfrak{P}(M)$ be countable in $\mathbf{V}$. Let $Y = X \cup R^p \cup R^q$. There is $e \in H_{\omega_1} \cap \Gamma$ s.t. $e \lhd \langle M, \mathring{C}^{\mathfrak{A}} \rangle$ in $\mathfrak{A}$ and $\pi_e^{\mathfrak{A}} \prec \langle M, A \rangle$ for all $A \in Y$. For $A \in Y$ set $\overline{A} = (\pi_e^{\mathfrak{A}})^{-1}{}''A$. Letting $\langle A_i \mid i < \omega \rangle$ be an enumeration of Y in $\mathbf{V}$, we see that $\langle \overline{A}_i \mid i < \omega \rangle \in H_{\omega_1}$. Hence $F \in \mathbf{V}$ where $F = \{\langle A, \overline{A} \rangle \mid A \in Y\} = \{\langle A_i, \overline{A}_i \rangle \mid i < \omega\}$.
Set $r = \langle e, F \rangle$. Then $\mathfrak{A} \models \mathcal{L}(r)$ and $p, q \leq r$. QED(Lemma 3.1)

Corollary 3.2 *p, q are compatible in* $\mathbb{P}$ *iff* $\mathcal{L}(p) \cup \mathcal{L}(q)$ *is consistent.*

Proof. $(\leftarrow)$ Lemma 3.1.
$(\rightarrow)$ If $r \leq p, q$, then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$. QED(Corollary 3.2)

Corollary 3.3 *Let* $p \in \mathbb{P}$, $X \subset \mathfrak{P}(M)$ *where* X *is countable. There is* $r \leq p$ *with* $X \subset R^r$.

Corollary 3.4 *Let* $p \in \mathbb{P}$, $u \subset M$, u *is countable. There is* $r \leq p$ *with* $u \subset \operatorname{rng}(\pi^r)$.

Lemma 3.5 *Let* $p \in \mathbb{P}$, $u \subset M_p$, u *finite. There is* $r \leq p$ *s.t.* $r_0 = p_0$ *and* $u \subset \operatorname{dom}(\pi^r)$.

Proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Set: $r_0 = p_0$, $F^r = F^p \cup (\pi_p^{\mathfrak{A}} \upharpoonright u)$. Then $\mathfrak{A} \models \mathcal{L}(r)$. QED(Lemma 3.5)

Using these extension lemmas we get:

Lemma 3.6 *Let G be $\mathbb{P}$-generic. For $p \in G$ set: $\pi_p^G = \bigcup\{\pi^q \mid q \in G \wedge p_0 = q_0\}$. Then:*

(a) *$\{p_0 \mid p \in G\}$ is a $\triangleleft$-directed system with limit $\langle M, C^G\rangle$, $\langle \pi_p^G \mid p \in G\rangle$, where $C^G = \bigcup_p \pi_p^G{}''C_p$.*

(b) *$\pi_p^G : \langle M_p, \overline{a}\rangle \prec \langle M, a\rangle$ for $\langle a, \overline{a}\rangle \in F^p$.*

Note $\pi_p^G : p_0 \triangleleft \langle M, C^G\rangle$ in $\mathbf{V}[G]$ by (a).

The proof is straightforward. Now let $\kappa > (2^\beta)$ be regular in $\mathbf{V}$. Then κ remains regular in $\mathbf{V}[G]$, since $\mathbb{P} \in H_\kappa$. $\langle H_\kappa^{\mathbf{V}[G]}, C^G\rangle$ then models all of the core axioms except possibly the axiom: $H_{\omega_1} = \underline{H_{\omega_1}}$.

We now state a condition called *revisability* which will guaratee that no reals are added – hence that all core axioms hold in $\langle H_\kappa^{\mathbf{V}[G]}, C^G\rangle$.

We first define:

Definition Let $N^* = \langle H_\delta, M, <, \ldots\rangle$ be a model of countable or finite type, where $\delta > 2^\beta$ is a cardinal and $<$ well orders H_δ. Let $p \in \mathbb{P}$. p *conforms* to N^* iff whenever $a_1, \ldots, a_n \in R^p$ $(n \geq 0)$ and $b \subset M$ is N^*-definable in $a_1, \ldots, a_n$, then $b \in R^p$.

Note If p conforms to N^* then $R^p \neq \emptyset$ and $F^p : D^p \leftrightarrow R^p$ by Lemma 2.2.

Note $\{p \mid p$ conforms to $N^*\}$ is dense in $\mathbb{P}$ by the extension lemmas.

Before defining revisability we prove a theorem:

Lemma 4 *Let p conforms to N^*. There is a unique $\overline{N}^* = \overline{N}^*(p, N^*)$ s.t.*

(i) *$\overline{N}^*$ is transitive and of the same type as N^*.*

(ii) *If $a_1, \ldots, a_n \in R^p$ $(n \geq 0)$ and $b \subset M$ is N^*-definable in $a_1, \ldots, a_n$, then $\overline{a}_1^p, \ldots, \overline{a}_n^p \in \overline{N}^*$ (where $\overline{a}_i^p = F^{-1}(a_i)$) and $\overline{b}^p$ is $\overline{N}^*$-definable in $\overline{a}_1^p, \ldots, \overline{a}_n^p$ by the same definition.*

(iii) *Each $x \in \overline{N}^*$ is $\overline{N}^*$-definable from parameters in $M_p \cup D^p$.*

Moreover, if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\pi_p^{\mathfrak{A}} \cup F^p$ extends uniquely to a $\pi \supset \pi_p^{\mathfrak{A}} \cup F^p$ s.t. $\pi : \overline{N}^* \prec N^*$.

Proof. We use the following:

Fact For any $X \subset M$ the following are equivalent:

(a) $X \prec \langle M, a\rangle$ for all $a \in R^p$.

(b) Let $Y =$ the smallest $Y \prec N^*$ s.t. $X \cup R^p \subset Y$. Then $Y \cap M = X$.

((b) $\rightarrow$ (a) is trivial. (a) $\rightarrow$ (b) follows from the fact that each $z \in Y$ is N^*-definable from parameters in $X \cup R^p$.)

Let $\tilde{Y} =$ the smallest $\tilde{Y} \prec N^*$ s.t. $M \cup R^p \subset \tilde{Y}$. Then $\tilde{Y}$ has cardinality β in $\mathbf{V}$. Hence, if – in some extension $\mathbf{V}[G]$ – $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\tilde{N}^* \in N \subset \mathfrak{A}$, where $\tilde{\pi} : \tilde{N}^* \overset{\sim}{\leftrightarrow} \tilde{Y}$ is the transitivation of $\tilde{Y}$. Working in $\mathfrak{A}$, we now form $Z =$ the smallest $Z \prec \tilde{N}^*$ s.t. $X \cup R^p \subset Z$, where $X = \text{rng}(\pi_p^{\mathfrak{A}})$. Transitivize Z to get $\overline{\pi} : \overline{N}^* \overset{\sim}{\leftrightarrow} Z$. Then $Z \in H_{\omega_1}^{\mathfrak{A}} = H_{\omega_1}^{\mathbf{V}}$ omit: where $X = \text{rng}(\pi_p^{\mathfrak{A}})$.

Claim 1 $\overline{N}^*$ satisfies (i)–(iii).

Proof. Let $\pi = \tilde{\pi}\overline{\pi} : \overline{N}^* \overset{\sim}{\leftrightarrow} Y =$ the smallest $Y \prec N^*$ s.t. $X \cup R^p \subset Y$. Then $\pi \upharpoonright M_p = \pi_p^{\mathfrak{A}}$, since $X = Y \cap M$ by the above Fact. For $a \in R^p$ we have $\pi^{-1}(a) = \pi^{-1}{}''(X \cap a) = (\pi_p^{\mathfrak{A}})^{-1}{}''(X \cap a) = \overline{a}^p$. Thus $\pi \supset \pi_p^{\mathfrak{A}} \cup F^p$. Using this, (i)–(iii) follow easily.

Claim 2 At most one $\overline{N}^*$ satisfies (i)–(iii).

Proof. Let $\overline{N}_0^*$, $\overline{N}_1^*$ be two different ones. Then

(1) Let $x_1, \ldots, x_n \in M_p$, $b_1, \ldots, b_m \in D^p$. Then $\overline{N}_0^* \models \varphi(\vec{x}, \vec{b}\,) \leftrightarrow \overline{N}_1^* \models \varphi(\vec{x}, \vec{b}\,)$.

Proof. Let $b_i = \overline{a}_i^p$, $a_i \in R^p$. Set: $c = \{\langle \vec{x}\,\rangle \in M \mid N^* \models \varphi(\vec{x}, \vec{a}\,)\}$. Then by (ii):
$\overline{c}^p = \{\langle \vec{x}\,\rangle \in M_p \mid \overline{N}_h^* \models \varphi(\vec{x}, \vec{b}\,)\}$. QED(1)

But it then follows straightforwardly that id $\upharpoonright (M_p \cup D^p)$ extends to a $\sigma : \overline{N}_0^* \overset{\sim}{\leftrightarrow} \overline{N}_1^*$. Hence $\sigma =$ id, since the models are transitive. QED(Claim 2)

In the proof of Claim 1, we have shown that, if $\mathfrak{A}$ is a solid model of $\mathcal{L}(p)$, then $\pi_p^{\mathfrak{A}} \cup F^p$ extends to a $\pi : \overline{N}^* \prec N^*$. It remains only to note that π is unique, since every $z \in \text{rng}(\pi)$ is N^*-definable from elements of $X \cup R^p = \text{rng}(\pi_p^{\mathfrak{A}} \cup F^p)$.
QED(Lemma 4)

Note Clearly $M_p = \overline{M}$, where $\overline{N}^* = \langle \overline{H}, \overline{M}, <, \ldots \rangle$.

We now define:

Definition $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ is *revisable* iff for sufficiently large cardinals $\Omega > 2^\beta$: Let $N^* = \langle H_\Omega, M, <, \mathbb{P}, \ldots \rangle$ where $<$ well orders H_Ω. Let p conform to N^* and set $\overline{N}^* = \overline{N}^*(p, N^*)$. Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}^*$, where $\overline{N}^* = \langle \overline{H}, \overline{M}, <, \overline{\mathbb{P}}, \ldots \rangle$. Then there is $q \in \mathbb{P}$ s.t. $M_q = M_p$, $C_q = C^{\overline{G}}$, and $F^q = F^p$.
(In other words $q = \langle \langle M_p, C^{\overline{G}} \rangle, F^p \rangle \in \mathbb{P}$.)

