§ 3 The conclusion

Letting $\mathcal{I}$, N, b_0, b_1 etc. be as in § 1 we prove:

Thm 1 Let ρ be the least $\rho > 0$ s.t. $J_\rho(N)$ is admissible. Then δ is Woodin in $J_\rho(N)$.

The proof proceeds by contradiction, so assume the theorem to be false.

Lemma 2 One of b_0, b_1 is of type B.

prf.

Suppose not. Then in M_{b_h} $(h = 0, 1)$ there is $\nu > \delta$ s.t. $E_\nu \neq \emptyset$, by § 1 (44). Since $J_\nu^{E^{M_{b_h}}} \models ZFC^-$, it follows that $\gamma < \nu$. Hence $J_\rho(N) \subset M_{b_h}$. Hence each $B \in \mathcal{P}(N) \cap J_\rho(N)$ is captured at some n. Hence δ is Woodin in $J_\rho(N)$. Contr!

QED (Lemma 2)

Now let b_h ($h = 0$ or 1) be a branch of type B.
Since N is a ZFC model and:
$\bar\pi_{j b_h} : N_j \rightarrow_{\Sigma_0} N$ cofinally for $j \in b_h \setminus \xi_h$.
we conclude that $\bar\pi_{j b_h}$ is elementary
and N_j is a ZFC model. Now let:

γ_h = the maximal γ s.t. N_{ξ_h} is regular in $J_\gamma(N_{\xi_h})$
and $J_\gamma(N_{\xi_h})$ is not admissible
for any $\nu \in (0, \gamma)$.

Set $M_h = J_{\gamma_h}(N_{\xi_h})$. Let

- $\langle \mathfrak{M}_{b_h}, \pi^*_{\xi_h b_h} \rangle$ be a good liftup of $\langle M_{\xi_h}, \bar\pi_{\xi_h b_h} \rangle$
- $\langle \mathfrak{M}_j, \pi^*_{\xi_h j} \rangle$ " " " " " $\langle M_{\xi_h}, \bar\pi_{\xi_h j} \rangle$

for $j \in b_h \setminus \xi_h$. Let $\pi^*_{j b_h}$, π^*_{jk} ($j \le k$ in $b_h \setminus \xi_h$)
be the <u>bridging maps</u> defined in §2
s.t. $\pi^*_{j b_h} \circ \pi^*_{\xi_h j} = \pi^*_{\xi_h b_h}$ and $\pi^*_{jk} \cdot \pi^*_{\xi_h j} = \pi^*_{\xi_h k}$
for $j \le k$ in $b_h \setminus \xi_h$. We recall that

$$\pi^*_{k b_h} \circ \pi^*_{jk} = \pi^*_{j b_h} \quad \text{and}$$

$$\pi^*_{kl} \circ \pi^*_{jk} = \pi^*_{jl} \quad \text{for } j \le k \le l \text{ in } b_h \setminus \xi_h.$$

For b_h of type A we simply set:
$\mathfrak{M}_{b_h} = M_{b_h}$, $\mathfrak{M}_j = M_j$, $\pi^*_{j b_h} = \bar\pi_{j b_h}$,
and $\pi^*_{jk} = \bar\pi_{jk}$ for $j \le k$ in $b_h \setminus \xi_h$.

Then $\langle \mathfrak{M}_i \mid i \in b_h \setminus \xi_h \rangle$, $\langle \pi^*_{ii'} \mid i \leq i' \text{ in } b_h \setminus \xi_h \rangle$ is a commutive system whose limit is $\mathfrak{M}_{b_h}$, $\langle \pi^*_{i b_h} \mid i \in b_h \setminus \xi_h \rangle$.

From this we easily get:

<u>Lemma 3</u> Let $B \subset N$, $B \in \mathfrak{M}_{b_0} \cap \mathfrak{M}_{b_1}$.

Then B is strongly captured at some n.

prf.

We first note: $N \in \operatorname{rng} \pi^*_{\xi_h, b_h}$ and $\pi^*_{\xi_h b_h}(N_{\xi_h}) = N$ for $h = 0, 1$. (If b_h is of type A, this is immediate. Otherwise $M_h = J_{\gamma_0}(N_{\xi_h})$ where $\gamma_0 \geq 1$ since N_{ξ_h} is a ZFC model.

Now let $B \in \operatorname{rng}(\pi_{\xi_n, b_n}) \cap \operatorname{rng}(\pi_{\xi_{n+1}, b_{n+1}})$.

Then if $j \in b_p$, $j \in b_p \setminus \xi_p$ for $p \geq n$, we have

$$\pi_{j, b_p}(\langle N_j, B_j \rangle) = \langle N, B \rangle$$

where $B_j = \pi^{-1}_{j, b_p}(B) = \pi^{-1}_{j, b_p} {}'' B$.

QED (Lemma 3)

Corollary 3.1 Let δ be maximal s.t. N is regular in $J_\delta(N)$ and $J_\nu(N)$ is not admissible for $0<\nu<\delta$. Then δ is Woodin in $J_\delta(N)$.

prf.
We show: $J_\delta(N) \subset \mathfrak{N}_{b_h}$ $(h=0,1)$.
If b_h is of type A, then
$$J_\delta(N) \subset J_\rho(N) \subset \mathfrak{N}_{b_h} = M_{b_h},$$
where ρ is least s.t. $\rho>0$ and $J_\rho(N)$ is admissible.
If not, then $J_\delta(N)$ is a segment of $\mathfrak{N}_{b_h}$ by § 2 Theorem 2.

QED (Cor 3.1)

But $J_\delta(N)$ is then not admissible by our assumption. Hence:

Cor 3.2 N is not regular in $J_{\delta+1}(N)$.

Cor 3.3 b_h is of type B and $\mathfrak{N}_h = J_\delta(N)$ is well founded $(h=0,1)$.

proof.
Suppose not. If b_h is of type A, then $J_{\delta+1}(N) \subset J_\rho(N) \subset \mathfrak{N}_{b_h} = M_{b_h}$.

where $\rho > 0$ is least s.t. $J_\rho(N)$ is admissible. But N is regular in M_{b_h}, hence in $J_{\gamma+1}(N)$. Contradiction!

Thus b_h is of type B. If $\mathfrak{A}_{b_h}$ were ill founded, then, letting $\delta = \mathrm{On} \cap \mathrm{wfc}(\mathfrak{A}_{b_h})$, we have $J_{\gamma+1}(N) \subset J_\delta(N) \subset \mathfrak{A}_{b_h}$, since $J_\delta(N)$ is admissible. But N is regular in $\mathfrak{A}_{b_h}$, hence in $J_{\gamma+1}(N)$. Contr!

Thus $\mathfrak{A}_{b_h}$ is transitive. Call it $M^*_{b_h}$.

Let $M^*_{b_h} = J_{\gamma^*}(N)$. Then $\gamma \leq \gamma^*$. But then $\gamma = \gamma^*$, since otherwise $J_{\gamma+1}(N) \subset M^*_{b_h}$, where N is regular in $M^*_{b_h}$.

QED (Corollary 3.3)

Thus every $\mathfrak{A}_i$ is transitive.

Set: $N^* = \mathfrak{A}_{b_h} = J_\gamma(N)$ $(h=0,1)$

$N^*_i = \mathfrak{A}_i$ for $i \in b_h \setminus \xi_h$, $h = 0,1$.

By §2 Lemma 2.1 π^*_{jk} is the good liftup of $\overline{\pi}_{jk}$ and $\pi^*_{i b_h}$ is the good liftup of $\overline{\pi}_{i b_h}$ for $i \leq k$ in $b_h \setminus \xi_h$ $(h=0,1)$.

We now prove:

Cor 3.4 N^* is Σ_1-reflecting wrt. N.

prf.

Trivially N^* is grounded wrt. N and $N^* \models \bar{\theta}$ is the largest cardinal.

Thus it suffices to show:

Claim Let p be a grounding parameter for N^* wrt. N. Let $p \in \mathrm{rng}(\pi^*_{\xi_m, b_{i_m}}) \cap \mathrm{rng}(\pi^*_{\xi_{m+1}, b_{i_{m+1}}})$

Let $h_{N^*}(i, \langle \xi, p \rangle) < \theta$ where $\xi < \kappa_{i_m}$. Then $h_{N^*}(i, \langle \xi, p \rangle) < \kappa_{i_m}$.

proof

Let $\zeta = h_{N^*}(i, \langle \xi, p \rangle)$. Then $\zeta \in \mathrm{rng}(\pi^*_{j, b_m})$ whenever $m = n$ or $n+1$, $j \in b_m \setminus \xi_m$, since $\pi^*_{j, b_m} \restriction \kappa_{i_m} = \mathrm{id}$. Thus, setting $\zeta_j = \pi^{*\,-1}_{j, b_m}(\zeta)$, we have: $\{\zeta_j\} \subset N_j$ and $\bar{\pi}_{j b_m} : \langle N_j, \{\zeta_j\} \rangle \to_{\Sigma_0} \langle N, \{\zeta\} \rangle$. Thus $\{\zeta\} \subset N$ is captured at n.

By §1 Corollary 4 we conclude:

$\langle N | \kappa_{i_m}, \{\zeta\} \cap \kappa_{i_m} \rangle | \kappa_{i_m} \prec \langle N, \{\zeta\} \rangle$.

Hence $\zeta < \kappa_{i_m}$. QED (Cor 3.4)

But then every N_j^* is Σ_1-reflecting and all of the maps $\pi^*_{j\,b_h}$ $(j \in b_h \setminus \xi_h,\ h = n, n+1)$ are Σ_1-liftups by §2 Lemma 1.5.