Lemma 5.1 *Let $\mathbb{P}$ be revisable. Then $\mathbb{P}$ adds no new reals.*

Proof. Let $\Vdash \overset{\circ}{f} : \omega \to 2$.

Claim $\Delta = \{p \mid \bigvee f \ p \Vdash \overset{\circ}{f} = \check{f}\}$ is dense in $\mathbb{P}$.

Let $r \in \mathbb{P}$. Pick Ω big enough to verify revisability and set $N^* = \langle H_\Omega, M, <, \mathbb{P}, \overset{\circ}{f}, r, \ldots \rangle$. Let p conforms to N^*. Set $\overline{N}^* = \overline{N}^*(p, N^*)$. Let $\overline{N}^* = \langle \overline{H}, \overline{M}, <, \overline{\mathbb{P}}, \overline{f}, \overline{r}, \ldots \rangle$. Let $\overline{G} \ni \overline{r}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}^*$. Let $f = \overline{f}^{\overline{G}}$. Let $q = \langle \langle \overline{M}, C^{\overline{G}} \rangle, F^p \rangle \in \mathbb{P}$.

Claim $q \leq r$ and $q \Vdash \overset{\circ}{f} = \check{f}$.

Proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q)$. Let $\sigma \supset \pi_q^{\mathfrak{A}} \cup F^q$ s.t. $\sigma : \overline{N}^* \prec N^*$.

(1) $q \leq r$.

Proof. Let $\overline{C} = C^{\overline{G}}$, $r_0 = \overline{r}_0 \lhd \langle \overline{M}, \overline{C} \rangle = q_0$ and $\pi_{r_0,q_0} = \pi^{\overline{G}}_{\overline{r}}$. But $R^r \subset R^q$, since r is $\overline{N}^*$-definable. Let $\langle a, \overline{a} \rangle \in F^r, \langle a, a' \rangle \in F^q$.

Claim $\pi_{r_0,q_0} : \langle M_s, \overline{a} \rangle \prec \langle M_q, a' \rangle$.
This is clear, since $a' = (F^q)^{-1}(a) = \sigma^{-1}(a)$ and hence $\langle \overline{a}, a' \rangle \in \sigma^{-1}{}''F^r = F^{\overline{r}}$).

QED(1)

(2) Let $\overline{s} \in \overline{G}, s = \sigma(\overline{s})$. Then $\mathfrak{A} \models \mathcal{L}(s)$.

Proof. $s_0 = \overline{r}_0 \lhd q_0 = \langle \overline{M}, \overline{C} \rangle \lhd \langle M, \dot{C}^{\mathfrak{A}} \rangle$, and $\pi^{\mathfrak{A}}_{s_0} = \sigma \circ \pi_{s_0,q_0}$.
Let $\langle a, \overline{a} \rangle \in F^s$. Then $a = \sigma(a')$, where $\langle a', \overline{a} \rangle \in F^{\overline{s}}$. Hence,

$$\pi^{\mathfrak{A}}_{s_0} : \langle M_s, \overline{a} \rangle \prec \langle M, a \rangle.$$

QED(2)

(3) $q \Vdash \overset{\circ}{f} = \check{f}$.

Suppose not. Then there is i s.t. $f(i) = h$ and $q \nVdash \overset{\circ}{f}(\check{i}) = \check{h}$. Let $q' \leq q$ s.t. $q' \Vdash \overset{\circ}{f}(\check{i}) \neq \check{h}$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q')$, hence of $\mathcal{L}(q)$. Let $\overline{s} \in \overline{G}$ s.t. $\overline{s} \Vdash_{\mathbb{P}} \overline{f}(\check{i}) = \check{h}$. Let σ be as above. Let $s = \sigma(\overline{s})$. Then $q \leq s$ and $s \Vdash \overset{\circ}{f}(\check{i}) = \check{h}$. Hence q', s are incompatible. But $\mathfrak{A} \models \mathcal{L}(q') \cup \mathcal{L}(s)$. Contradiction! by Lemma 3.1.

QED(Lemma 5.1)

Now let $\mathcal{L}^c$ be $\mathcal{L}$ with its axioms reduced to the core axioms. (Thus $\mathcal{L}^c$ is uniquely determined by Γ, Π.) By Lemma 5.1 we have:

Lemma 5.2 *Let $\mathbb{P}$ be revisable. Let G be $\mathbb{P}$-generic. Let $p \in G$. Set: $\mathfrak{A} = \langle H^{\mathbf{V}[G]}_\kappa, C^G \rangle$, where $\kappa > 2^\beta$ is regular. Then $\mathfrak{A}$ models $\mathcal{L}^c(p)$.*

An examination of the proof of Lemma 4 shows, however, the proof of the final clause in that Lemma used only that $\mathfrak{A}$ models $\mathcal{L}^c(p)$. Hence:

Corollary 5.3 *Let $\mathbb{P}$ be revisable. Let G be $\mathbb{P}$-generic. Let $p \in G$ where p conforms to $N^* = \langle H_\Omega, M, <, \ldots \rangle$. Let $\overline{N}^* = \overline{N}^*(p, N^*)$. There is a unique $\sigma \supset \pi^G_p \cup F^p$ s.t. $\sigma : \overline{N}^* \prec N^*$.*

Proof. $\pi^G_p = \pi^{\mathfrak{A}}_p$ where $\mathfrak{A}$ is as in Lemma 5.2. QED(Corollary 5.3)

Combining this with the proof of Lemma 5.1 we get:

Lemma 5.4 *Let $\mathbb{P}$ be revisable. Let $N^* = \langle H_\Omega, M, <, \mathbb{P}, \ldots \rangle$ where Ω verifies revisability. Let p conform to N^*. Set:*

$$\overline{N}^* = \overline{N}^*(p, N^*) = \langle \overline{H}, \overline{M}, <, \overline{\mathbb{P}}, \ldots \rangle.$$

Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}^*$ and set: $q = \langle\langle M_p, C^{\overline{G}}\rangle, F^p\rangle$. Let $G \ni q$ be $\mathbb{P}$-generic. Let $\sigma \supset \pi_p^G \cup F^p$ s.t. $\sigma : \overline{N}^* \prec N^*$. Then $\sigma''\overline{G} \subset G$. (Hence σ extends uniquely to $\sigma^* : \overline{N}^*[\overline{G}] \prec N^*[G]$ with $\sigma^*(\overline{G}) = G$.)

Proof. The proof of (2) in Lemma 5.1 made use of a solid model $\mathfrak{A}$ of $\mathcal{L}(q)$. An examination of this proof shows, however, that it is enough that $\mathfrak{A}$ models $\mathcal{L}^c(q)$. Hence we can take $\mathfrak{A} = A$, where $A = \langle H_\kappa^{\mathbf{V}[G]}, C^G\rangle$ is as above. Hence $\pi_q^A = \pi_q^G$ and if $\sigma \supset \pi_q^G \cup F^q$ is s.t. if $\sigma : \overline{N}^* \prec N^*$ then $A \models \mathcal{L}(s)$ whenever $\overline{s} \in \overline{G}$ and $s = \sigma(\overline{s})$. If $s \notin G$, there would be $p \in G$ incompatible with s. But $A \models \mathcal{L}(p) \cup \mathcal{L}(s)$. Contradiction!

QED(Lemma 5.4)

We say that $\mathcal{L}$ is *modest* if all of its axioms can be forced by $\mathbb{P}_{\mathcal{L}}$-more precisely:

Definition Let $\mathcal{L}$ satisfy the core axioms. $\mathcal{L}$ is *modest* iff whenever G is $\mathbb{P}_{\mathcal{L}}$-generic there is a regular $\kappa > 2^\beta$ s.t. $A = \langle H_\kappa^{\mathbf{V}[G]}, C^G\rangle$ satisfies $\mathcal{L}$.

Lemma 5.2 says that $\mathcal{L}^c$ is modest. Assuming modesty, we have a simple criterion for deciding whether a given condition lies in a generic set G:

Lemma 5.5 *Let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ where $\mathcal{L}$ is modest. Let G be $\mathbb{P}$-generic. Let $p \in \mathbb{P}$. Then $p \in G$ iff the following hold:*

- $p_0 \lhd \langle M, C^G\rangle$,
- $\pi_{p_0}^G : \langle M_p, \overline{a}\rangle \prec \langle M, a\rangle$ *whenever* $\langle a, \overline{a}\rangle \in F^p$.

Proof. $(\rightarrow)$ is trivial. We prove $(\leftarrow)$. Let κ be regular s.t. $\kappa > 2^\beta$ and $A = \langle H_\kappa, C^G\rangle$ satisfies $\mathcal{L}$. Then $A \vDash \mathcal{L}(p)$. Iff $p \notin G$ there would be a $q \in \mathbb{P}$ s.t. p, q are incompatible. But $A \vDash \mathcal{L}(p) \cup \mathcal{L}(q)$. QED(Lemma 5.5)

Note In [LF], §4 we have shown that the assumption of modesty can be omitted from Lemma 5.5 assuming that $\mathbb{P}$ adds no reals. This is because $\mathbb{P} = \mathbb{P}_{\mathcal{L}^*}$, where $\mathcal{L}^*$ is the set of $\mathcal{L}$ statements forced to hold in $A = \langle H_\kappa^{\mathbf{V}[G]}, C^G\rangle$, where $\kappa > 2^\beta$ is regular. We shall not use that here, however, since our languages will always be modest. (We are unlikely to adopt an axiom without the expectation that it will be forced.)

Finally, we note that there is an apparently weaker notion of revisability relative to a parameter:

Definition $\mathbb{P}$ is *weakly revisable* iff there exist a cardinal $\Omega > 2^\beta$ and an $s \in H_\Omega$ s.t. whenever $N^* = \langle H_\Omega, M, <, \mathbb{P}, s, \ldots\rangle$ and p conforms to N^*, then, letting $\overline{N}^* = \overline{N}^*(p, N^*) = \langle \overline{H}, \overline{M}, <, \overline{\mathbb{P}}, \overline{s}, \ldots\rangle$, we have: Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}^*$. Then $q = \langle\langle \overline{M}, C^{\overline{G}}\rangle, F^p\rangle \in \mathbb{P}$.

It turns out that this is equivalent to full revisability. This fact is useful (and may be used tacitly) in verifying revisability.

Lemma 5.6 *Let $\mathbb{P}$ be weakly revisable. Then it is fully revisable.*

Proof. Let Ω be the smallest cardinal verifying weak revisability. Let $\Omega' > \overline{\overline{H}}_\Omega$ be a cardinal. Let $N'^* = \langle H_{\Omega'}, M, <', \mathbb{P}, \ldots \rangle$. Let p conform to N'^* and let $\overline{N}'^* = \overline{N}^*(p, N'^*) = \langle \overline{H}', \overline{M}', <, \overline{\mathbb{P}}', \ldots \rangle$. Let $\overline{G}$ be $\overline{\mathbb{P}}'$-generic over $\overline{N}'^*$.

Claim $q = \langle \langle \overline{M}', C^{\overline{G}} \rangle, F^p \rangle \in \mathbb{P}$.

Note that Ω, s are N'^*-definable, where $s =$ the $<'$-least s s.t. $\langle \Omega, s \rangle$ verifies weak revisability. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$ and let $\sigma' \supset \pi_p^{\mathfrak{A}} \cup F^p$ s.t. $\sigma' : \overline{N}'^* \prec N'^*$. Let

(1) $\sigma'(\overline{\Omega}, \overline{s}) = \Omega, s$.

Set: $N^* = \langle H_\Omega, M, <, \mathbb{P}, s, \ldots \rangle$ where $< = <' \cap H_\Omega^2$. Then p conforms to N^*. Set: $\overline{N}^* = \overline{N}^*(p, N^*) = \langle \overline{H}, \overline{M}, <, \overline{\mathbb{P}}, \overline{s}, \ldots \rangle$. Let $\sigma \supset \pi_p^{\mathfrak{A}} \cup F^p$ s.t. $\sigma : \overline{N}^* \prec N^*$. Then each $x \in \mathrm{rng}(\sigma)$ is N^*-definable in parameters from $\mathrm{rng}(\pi_p^{\mathfrak{A}} \cup F^p)$. Hence it is N'^*-definable in these parameters. Hence:

(2) $\mathrm{rng}(\sigma) \subset \mathrm{rng}(\sigma')$.

But:

(3) $\overline{M} = M_p = \overline{M}'$; $\sigma \upharpoonright \overline{M} = \pi_p^{\mathfrak{A}} = \sigma' \upharpoonright \overline{M}$.

Moreover, each $a \in \mathfrak{P}(\overline{M}) \cap \overline{N}^*$ is $\langle \overline{M}, b \rangle$-definable from parameters from $\overline{M}$, where $b \in D^p$. Similarly for $\mathfrak{P}(\overline{M}) \cap \overline{N}^*$. Hence:

(4) $\mathfrak{P}(\overline{M}) \cap \overline{N}^* = \mathfrak{P}(\overline{M}) \cap \overline{N}'^*$.

Since $\sigma \upharpoonright D^p = F^p = \sigma' \upharpoonright D^p$ and $\sigma \upharpoonright \overline{M} = \sigma' \upharpoonright \overline{M}'$, we conclude

(5) $\sigma \upharpoonright \mathfrak{P}(\overline{M}) = \sigma' \upharpoonright \mathfrak{P}(\overline{M})$.

$\overline{\mathbb{P}} = \langle |\overline{\mathbb{P}}|, \leq_{\overline{\mathbb{P}}} \rangle$ is canonically codable as a subset of $\mathfrak{P}(\overline{M})$. Similarly for $\overline{\mathbb{P}}'$. But $\sigma(\overline{\mathbb{P}}) = \sigma'(\overline{\mathbb{P}}') = \mathbb{P}$. It follows easily that.

(6) $\overline{\mathbb{P}} = \overline{\mathbb{P}}'$ and $\sigma \upharpoonright \mathbb{P} = \sigma' \upharpoonright \overline{\mathbb{P}}'$.

But if $\Delta \in \mathfrak{P}(\overline{\mathbb{P}}) \cap \overline{N}^*$, then $\Delta = (\sigma^{-1}) \cdot \sigma(\Delta) \in \overline{N}'^*$. Hence:

(7) $\mathfrak{P}(\overline{\mathbb{P}}) \cap \overline{N}^* \subset \overline{N}'^*$.

Hence $\overline{G}$ is generic over $\overline{N}^*$ and we conclude:

(8) $q = \langle \langle \overline{M}, C^{\overline{G}} \rangle, F^p \rangle \in \mathbb{P}$. QED(Lemma 5.6)

In conclusion we say a few words about the difference between the present approach and that taken in [LF]. There too we approximated $M = L_\tau^A$ s.t. $L_\tau^A = H_\beta$ for some $\beta > \omega_1$. Our intention, however, was simply to make M the limit of a tower of countable models. In place of an approximation system Γ, Π we worked with a collection T of *tower segments* $\langle \langle M_i \mid i \leq \alpha \rangle, \langle \pi_{ij} \mid i \leq j \leq \alpha \rangle \rangle$ satisfying:

- $M_i = L_{\beta_i}^{A_i}$, $i \leq \omega_1^{M_i}$, $M_h \in H_{\omega_1}^{M_i}$ for $h < i$.

- $\pi_{ij} : M_i \prec M_j \ (i \leq j)$ with $\pi_{ii} = \text{id}$.
- $\pi_{ij}\pi_{hi} = \pi_{hj}$.
- If $\lambda \leq \alpha$ is a limit ordinal, then $M_\lambda = \bigcup_{i<\lambda} \text{rng}(\pi_{i\lambda})$.

We sometimes imposed further requirements on T, but T was always primitive recursive. For $t \in T$ we set:

$$t = \langle\langle M_i^r \mid i \leq \alpha_t\rangle, \langle \pi_{ij}^t \mid i \leq j \leq \alpha_t\rangle\rangle.$$

Call t a segment of s iff $\alpha_t \leq \alpha_s$ and

$$M_i^t = M_i^s, \quad \pi_{ij}^t = \pi_{ij}^s \quad \text{for } \ i \leq j \leq \alpha_t.$$

Our language contained a single constant $\overset{\circ}{t}$ in addition to $\underline{x}$ $(x \in N)$ and the core axioms:

$$\text{ZFC}^-, \quad H_{\omega_1} = \underline{H}_{\omega_1}, \quad \bigwedge v\Big(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} w = \underline{z}\Big), \quad \overset{\circ}{t} \in T,$$
$$\alpha_{\overset{\circ}{t}} = \underline{\omega}_1, \quad M_{\underline{\omega}_1}^t = \underline{M}, \quad \bigwedge i < \underline{\omega}_1 \ M_i^t \in H_{\omega_1}.$$

We now show how to convert this approach into our present one. For each $t \in T$ set:

$$e_t = \langle M_{\alpha_t}, \{\langle y, x, i\rangle \mid i < \alpha_t \wedge \pi_{i\alpha_t}(x) = y\}\rangle.$$

Set $\Gamma = \{e_t \mid t \in T\}$. Note that t is uniquely recoverable from e_t. We set:

$$e_t \lhd e_s \ \text{ iff } \ e_t \quad \text{is a segment of } \ e_1,$$
$$\pi : e_t \lhd e_s \ \text{ iff } \ (e_t \lhd e_s \ \text{ and } \ \pi = \pi_{\alpha_t,\alpha_s}^s).$$

Then Γ, Π is an approximation system and the above core axioms translate into our usual core axioms.

Chapter 6

Examples

We now display some specific examples of $\mathcal{L}$-forcing. All of them are revisable and will turn out to be subcomplete as well.

6.1 Example 1

Assume CH and $2^{\omega_1} = \omega_2$. Without adding reals we wish to make ω_2 become ω-cofinal. We first define our approximation system:

Definition Γ = the set of $\langle M, C\rangle$ s.t.

- $M = L^A_\tau$ models ZFC$^-$ and "ω_1 is the largest cardinal".
- C is a cofinal subset of On_M of order type ω.

Definition For $u \in \Gamma$ set $u = \langle M_u, C_u\rangle$.

Definition Π = the set of $\langle \pi, u, v\rangle$ s.t. $u, v \in \Gamma$, $\pi : M_u \prec M_v$, $\pi{}''C_u = C_v$.

We again write $\pi : u \lhd v$ for $\langle \pi, u, v\rangle \in \Pi$ and $u \lhd v$ for $\bigvee \pi\ \pi : u \lhd v$.

Definition $\alpha_u = \omega_1^{M_u}$ for $u \in \Gamma$.

We note that:

(1) Let $v \in \Gamma$. Let $\alpha \le \alpha_v$. There is at most one $u \in \Gamma$ s.t. $u \lhd v$ and $\alpha = \alpha_u$. Thus $\lhd$ is a tree.

Now let $M = L^A_{\omega_2}$, where $L_{\omega_2}[A] = H_{\omega_2}$. Set: $N = \langle H_{\omega_3}, M, <, \ldots\rangle$ where $<$ well orders H_{ω_3}.

Let $\mathcal{L}$ be the language on N constaining exactly the core axioms (wrt. Γ, Π).

Lemma 1 *$\mathcal{L}$ is consistent.*

Proof. Let $\theta > 2^\beta$ be a regular cardinal. Let $H = H_\theta$ and $\sigma : \overline{H} \prec H$, where $\overline{H}$ is countable and transitive. Let $\sigma(\overline{M}, \overline{N}) = M, N$, $\sigma(\tilde{\mathcal{L}}) = \mathcal{L}$. Set $\tilde{M} = \bigcup_{u\in\overline{M}} \sigma(u)$. Then $\sigma \restriction \overline{M} : \overline{M} \prec \tilde{M}$ cofinally. Let $\langle \tilde{H}, \tilde{\sigma}\rangle$ be the liftup of $\langle \overline{M}, \sigma \restriction \overline{M}\rangle$. Then

$\tilde{\sigma} : \overline{H} \prec \tilde{H}$ $\omega_2^{\overline{M}}$-cofinally. Let $\tilde{k} : \tilde{H} \prec H$ s.t. $k\tilde{\sigma} = \sigma$, $k \upharpoonright \omega_2^{\tilde{H}} = \text{id}$. Let $k(\tilde{\mathcal{L}}) = \mathcal{L}$. Then $\tilde{\mathcal{L}}$ is a language on $\tilde{N}$ and it suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent.

Proof. Let $\overline{C} \subset \overline{M}$ be cofinal in $On_{\overline{M}}$ with order type ω. Set $\tilde{C} = \sigma''\overline{C}$. Then $\langle H_{\omega_2}^{\tilde{H}}, \tilde{C}\rangle$ models $\tilde{\mathcal{L}}$. QED(Lemma 1)

Now let $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$. We show that $\mathbb{P}$ satisfies a particularly strong form of revisability.

Lemma 2 *Let $p \in \mathbb{P}$. Let C be cofinal in On_{M_p} with order type ω. Then $q = \langle\langle M_p, C\rangle, F^p\rangle \in \mathbb{P}$.*

Proof. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. We shall "resection" $\mathfrak{A}$ to get a solid model $\mathfrak{A}'$ of $\mathcal{L}(q)$. Let $\mathfrak{A} = \langle|\mathfrak{A}|, C^{\mathfrak{A}}\rangle$. Set $\mathfrak{A}' = \langle|\mathfrak{A}|, C'\rangle$ where $C' = \pi_p^{\mathfrak{A}}{}''C$. Since C' is defined in $\mathfrak{A}$, we have $\mathfrak{A}' \models (\text{ZFC}^- \wedge \underline{H}_{\omega_1} = H_{\underline{\omega}_1})$. Since $H_{\omega_1}^{\mathfrak{A}} = H_{\omega_1}$ it follows easily that whenever $X \subset M$ is countable in $\mathfrak{A}$, then there is $u \lhd \langle M, C'\rangle$ s.t. $u \in H_{\omega_1}$ and $X \subset \text{rng}(\pi_u^{\mathfrak{A}})$. Hence all core axioms hold. QED(Lemma 2)

An immediate corollary is:

Corollary 2.1 $\mathbb{P}$ *is revisable.*

Thus, if G is $\mathbb{P}$-generic and $\kappa > 2^\beta$ is regular, $\langle H_\kappa^G, C^c\rangle$ satisfies all core axioms. But these are exactly the axioms of $\mathcal{L}$. Hence $\mathcal{L}$ is modest.

Making use of Lemma 2 we now prove:

Lemma 3 $\mathbb{P}$ *is subcomplete.*

Proof. Let $\theta > 2^{2^{\omega_2}}$. Let $W = L_\tau^A$ be a ZFC$^-$ model s.t. $H_\theta \subset W$ and $\theta < \tau$. Let $\pi : \overline{W} \prec W$, where $\overline{W}$ is countable and full. Let $\pi(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$. Since $\omega_2 \leq \delta(\mathbb{P})$ it suffices to show:

Claim Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{W}$. There is $q \in \mathbb{P}$ s.t. whenever $G \ni q$ is $\mathbb{P}$-generic, then there is $\sigma \in \mathbf{V}[G]$ s.t.

(a) $\sigma : \overline{W} \prec W$
(b) $\sigma(\overline{\mathbb{P}}, \overline{\theta}, \overline{s}) = \mathbb{P}, \theta, s$
(c) $C_{\omega_2}^W(\text{rng}(\sigma)) = C_{\omega_2}^W(\text{rng}(\pi))$
(d) $\sigma''\overline{G} \subset G$.

Now let $C = C_{\omega_2}^W(\text{rng}(\pi))$, $k : \tilde{W} \overset{\sim}{\leftrightarrow} C$, where $\tilde{W}$ is transitive. Set $\tilde{\pi} = k^{-1} \cdot \pi$. Then $\tilde{\pi} : \overline{W} \prec \tilde{W}$ is $\omega_3^{\overline{W}}$-cofinal. If σ satisfies (a)–(d) and we set: $\tilde{\sigma} = k^{-1}\sigma$, then $\tilde{\sigma} : \overline{W} \prec \tilde{W}$ is also $\omega_3^{\overline{W}}$-cofinal. But since $\tilde{\sigma}$ takes $\omega_2^{\overline{W}}$ cofinally to $\omega_2 = \omega_2^{\tilde{W}}$, it follows that $\tilde{\sigma}$ is ω_2-cofinal.

The following lemma hints at the possibility of such a $\tilde{\sigma}$: Let $\tilde{\pi}(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \tilde{\theta}, \tilde{\mathbb{P}}, \tilde{s}$.

Sublemma 3.1 *Let $\delta = \delta_{\tilde{W}}$ = the least δ s.t. $L_\delta(\tilde{W})$ is admissible. Then the following language $\tilde{\mathcal{L}}$ on $L_\delta(\tilde{W})$ is consistent:*

Predicate: $\in$
Constants: $\mathring{\sigma}$, $\underline{x}$ $(x \in L_\delta(\tilde{W}))$
Axioms: ZFC$^-$, $\bigwedge v(v \in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$, $\mathring{\sigma} : \underline{\overline{W}} \prec \underline{\tilde{W}}$ $\underline{\omega_2^{\tilde{W}}}$*-cofinally,*
$\sigma(\underline{\overline{\mathbb{P}}}, \underline{\overline{\theta}}, \underline{\overline{s}}) = \tilde{\mathbb{P}}, \tilde{\theta}, \tilde{s}$.

Proof. Let $\langle \hat{W}, \hat{\pi} \rangle$ be the liftup of $\langle \overline{W}, \pi \restriction \overline{H} \rangle$, where $\overline{H} = (H_{\omega_2})^{\overline{W}}$. (Hence $\pi \restriction \overline{H} = \tilde{\pi} \restriction \overline{H}$.) Let $\hat{k} : \hat{W} \prec \tilde{N}$ s.t. $\hat{k}\hat{\pi} = \tilde{\pi}$, $\hat{k} \restriction \omega_2^{\hat{W}} = \text{id}$. Then $\hat{k}$ is cofinal in $\tilde{W}$. Let $\hat{\delta} = \delta_{\hat{W}}$ be least s.t. $L_{\hat{\delta}}(\hat{W})$ is admissible. Let $\hat{\mathcal{L}}$ be defined on $L_{\hat{\delta}}(\hat{W})$ as $\tilde{\mathcal{L}}$ was defined on $L_\delta(\tilde{W})$ with $\hat{W}$, $\hat{\mathbb{P}}$, $\hat{\theta}$, $\hat{s}$ in place of $\tilde{W}$, $\tilde{\mathbb{P}}$, $\tilde{\theta}$, $\tilde{s}$, where $\hat{\pi}(\overline{\mathbb{P}}, \overline{\theta}, \overline{s}) = \hat{\mathbb{P}}, \hat{\theta}, \hat{s}$. It suffices to show:

Claim $\hat{\mathcal{L}}$ is consistent.
This is trivial, however, since $\langle \hat{W}, \hat{\pi} \rangle$ models $\hat{\mathcal{L}}$. QED(Sublemma 3.1)

Now let $\Omega > 2^\beta$ be a cardinal. Set: $N^* = \langle H_\Omega, M, W, \mathbb{P}, \theta, s, \pi, \ldots \rangle$. Let p conform to N^* and set:

$$\overline{N}^* = \overline{N}^*(p, N^*) = \langle H', M', W', \mathbb{P}', \theta', s', \pi', \ldots \rangle.$$

Let $\tilde{W}'$, $\tilde{\pi}'$, $\tilde{\mathcal{L}}'$ be defined in $\overline{N}^*$ as $\tilde{W}, \tilde{\pi}$, $\tilde{\mathcal{L}}$ were defined in N^*. Since $\overline{N}^*$ is countable, there is a solid model $\mathfrak{A}$ of $\tilde{\mathcal{L}}$. Set $\tilde{\sigma} = \mathring{\sigma}^{\mathfrak{A}}$. Then

$$\tilde{\sigma} : \overline{W} \prec \tilde{W}' \quad \omega_2^{\overline{W}}\text{-cofinally.}$$

Hence $\tilde{W}' = C^{\tilde{W}'}_{\omega_2^{W'}}(\text{rng}(\tilde{\sigma}))$. Set: $\overline{C} = C^{\overline{G}}$, $C' = \pi'{}''\overline{C}$. Then C' is cofinal in $\omega_2^{W'}$ and has order type ω. Set: $q = \langle\langle M', C' \rangle, F^p \rangle$. Then $q \in \mathbb{P}$ by the strong revisability lemma. Let $G \ni q$ be $\mathbb{P}$-generic. Let $\pi^* \supset \pi_q^G \cup F^q$ s.t. $\pi^* : \overline{N}^* \prec N^*$. Let $\pi^*(k') = k$. Set: $\sigma' = k'\tilde{\sigma}$, $\sigma = \pi^*\sigma'$. Then $\sigma \in \mathbf{V}[G]$.

Claim σ satisfies (a)–(d).

Proof. (a), (b) are trivial. We prove (c). Set $\omega_2' = \omega_2^{W'}$.

(1) $C^{W'}_{\omega_2'}(\text{rng}(\sigma')) = C^{W'}_{\omega_2'}(\text{rng}(\pi'))$,

since $k'{}''\tilde{W}' = C^{W'}_{\omega_2'}(\text{rng}(\pi'))$ by definition and $k'{}''\tilde{W}' = C^{W'}_{\omega_2'}(\text{rng}(\sigma'))$, since $\tilde{W}' = C^{\tilde{W}'}_{\omega_2'}(\text{rng}(\tilde{\sigma}))$, $k'{}''\text{rng}(\tilde{\sigma}) = \text{rng}(\sigma')$, and $k' \restriction \omega_2' = \text{id}$.

(2) $C^W_{\omega_2}(\text{rng}(\sigma)) \subset C^W_{\omega_2}(\text{rng}(\pi))$,

since $\text{rng}(\sigma) = \pi^*{}''\text{rng}(\sigma') \subset \pi^*{}''C^{W'}_{\omega_2'}(\text{rng}(\pi')) \subset \pi^*(C^{W'}_{\omega_2'}(\text{rng}(\pi')) = C^W_{\omega_2}(\text{rng}(\pi))$.

(3) $C^W_{\omega_2}(\text{rng}(\pi)) \subset C^W_{\omega_2}(\text{rng}(\sigma))$,

since $\text{rng}(\pi) = \pi^*{}''\text{rng}(\pi') \subset \pi^*{}''C^{W'}_{\omega_2}(\text{rng}(\sigma')) \subset C^W_{\omega_2}(\text{rng}(\sigma))$, since $\pi^*{}''\text{rng}(\sigma') = \text{rng}(\sigma)$ and $\pi^*{}''\omega_2' \subset \omega_2$. QED(c)

We now prove (d). Since $\mathcal{L}$ is modest we have:

Sublemma 3.2 *Let $C = C^G$. Then $G = G^C =$ the set of $p \in \mathbb{P}$ s.t. $p_0 \lhd \langle M, C \rangle$ and $\pi : \langle M_p, \overline{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \overline{a} \rangle \in F^p$, when $\pi = \pi_{p_0, \langle M, a \rangle}$.*

Now let $\overline{r} \in \overline{G}$, $r = \sigma_0(\overline{r})$. Then $r_0 = \overline{r}_0 \lhd \langle \overline{M}, \overline{C} \rangle$ and $\pi_{\overline{r}_0, \langle \overline{M}, \overline{C} \rangle} = \pi_{\overline{r}}^{\overline{G}}$, where $\overline{C} = C^{\overline{G}}$. Obviously,

$$\sigma' \upharpoonright \overline{M} : \langle \overline{M}, \overline{C} \rangle \lhd \langle M', C' \rangle$$

and

$$\pi_q^G : \langle M', C' \rangle \lhd \langle M, C \rangle,$$

where $\pi_q^G = \pi^* \upharpoonright M'$. Hence

$$\sigma \upharpoonright \overline{M} : \langle \overline{M}, \overline{C} \rangle \lhd \langle M, C \rangle.$$

Let $r = \sigma(\overline{r})$. Then $r_0 = \overline{r}_0$ and $F^r = \{\langle \pi^*(a), \overline{a} \rangle \mid \langle a, \overline{a} \rangle \in F^{\overline{r}}\}$. Clearly $r_0 \lhd \langle M, C \rangle$ and: $\pi_{r_0, \langle M, C \rangle} = \sigma \circ \pi_{\overline{r}}^{\overline{G}}$. Now let $\langle a, \overline{a} \rangle = \langle \pi^*(a'), \overline{a} \rangle \in F^r$. Then $\pi_r^{\overline{G}} : \langle M_r, \overline{a} \rangle \prec \langle M', a' \rangle$ and $\sigma(\langle M', a' \rangle) = \langle M, a \rangle$. QED(Lemma 3)

Note We could in this case have omitted the predicate C and simply taken Γ as the set of $\langle M, \emptyset \rangle$ s.t. $M = L_\tau^A$ models ZFC$^-$ and "ω_1 is the largest cardinal". Π would then be defined as the set of $\langle \pi, u, v \rangle$ s.t. $u, v \in \Gamma$, $\pi'' M_u \prec M_v$ cofinally. If we call $\mathbb{P}'$ the resulting set of conditions, then it is the "same" as $\mathbb{P}$ in the sense that $\mathrm{BA}(\mathbb{P}) \simeq \mathrm{BA}(\mathbb{P}')$.

Note $\mathbb{P}$ is, in fact, equivalent to Namba forcing in the sense that $\mathrm{BA}(\mathbb{P}) \simeq \mathrm{BA}(\mathbb{N})$. This is surprising, since $\mathbb{P}$ not only looks different and has a different motivation, but the combinatorics involved in the proofs are quite different.

6.2 Example 2

Now let $\beta > \omega_2$ be a cardinal and assume: $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\underset{\smile}{\beta}} = \beta$. We shall develop a forcing very much like the previous forcing which, however, gives cofinality ω not only to ω_2 but to every regular $\tau \in [\omega_2, \beta]$. There will be some variation in the definition of the forcing, depending on whether $\mathrm{cf}(\beta) = \omega_1$. Thus, in this example, we assume $\mathrm{cf}(\beta) = \omega_1$. In Example 3 we shall then detail the changes which must be made if $\mathrm{cf}(\beta) \neq \omega_1$. Let $M = L_\beta^A$ where $H_{\omega_2} = L_{\omega_2}[A]$ and $H_\beta = L_\beta[A]$. M is then *smooth* in the sense defined in Chapter 3.2.

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Definition Relable the classes Γ, $\lhd$ defined in Example 1 as Γ_0, $\lhd_0$. Set:
$\Gamma =$ the collection of $\langle M, C \rangle$ s.t.

- $M = L_\beta^A$ is smooth.
- $\gamma = \omega_2^M$ exists and $L_\gamma[A] = H_{\omega_2}$ in M.
- $C \subset \gamma$, $\sup C = \gamma$, $\mathrm{otp}\, C = \omega$.

For $u = \langle M_u, C_u \rangle \in \Gamma$ set:

$\alpha_u = \alpha_{M_u} = \omega_1^{M_u}$, $\gamma_u = \gamma_{M_u} = \omega_2^{M_u}$, $M_u = L_{\beta_u}^{A_u}$, $M_u^0 = L_{\gamma_u}^{A_u}$, $u^0 = \langle M_u^0, C_u \rangle$, Hence: $u^0 \in \Gamma_0$.

Definition Let $u, v \in \Gamma$. $\pi : u \lhd v$ iff the following hold:

- $\pi^0 : u^0 \lhd_0 v^0$ where $\pi^0 = \pi \upharpoonright M_u^0$.
- $\pi : M_u \prec M_v$.
- Let $\pi : M_u \to_{\Sigma_0} M_{uv}$ cofinally. Then $\langle M_{uv}, \pi \rangle$ is the liftup of $\langle M_u, \pi^0 \rangle$. (In other words $\pi : M_u \to_{\Sigma_0} M_{uv} \quad \gamma_u$-cofinally.)

$\langle \Gamma, \Pi \rangle$ is easily seen to be an approximation system.

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We return to $M = L_\beta^A$ as stated at the outset. Let $N = \langle H_{\beta^+}, M, <, \ldots \rangle$ where $<$ well orders H_{β^+}. Let $\mathcal{L}$ be the language on N containing only the core axioms (wrt. Γ, Π).

Lemma 4 *$\mathcal{L}$ is consistent.*

Proof. Let $\theta > 2^\beta$ be a regular cardinal. Let $\pi : \overline{H} \prec H_\theta$ s.t. $\overline{H}$ is countable and transitive and $\pi(\overline{M}, \overline{N}, \overline{\mathcal{L}}) = M, N, \mathcal{L}$. Let $\hat{H} = H_{\omega_2}^{\overline{M}}$ and set

$$\langle \tilde{H}, \tilde{\pi} \rangle = \text{ the liftup of } \langle \overline{H}, \pi \upharpoonright \hat{H} \rangle.$$

Let $k : \tilde{H} \prec H_\theta$ s.t. $k\tilde{\sigma} = \sigma$. $k \upharpoonright \omega_2^{\tilde{H}} = \text{id}$. Set $\tilde{M}, \tilde{N}, \tilde{\mathcal{L}} = \tilde{\pi}(\overline{M}, \overline{N}, \overline{\mathcal{L}})$. Then $k(\tilde{\mathcal{L}}) = \mathcal{L}$ and it suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent.

Let $\overline{C} \subset \omega_0^{\overline{M}}$ cofinally s.t. $\text{otp}(\overline{C}) = \omega$. Set $\tilde{C} = \pi''\overline{C} = \tilde{\pi}''\overline{C}$. We prove:

Claim $\langle H_{\omega_2}, \tilde{C} \rangle$ models $\tilde{\mathcal{L}}$.

Proof. All axioms are trivial except for the last one. We show that if $X \subset M$ is countable, then there is $u \in \Gamma \cap H_{\omega_1}$ s.t. $u \lhd \langle M, \tilde{C} \rangle$ and $X \subset \text{rng}(\pi_{u, \langle M, \tilde{C} \rangle})$. We construct such a u: Let $Z \prec \tilde{H}$ be countable s.t. $X \cup \text{rng}(\tilde{\pi}) \subset Z$. Let $\pi' : H' \overset{\sim}{\leftrightarrow} Z$. Set: $M' = \pi'^{-1}(\tilde{M})$, $C' = \pi'^{-1}{}''\tilde{C}$, $\pi'' = \pi' \upharpoonright M'$. Then $X \subset \text{rng}(\pi')$ and it suffices to show:

Claim $\pi'' : \langle M', C' \rangle \lhd \langle \tilde{M}, \tilde{C} \rangle$.

$\pi' \upharpoonright M^0 : \langle M^0, C' \rangle \lhd^0 \langle \tilde{M}^0, \tilde{C} \rangle$ is obvious. We therefore need only to show:

Claim Let $\pi'' : M' \to_{\Sigma_0} M^*$ cofinally. Then the map π'' is $\omega_2 M'$-cofinal into M^*.

Proof. First note that $\pi' : H' \prec \tilde{H} \quad \omega_2^{H'}$-cofinally, since if $x \in \tilde{H}$, then $x \in \tilde{\pi}(u)$, where $\overline{\overline{u}} < \omega_2$ in $\overline{H}$. Set $u' = \pi'^{-1}\tilde{\pi}(u)$. Then $x \in \pi'(u')$, $\overline{\overline{u}}' < \omega_2$ in H'. Now let $x \in M^*$. By cofinality there is $v \in M'$ s.t. $x \in \pi''(v)$. Let $u \in \tilde{H}$ s.t. $x \in \pi'(u)$ and $\overline{\overline{u}} < \omega_2$ in H'. Set: $w = u \cap v$. Then $x \in \pi''(w)$ where $\overline{\overline{w}} < \omega$ in M', since $M' = H_{\beta'}$ in H', where $\beta' = (\pi')^{-1}(\tilde{\beta})$. QED(Lemma 4)

We then define $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ as before. Exactly as before we get:

Lemma 5 *Let $p \in \mathbb{P}$. Let $C \subset \gamma_p$ be cofinal in γ_p with order type ω, where $\gamma_p =_{\text{Df}} \gamma_{p_0} = \omega_2^{M_p}$. Then $q = \langle \langle M_p, C \rangle, F^p \rangle \in \mathbb{P}$.*

Hence:

Corollary 5.1 $\mathbb{P}$ *is revisable.*

Hence $\langle H_\kappa[G], C^G\rangle$ models the core axioms whenever G is $\mathbb{P}$-generic and $\kappa > 2^\beta$. But $\mathcal{L}$ has only the core axioms and is, therefore, modest. Using this we obtain:

Lemma 6 $\mathbb{P}$ *is subcomplete.*

The proof is virtually identical to that of Lemma 3. However, in the verification of (d) at the end of the proof we need additional justification for:

$$\sigma' \upharpoonright \overline{M} : \langle \overline{M}, \overline{C}\rangle \lhd \langle M', C'\rangle.$$

Letting $\sigma' \upharpoonright \overline{M}$ map $\overline{M}$ cofinally to M^*, we must show:

$$\sigma' \upharpoonright \overline{M} : \overline{M} \longrightarrow_{\Sigma_0} M^* \quad \omega_2^{\overline{M}}\text{-cofinally.}$$

This follows from

$$\sigma' : \overline{N} \prec \tilde{N}' \quad \omega_2^{\overline{N}}\text{-cofinally}$$

by the argument used in Lemma 4 to get:

$$\pi'' : M' \longrightarrow_{\Sigma_0} M^* \quad \omega_2^{M'}\text{-cofinally}$$

from: $\pi' : H' \prec \tilde{H}' \quad \omega_2^{H'}$-cofinally. QED(Lemma 6)

$\mathbb{P}$ obviously collapses β to ω_1. We now show that its successor is not collapsed:

Lemma 7 *Let G be $\mathbb{P}$-generic. Then β^+ is regular in $\mathbf{V}[G]$.*

This is immediate from:

Sublemma 7.1 $\mathbb{B} = \mathrm{BA}(\mathbb{P})$ *has a dense subset of size β.*

Proof. We defined a collection S of statements in the forcing language s.t. $\overline{\overline{S}} \leq \beta$ (in $\mathbf{V}$), and for each $p \in \mathbb{P}$ there is a $\psi \in S$ s.t. $0 \neq [[\psi]] \subset [p]$. ($[p]$ being the smallest $a \in \mathbb{B}$ s.t. $p \in a$.) Let $\mathring{C}$ be the canonical term s.t. $\mathring{C}^G = C^G$ for $\mathbb{P}$-generic G. For each triple $\langle u, \overline{a}, a\rangle$ s.t.

$$u = \langle M_u, C_u\rangle \in \Gamma \cap H_{\omega_1}, \quad \overline{a} : \omega \to \mathfrak{P}(M_u), \quad a : \omega \to M,$$

let $\psi_{u\overline{a}a}$ be the statement:

$$\check{u} \lhd \langle \check{M}, \mathring{C}\rangle \wedge \bigwedge i < \omega \bigwedge z (z \in \check{\overline{a}}(i) \longleftrightarrow \mathring{\pi}(z) \in \check{a}(i))$$

where $\mathring{\pi} = \pi_{\check{u}\langle \check{M}, \mathring{C}\rangle}$. All such triples are elements of M, so the set S of such statements has at most cardinality β. We now show that for each $p \in \mathbb{P}$ there is $\psi \in S$ with $0 \neq [[\psi]] \subset [p]$. It suffices to prove this for a dense subset of $\mathbb{P}$, so assume w.l.o.g. that p conforms to $N^* = \langle H_{(2^\beta)^+}, M, <\rangle$. Let $G \ni p$ be $\mathbb{P}$-generic. Let $\tilde{\beta} = \sup \pi_p^G{}''\beta_p$. Then $\tilde{\beta} < \beta$. Set $\tilde{M} = L^A_{\tilde{\beta}}$, where $M = L^A_\beta$. For each $a \in R^p$ set $\tilde{a} = a \cap \tilde{M}$. Let $\langle\langle a_i, \overline{a}_i\rangle \mid i < \omega\rangle$ enumerate F^p in $\mathbf{V}$. Set $\overline{a} = \langle \overline{a}_i \mid i < \omega\rangle$,

$\tilde{a} = \langle \tilde{a}_i \mid i < \omega \rangle$. Let $\psi = \psi_{p_0,\overline{a},\tilde{a}}$. Then $[[\psi]] \neq 0$, since ψ is true in $\mathbf{V}[G]$. We claim that $[[\psi]] \subset [p]$, or equivalently:

Claim Let G be $\mathbb{P}$-generic. Then $G \cap [[\psi]] \neq \emptyset \to p \in G$.

Then $p_0 \lhd \langle M, C \rangle$, since ψ is true in $\mathbf{V}[G]$. Let $\langle a, \overline{a} \rangle \in F^p$. We must show:

Claim $\pi : \langle M_p, \overline{a} \rangle \prec \langle M, a \rangle$, where $\pi = \pi_{p_0,\langle M,C \rangle}$.

Set: $b = \{\langle z_1, \ldots, z_n \rangle \mid \langle M, a \rangle \models \mathcal{X}(z_1, \ldots, z_n)\}$. Then $b \in R^p$, since p conforms to N^*. Moreover $\langle b, \overline{b} \rangle \in F^p$ where $\overline{b}$ has the same definition over $\langle M_p, \overline{a} \rangle$. Hence:

$$\begin{aligned} \langle M_p, \overline{a} \rangle \models \mathcal{X}(z_1, \ldots, z_n) &\longleftrightarrow \langle z_1, \ldots, z_n \rangle \in \overline{b} \longleftrightarrow \pi(\langle z_1, \ldots, z_n \rangle) \in \tilde{b} = \tilde{M} \cap b \\ &\longrightarrow \pi(\langle z_1, \ldots, z_n \rangle) \in b \longrightarrow \langle M, a \rangle \models \mathcal{X}(\pi(z_1), \ldots, \pi(z_n)). \end{aligned}$$

Since this holds for all $\mathcal{X}$ we have: $\pi : \langle M_p, \overline{a} \rangle \prec \langle M, a \rangle$. QED(Lemma 7)

6.3 Example 3

We now assume $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\beta} = \beta$, and $\mathrm{cf}(\beta) \neq \omega_1$. We again want to give cofinality ω to all regular cardinals $\tau \in [\omega_2, \beta]$. It is clear that β will also acquire cofinality ω, since it either already has cofinality ω, or its cofinality lies in $[\omega_2, \beta)$. The simplest way of handling this is to revise the definition of $\lhd$ to:

Definition Let $u, v \in \Gamma$. $\pi : u \lhd v$ iff the following hold:

- $\pi^0 : u^0 \lhd_0 v^0$ where $\pi^0 = \pi \upharpoonright M_u^0$.
- $\pi : M_u \prec M_v$ γ_u-cofinally.

Let $M = L_\beta^A$ where $L_\beta[A] = H_\beta$. As before set $N = \langle H_{\beta^+}, <, M, \ldots \rangle$. Let $\mathcal{L}$ be the language on N with only the core axioms. Exactly as before we prove:

Lemma 8 *$\mathcal{L}$ is consistent.*

(Note that if $\overline{N}$ is countable and transitive, $\pi : \overline{N} \prec N$, $\pi(\overline{M}) = M$, and $\overline{M}^0 = H_{\omega_2}^{\overline{M}}$, then if $\langle \tilde{N}, \tilde{\pi} \rangle$ is the liftup of $\langle \overline{N}, \pi \upharpoonright \overline{M}^0 \rangle$, then $\pi \upharpoonright \overline{M} : \overline{M} \prec \tilde{M}$ $\omega_2^{\overline{M}}$-cofinally, where $\tilde{M} = \tilde{\pi}(\overline{M})$.)

We then set $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$. Exactly as before we get:

Lemma 9 $\mathbb{P}$ *is strongly revisable.*

Corollary 9.1 $\mathbb{P}$ *is revisable.*

Hence $\mathcal{L}$ is modest, since it has only the core axioms. Exactly as before we get:

Lemma 10 $\mathbb{P}$ *is subcomplete.*

Lemma 7 does *not* go through, however. In fact 2^β acquires cardinality ω_1. This follows from the very general theorem:

Lemma 11 *Let W be an inner model of* ZFC *and* CH. *Let $H_{\omega_1} = H_{\omega_1}^W$. Let $\beta > \omega_1$ s.t. $2^{\overline{\overline{\beta}}} = \beta$ in W. Suppose that* $\mathrm{cf}(\beta) = \omega$ *and $\overline{\overline{\beta}} = \omega_1$ in* $\mathbf{V}$. *Then* $\mathrm{card}(2^\beta) = \omega_1$ *in* $\mathbf{V}$.

Proof. Let $M = L_\beta^A$ where $L_\beta[A] = H_\beta$ in W. Let f map ω_1 onto M in $\mathbf{V}$. Let $\langle \beta_i \mid i < \omega \rangle \in \mathbf{V}$ be cofinal in β. Set: $X_\alpha = f''\alpha$ for $\alpha < \omega_1$. Set:

$$C = \{\alpha < \omega_1 \mid \alpha = \omega_1 \cap X_\alpha \wedge X_\alpha \prec M \wedge \{\beta_i \mid i < \omega\} \subset X_\alpha\}.$$

For $\alpha \in C$ set $\pi_\alpha : M_\alpha \overset{\sim}{\leftrightarrow} X_\alpha$, where M_α is transitive. Then $M_\alpha \in H_{\omega_1}$. For any $B \subset \beta$, there is $\alpha \in C$, s.t. $B \cap \beta_i \in X_\alpha$ for $i < \omega$. Set:

$$\overline{B} = \bigcup \{\pi_\alpha^{-1}(B \cap \beta_i) \mid i < \omega\}.$$

Then $\langle \alpha, \overline{B} \rangle \in H_{\omega_1}$ and B is recoverable from $\langle \alpha, \overline{B} \rangle$ by:

$$\tilde{\pi}(\alpha, \overline{B}) = \bigcup_{u \in M_\alpha} \pi_\alpha(u \cap \overline{B}).$$

Thus $\tilde{\pi}$ maps a subset of H_{ω_1} onto $\mathfrak{P}(\beta)$. QED(Lemma 11)

6.4 The extended Namba problem

Shelah was the first to show that Namba forcing can be iterated without adding reals. If we iterate it out to a strongly inaccessible κ, then κ becomes the new ω_2 and arbitrarily large regular cardinals below κ become ω-cofinal. However, many regular cardinals become ω_1-cofinal. The "extended Namba problem" asks whether, without adding reals, one can make κ become ω_2 while giving *all* of the regular cardinals in the interval (ω_1, κ) cofinality ω. This problem seemed so difficult that at one point we conjectured a provably negative answer in ZFC for all κ. Moti Gitik then disproved this conjecture by constructing a ZFC model in which the extended Namba problem had a positive solution for some κ. His model was a generic extension of a universe containing a supercompact cardinal. Following Gitik's breakthrough we then obtained a positive solution in ZFC for all κ. It is impossible to give the full proof of that result in these notes, but we shall endeaver to give some account of the methods used. We may assume w.l.o.g. that GCH holds below κ, since we may achieve this by a prior forcing in which all collapsed regular cardinals acquire a cofinality $\geq \omega_2$. If we then give the surviving regular cardinals in (ω_1, κ) the cofinality ω, the collapsed ones will also become ω-cofinal.

It is natural to try to solve this problem by an iteration $\langle \mathbb{B}_i \mid i \leq \kappa \rangle$. We ask now what the initial steps of this iteration should look like. We follow the convention that $\mathbb{B}_0 = 2$. Thus $\mathbb{B}_1$ is the first stage which "does something". We certainly expect it to give ω_2 the cofinality ω without adding reals. By Lemma 11 it follows that ω_3 will be collapsed, so ω_3 must acquire cofinality ω. But then ω_4 is collapsed etc. Thus every ω_n must be collapsed with cofinality ω. By Lemma 11, it then follows that $\omega_{\omega+1}$ is collapsed etc. This chain of implications does not break down until we reach ω_{ω_1}. There, however, it does break down, since we can use the $\mathbb{P}$ of Example 2 with $\beta = \omega_{\omega_1}$. All regular cardinals in $(\omega_1, \omega_{\omega_1})$ acquire cofinality ω and ω_{ω_1+1} is not collapsed, thus becoming the new ω_2. We take $\mathbb{B}_1 \simeq \mathrm{BA}(\mathbb{P})$. We can then repeat the process, getting $\mathbb{B}_2 \supseteq \mathbb{B}_1$ which collapses $\omega_{\omega_1 \cdot 2}$ to ω_1 etc. This gives us the first ω stages $\langle \mathbb{B}_i \mid i < \omega \rangle$. Our job now is to find an appropriate limit $\mathbb{B}_\omega$. Since each $\mathbb{B}_i$ is subcomplete, the inverse limit $\mathbb{B}^*$ is also subcomplete. However, a bit of reflection shows that $\mathbb{B}^*$ is too small to do the job: At the limit stage $\omega_{\omega_1 \cdot \omega}$ will be collapsed to ω_1. Hence by Lemma 11 $\omega_{(\omega_1 \cdot \omega)+1}$ will be collapsed and hence must acquire cofinality ω etc.

Proceeding in this fashion we see that $\omega_{\omega_1 \cdot (\omega+1)}$ must be collapsed. Thus our limit algebra must be large, not containing any dense set of size less than $\omega_{\omega_1(\omega+1)}$. At the same time it should have a dense subset of size $\omega_{\omega_1(\omega+1)}$ in order that the successor is preserved. It turns out that a limit with the requisite properties can be obtained by a construction rather like that of Example 2. We shall now sketch that construction, but a full verification of its properties is beyond the purview of these notes.

Let $M^0 = L^A_\gamma$ where $\gamma = \omega_{\omega_1 \omega}$, $L_\gamma[A] = H_\gamma$, and A canonically codes $\langle \mathbb{B}_i \mid i < \omega \rangle$. We define Γ_0, Π_0 as follows:

$\Gamma_0 =$ the set of $u = \langle M_u, B_u \rangle$ s.t.

- $M = L^{A_u}_{\gamma_u}$ where M_u models Zermelo set theory and A_u canonically codes a sequence $\langle \mathbb{B}^u_i \mid i < \omega \rangle$ of complete Boolean algebras in the sense of M with $\mathbb{B}^u_i \subseteq \mathbb{B}^u_j$ $(i \le j < \omega)$.
- $B_u \subset \bigcup_i \mathbb{B}^u_i$ s.t. $B_u \cap \mathbb{B}_i$ is $\mathbb{B}_i$-generic over M for $i < \omega$.
- $\sup\{\delta(\mathbb{B}_i) \mid i < \omega\} = \beta$ and $\mathbb{B}_i$ collapses $\delta(\mathbb{B}_i)$ to ω_1^M for $i < \omega$.

$\Pi_0 =$ the set of $\langle \pi, u, v \rangle$ s.t. $u, v \in \Gamma_0$, $\pi : M_u \prec M_v$ and $\pi''B_u \subset B_v$.

We write $\pi : u \lhd_0 v$ for $\langle \pi, u, v \rangle \in \Pi_0$. Setting $B^i_u = B_u \cap \mathbb{B}^u_i$, we see that π has a unique extension π^i s.t. $\pi^i : M_u[B^i_u] \prec M_v[B^i_v]$ and $\pi^i(B^i_u) = B^i_v$. Set: $M^*_u = \bigcup_i M_u[B^i_u]$ and $\pi^* = \bigcup_i \pi^i$. Then $\pi^* : M^*_u \to M^*_v$ cofinally.

Letting f^i_u be the canonical map of ω_1 onto $L^{A_u}_{\delta(\mathbb{B}^i_u)}$, we see that π is uniquely characterized by: $\pi \circ f^i_u = f^i_v$ for $i < \omega$. It follows easily that $\pi = \pi_{uv}$ is the unique $\pi : u \lhd_0 v$ and that Γ^0, Π^0 is an approximation system. Now let $M = L^A_\beta$ where $\beta = \omega_{\omega_1(\omega+1)}$, $L_\beta[A] = H_\beta$ and $M^0 = L^A_\gamma$ $(\gamma = \omega_{\omega_1 \cdot \omega})$. Set:

$$\Gamma = \text{ the set of } u = \langle M_u, B_u \rangle \text{ s.t. } M_u \text{ is smooth and there is } \gamma = \gamma_u \in M_u \text{ s.t. } u^0 = \langle M^0_u, B_u \rangle \in \Gamma_0, \text{ where } M_u = L^{A_u}_\beta \text{ and } M^0_u = L^{A_u}_\gamma.$$

We then set: $\Pi =$ the set of $\langle \pi, u, v \rangle$ s.t. $u, v \in \Gamma$ and:

- $\pi^0 : u^0 \lhd v^0$ where $\pi^0 = \pi \upharpoonright M^0$.
- $\pi : M_u \prec M_v$.

- Let $\pi : M_u \to M_{u,v}$ cofinally. Then $\langle M_{u,v}, \pi\rangle$ is the liftup of $\langle M_u, \pi^0\rangle$.

We again set:

$$\pi : u \lhd v \quad \text{iff} \quad \langle \pi, u, v\rangle \in \Pi.$$

Thus $\langle \Gamma, \Pi\rangle$ is an approximation system which is related to $\langle \Gamma^0, \Pi^0\rangle$ exactly as in Chapter 6.2.

Again, letting $M = L^A_\beta$ be as above, and $N = \langle H_{\beta+}, <, M, \ldots\rangle$, we form the language $\mathcal{L}$ on N containing only the core axioms.

Lemma 12 *$\mathcal{L}$ is consistent.*

Proof. Let $\mathbb{B}^*$ = the inverse limit of $\langle \mathbb{B}_i \mid i < \omega\rangle$. Then $\mathbb{B}^*$ is subcomplete. Let B^* be $\mathbb{B}^*$-generic. We prove the consistency of $\mathcal{L}$ in $\mathbf{V}[B^*]$. Let $B_i = B^* \cap \mathbb{B}_i$, $B = \bigcup_{i<\omega} B_i$. Let $H = H_{(2^\beta)^+}$ in $\mathbf{V}$. Let $\pi : \overline{H} \prec H$ in $\mathbf{V}[B^*]$ s.t. $\overline{H}$ is countable and transitive. Let:

$$\pi(\overline{N}, \overline{M}, \overline{M}^0, \langle \overline{\mathbb{B}}_i \mid i < \omega\rangle) = N, M, M^0, \langle \mathbb{B}_i \mid i < \omega\rangle.$$

Set $\overline{B}_i = \pi^{-1}{}''B_i$ for $i < \omega$. Since we are working in $\mathbf{V}[B^*]$ we may assume that $\overline{B}_i$ is $\overline{\mathbb{B}}_i$-generic over $\overline{M}$ for $i < \omega$. Clearly π takes $\overline{M}^0$ to M^0 cofinally. Moreover:

$$\pi \upharpoonright \overline{M}^0 : \langle \overline{M}^0, \overline{B}\rangle \lhd_0 \langle M^0, B\rangle.$$

Now let $\langle \tilde{H}, \tilde{\pi}\rangle$ be the liftup of $\langle \overline{H}, \pi \upharpoonright \overline{M}^0\rangle$. Let: $\tilde{\pi}(\overline{M}, \overline{N}, \overline{\mathcal{L}}) = \tilde{M}, \tilde{N}, \tilde{\mathcal{L}}$, where $\pi(\overline{\mathcal{L}}) = \mathcal{L}$. Since there is $k : \tilde{H} \prec H$ with $k(\tilde{\mathcal{L}}) = \mathcal{L}$, it suffices to prove that $\tilde{\mathcal{L}}$ is consistent. We claim:

Claim $\langle H_\kappa, B\rangle$ models $\tilde{\mathcal{L}}$, where $\kappa > 2^\beta$ is regular in $\mathbf{V}$.

Proof. The only problematical case is: Let $X \subset \tilde{M}$ be countable. There is $u \in \Gamma \cap H_{\omega_1}$ s.t. $u \lhd \langle \tilde{M}, B\rangle$ and $X \subset \operatorname{rng}(\pi_{u,\langle \tilde{M},B\rangle})$. Let $Y \prec \tilde{H}$ be countable s.t. $\operatorname{rng}(\tilde{\kappa}) \cup X \subset Y$ and whenever $\Delta \in Y$ is dense in $\mathbb{B}_i$ $(i < \omega)$, then $\Delta \cap B \neq \emptyset$. Let:

$$\pi' : H' \overset{\sim}{\leftrightarrow} Y, \quad \pi'(M^{0'}, M', \langle \mathbb{B}'_i \mid i < \omega\rangle) = M^0, \tilde{M}, \langle \mathbb{B}_i \mid i < \omega\rangle.$$

Set: $B' = \pi'^{-1}{}''B_i$, $\pi'' = \pi' \upharpoonright M'$.

Claim $\pi'' : \langle M', B'\rangle \lhd \langle \tilde{M}, B\rangle$.

Clearly: $\pi'' \upharpoonright M^{0\prime} : \langle M^{0\prime}, B'\rangle \lhd_0 \langle M^0, B\rangle$. Since $\pi'' : M' \prec \tilde{M}$, it suffices to show that: If $\pi'' : M' \to M^*$ cofinally, then $\langle M^*, \pi''\rangle$ is the liftup of $\langle M', \pi'' \upharpoonright M^{0\prime}\rangle$ – i.e. that π'' takes M' γ'-cofinally to M^* wheres $\gamma' = (\omega_1 \cdot \omega)^{M'}$. This follows by the usual argument. QED(Lemma 12)

The strong revisability lemma reads:

Lemma 13 *For sufficiently large $\theta > 2^\beta$ we have: Let $N^* = \langle H_\theta, M, \mathbb{P}, <, \ldots\rangle$. Let p conform to N^* and set: $\overline{N}^* = \overline{N}^*(N^*, p) = \langle \overline{H}, \overline{M}, \overline{\mathbb{P}}, <, \ldots\rangle$. Let $\overline{B} \subset \bigcup_{i<\omega} \mathbb{B}_i^{\overline{M}}$ s.t. $\overline{B} \cap \mathbb{B}_i^{\overline{M}}$ is $\mathbb{B}_i^{\overline{M}}$-generic over $\overline{M}$ for $i < \omega$. Then $q = \langle\langle \overline{M}, \overline{B}\rangle, F^p\rangle \in \mathbb{P}$.*

We must forego the proof of Lemma 13, since it is very long and involves properties of the algebras $\mathbb{B}_i$ which we have not developed here.

An immediate corollary is:

Corollary 13.1 *$\mathbb{P}$ is revisable, since revisability says that the above holds when $\overline{B} = B^{\overline{G}}$ for a $\overline{G}$ which is $\overline{\mathbb{P}}$-generic over $\overline{N}^*$.*

Since $\mathcal{L}$ has only the core axioms, it is then modest. But then we get:

Lemma 14 *$\mathbb{P}$ is subcomplete.*

We sketch briefly the proof of Lemma 14, which is largely the same as before. Let θ be big enough to verify the subcompleteness of $\mathbb{B}_i$ for $i < \omega$. Let $W = L^A_\tau$ be a ZFC$^-$ model with $H_\theta \subset W$ and $\theta < \tau$. Let $\pi : \overline{W} \prec W$ where $\overline{W}$ is countable and full. Let $\pi(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$.

Claim There is $q \in \mathbb{P}$ s.t. if $G \ni q$ is $\mathbb{P}$-generic, there is $\sigma \in \mathbf{V}[G]$ with:

(a) $\sigma : \overline{W} \prec W$
(b) $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$
(c) $C^W_\gamma(\text{rng}(\sigma)) = C^W_\gamma(\text{rng}(\pi))$, where $\gamma = On \cap M^0 = \sup_{i<\omega} \delta(\mathbb{B}_i)$.
(d) $\sigma''\overline{G} \subset G$.

(**Note** $\gamma \le \delta(\mathbb{P})$, since otherwise γ would not be collapsed.)

Let $\Omega > \theta$ be big enough to verify the strong revisability of $\mathbb{P}$. Set:

$$N^* = \langle H_\Omega, <, M, N, \mathbb{P}, W, \pi, \ldots \rangle.$$

Let p conform to N^*. Set: $\overline{N}^* = \overline{N}^*(N^*, p) = \langle H', M', N', \mathbb{P}', W', \pi', \ldots \rangle$. Set: $\mathbb{B}'_i = \mathbb{B}^{M'}_i$ $(i < \omega)$. Set $\theta', \mathbb{P}', s' = \pi'(\overline{\theta}, \overline{\mathbb{P}}, \overline{s})$. Set $\gamma' = \pi'(\overline{\gamma})$, where $\pi(\overline{\gamma}) = \gamma = On \cap M^0$. Noting that W' is countable and imitating the proof of Chapter 4, Theorem 2 we get:

Sublemma 14.1 *There are σ' and $B' \subset \bigcup_{i<\omega} \mathbb{B}'_i$ s.t. $B'_i = B' \cap \mathbb{B}'_i$ is $\mathbb{B}'_i$-generic over W' for $i < \omega$ and:*

(a) $\sigma' : \overline{W} \prec W'$
(b) $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta', \mathbb{P}', s'$
(c) $C^{W'}_{\gamma'}(\text{rng}(\sigma')) = C^{W'}_{\gamma'}(\text{rng}(\pi'))$
(d) $\sigma'{}''\overline{B} \subset B'$, *where* $\overline{B} = B^{\overline{G}}$.

To get this we successively define $\mathring{\sigma}_i, b_i \in \mathbb{B}'_i$ s.t. whenever $B'_i \ni b_i$ is $\mathbb{P}'$-generic over W' and $\sigma'_i = \mathring{\sigma}_i{}^{B'_i}$, then σ'_i satisfies (a)–(c) and: $\sigma'_i{}''\overline{B}_i \subset B'_i$ (where $\overline{B}_i = \overline{B} \cap \overline{\mathbb{B}}_i$). We ensure $h_i(b_{i+1}) = b_i$ for $i < \omega$. We then successively choose $B'_i \ni b_i$ with: B'_i is $\mathbb{B}'_i$-generic over W' and $B'_i \supset B'_\ell$ for $\ell < i$. We set: $B' = \bigcup_i B'_i$ and let σ' be the 'limit' of $\sigma'_i = \mathring{\sigma}_i{}^{B'_i}$ $(i < \omega)$ exactly as in the proof of Chapter 4, Theorem 2.

QED(Sublemma 14.1)

By the strong revisability lemma we have: $q = \langle\langle M', B'\rangle, F^p\rangle \in \mathbb{P}$. Let $G \ni q$ be $\mathbb{P}$-generic. Then $\pi_q^G \cup F^q$ extends uniquely to: $\sigma^* : \overline{N}^* \prec N^*$. Set $\sigma = \sigma^* \cdot \sigma'$. It follows by a virtual repetition of previous proofs that σ has the desired properties.

QED(Lemma 14)

Now let $\mathbb{B}' = \text{BA}(\mathbb{P})$. We define a map $\mu : \bigcup_{i<\omega} \mathbb{B}_i \to \mathbb{B}'$ by:

$$\mu(b) = [[\check{b} \in \overset{\circ}{\mathbb{B}}]] \quad \text{where} \quad \overset{\circ}{\mathbb{B}}^G = B^G \quad \text{for all generic } G.$$

Then:

(1) μ is injective.

Proof. It suffices to show: $\mu(b) = 0 \to b = 0$. Let $b \neq 0$. Then $\mathcal{L} + \underline{b} \in \overset{\circ}{\mathbb{B}}$ is consistent by the proof that $\mathcal{L}$ is consistent. Hence there is $p \in \mathbb{P}$, $\overline{b} \in B_p$ s.t. $\pi^p(\overline{b}) = b$. Hence $p \vDash \check{b} \in \overset{\circ}{\mathbb{B}}$ – i.e. $p \in \mu(b)$. QED(1)

(2) $\mu \upharpoonright \mathbb{B}_i$ is a complete embedding.

Proof. $\mu\Big(\bigcap_{i\in X} b_i\Big) = \Big[\Big[\check{\bigcap_{i\in X}} b_i \in \overset{\circ}{\mathbb{B}}\Big]\Big]$ QED(2)

Hence we can take $\mathbb{B} \supset \bigcup_{i<\omega} \mathbb{B}_i$ s.t. for some k, $k : \mathbb{B}' \overset{\sim}{\leftrightarrow} \mathbb{B}$ and $k\mu = \text{id}$. $\mathbb{B}$ is then a limit of $\langle \mathbb{B}_i \mid i < \omega\rangle$ which collapses $\varrho = \omega_{\omega_1(\omega_1+1)}$ to ω_1 while making all regular $\tau \in (\omega_1, \varrho)$ become ω-cofinal. A proof like that of Lemma 7 shows that ϱ^+ is not collapsed, becoming the new ω_2. Hence we apply Example 2 at the next stage to collapse $\varrho^{(\omega_1)}$ = the ω_1-th successor of ϱ to ω_1. We continue in this fashion. We define an iteration $\langle \mathbb{B}_i \mid i \leq \kappa\rangle$ and a sequence $\langle \varrho_i \mid i \leq \kappa\rangle$ as follows: $\varrho_0 = \omega_1$, $\mathbb{B}_0 = 2$.

$\varrho_{i+1} = \varrho_i^{(\omega_1)}$ and $\mathbb{B}_{i+1}$ is constructed using Example 2 so as to collapse all regular $\tau \in (\omega_1, \varrho_{i+1})$ without collapsing ϱ_{i+1}^+. For limit λ we proceed as follows:

Case 1 λ has cofinality ω or has acquired cofinality ω at an earlier stage (i.e. $\text{cf}(\lambda) < \lambda \wedge \text{cf}(\lambda) \neq \omega_1$ in $\mathbf{V}$).

By essentially the above construction we form a limit $\mathbb{B}_\lambda$ which collapses $\varrho_\lambda = \Big(\sup_{i<\lambda} \varrho_i\Big)^{(\omega_1)}$ without collapsing ϱ^+.

Case 2 Case 1 fails.

We set $\varrho_\lambda = \sup_{i<\lambda} \varrho_i$ and let $\mathbb{B}_\lambda$ be the direct limit of $\langle \mathbb{B}_i \mid i < \lambda\rangle$. If $\text{cf}(\lambda) = \omega_1$ in $\mathbf{V}$, then ϱ_λ^+ becomes the new ω_2. Otherwise $\lambda = \varrho_\lambda$ is inaccessible. Using the fact that we took the direct limit stationarily often below λ it follows that $\mathbb{B}_\lambda$ satisfies the λ-chain condition. Hence λ is the new ω_2.

⋆ ⋆ ⋆ ⋆ ⋆

By induction on i we verify that $\mathbb{B}_i$ is subcomplete for $i \leq \kappa$, using Chapter 4, Theorem 4 for Case 2 above. We stress, however, that in order to carry out the induction we must also verify many other properties of the $\mathbb{B}_i$ which have not been dealt with here. These include some strong symmetry properties.

Given that GCH holds below κ, we can modify the above construction by making selective regular $\tau \in (\omega_1, \kappa)$ ω_1-cofinal. The set of such τ can be chosen arbitrarily in advance. Hence:

Theorem *Let κ be inaccessible. Let* GCH *hold below κ. Let $A \subset \kappa$. There is a set of conditions $\mathbb{P} \subset \mathbf{V}_\kappa$ s.t. whenever G is $\mathbb{P}$-generic, then in $\mathbf{V}[G]$ we have:*

- *κ is ω_2.*
- *If $\tau \in (\omega_1, \kappa)$ is regular in $\mathbf{V}$, then* $\mathrm{cf}(\tau) = \begin{cases} \omega_1 & \text{if } \tau \in A, \\ \omega & \text{if not.} \end{cases}$

Bibliography

[ASS] J. Barwise. *Admissible Sets and Structures*, Perspectives in Math. Logic Vol. 7, Springer Verlag, 1976.

[NA] H. Friedman, R. Jensen. *A Note on Admissible Ordinals*, in: The Syntax and Semantics of Infinitary Languages. Springer Lecture Notes in Math. Vol. 72, 1968.

[PR] R. Jensen, C. Karp. *Primitive Recursive Set Functions*, in Axiomatic Set Theory, AMS Proceedings of Symposia in Pure Math. Vol. XIII, Part 1, 1971.

[PF] S. Shelah. *Proper and Improper Forcing*, Perspectives in Math. Logic, Springer Verlag, 1998.

[AS] R. Jensen: *Admissible Sets**.

[LF] R. Jensen. *$\mathcal{L}$-Forcing**

[SPSC] R. Jensen. *Subproper and Subcomplete Forcing**

[ENP] R. Jensen. *The Extended Namba Problem**

[ITSC] R. Jensen. *Iteration Theorems for Subcomplete and Related Forcings**

[DSP] R. Jensen. *Dee-Subproper Forcing**

[FA] R. Jensen. *Forcing Axioms Compatible with* CH*

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