Now let $B \subset N$ be $\Sigma_1(N^*)$ in q. Let $q \in \mathrm{rng}(\pi_{\xi_n, b_n}) \cap \mathrm{rng}(\pi_{\xi_{n+1}, b_{n+1}})$.

For $j \in b_m \setminus \xi_m$ $(m = n \text{ or } n+1)$ let B_j have the same Σ_1 definition in $q_j = (\pi^*_{j\,b_m})^{-1}(q)$. By §2 Lemma 1.3 (c) we then have:

- $\langle N, B \rangle$ is amenable
- $\pi^*_{j\,b_m} : \langle N_j, B_j \rangle \to_{\Sigma_0} \langle N, B \rangle$ cofinally.

Hence B is captured at n. Hence by §1 Cor 4. we conclude:

$$\langle N, B \rangle | \kappa_{i_n} \prec \langle N, B \rangle.$$

But κ_{i_n} is inaccessible in N. Hence $\langle N, B \rangle | \kappa_{i_n} = \langle J^{E^N}_{\kappa_{i_n}}, B \cap J^{E^N}_{\kappa_{i_n}} \rangle$ is a ZFC model. Hence so is $\langle N, B \rangle$.

By §1 Fact 3 we conclude:

N is definably regular in $N^* = J_\gamma(N)$.

Hence N is regular in $J_{\gamma+1}(N)$, contradicting the maximality of γ.

Contr! QED Thm 1

Finally we note:

<u>Thm 2</u> Let $M = J_\rho(N)$ where ρ is as in Thm 1. Then $\rho^1_M = \delta$.

proof

$\leq$ is given. We prove $\geq$. We must show that if B is $\Sigma_1(M)$, $B \subset N$, then $\langle N, B\rangle$ is amenable.

Suppose not. We derive a contradiction. b_h is of type B for $h = 0, 1$, since otherwise $\rho \in M_{b_h}$ and $N = V_\delta$ in M_{b_h}, where δ is a cardinal in M_{b_h}.

For $h = 0, 1$ we define $\xi_h, \mathfrak{M}_{b_h}, M_h$, $\pi^*_{\xi_h b_h} : M \to \mathfrak{M}_{b_h}$ and $\pi^*_{\xi_h j} : M \to \mathfrak{M}_j$ for $j \in b_h \setminus \xi_h$ exactly as before. The bridging map $\pi^*_{j b_h} : \mathfrak{M}_j \to \mathfrak{M}_{b_h}$ and $\pi^*_{jk} : \mathfrak{M}_j \to \mathfrak{M}_k$ ($j \leq k$ in b_h) are defined as before. Then $M \subset \mathfrak{M}_{b_0} \cap \mathfrak{M}_{b_1}$.

For $j \in b_h \setminus \xi_h$ set: $\rho_j = \pi^{*-1}_{j b_h}{}''\rho$, $M_j = J_{\rho_j}(N_j) = \pi^{*-1}_{j b_h}{}'' M$. Then M_j is transitive and

$\overline{\pi}_{jk} : M_j \to_{\Sigma_0} M_k$, $\overline{\pi}_k : M_k \to_{\Sigma_0} M$, where $\overline{\pi}_{ik} = \pi^*_{ik} \upharpoonright M_i$, $\overline{\pi}_h = \pi^*_{k b_h} \upharpoonright M_k$ for $i \leq k$ in b_h.

<u>Claim</u>: $\overline{\pi}_i$ is Σ_1 preserving (hence so is $\overline{\pi}_{ik}$ for $i \leq h$ in b_h)

proof.

<u>Case 1</u> $\mathfrak{M}_i = M_i$.

Let φ be Σ_1. Then

$$M \models \varphi(\overline{\pi}_i(x)) \rightarrow \mathfrak{M}_{b_h} \models \varphi(\overline{\pi}(x))$$
$$\rightarrow \underset{\substack{\shortparallel \\ M_i}}{\mathfrak{M}_i} \models \varphi(x)$$

since $\overline{\pi}_i = \pi^*_{i b_h}$ is Σ_1-preserving.

<u>Case 2</u> $\rho_i \in wfc(\mathfrak{M}_i)$.

Then

$$M \models \varphi(\overline{\pi}_i(x)) \rightarrow \mathfrak{M}_{b_h} | \pi(\rho_i) \models \varphi(\overline{\pi}(x))$$
$$\rightarrow M_i \models \varphi(x),$$

since $\pi_i : M_i \prec \mathfrak{M}_{b_h} | \pi(\rho_i)$.

<u>Case 3</u> The above fail.

Then $\rho_i = On \cap wfc(\mathfrak{M}_i)$. Let $\xi \in On_{\mathfrak{M}_i} \setminus \rho_i$. Then $\pi^*_{i b_h}(\xi) \in On_{\mathfrak{M}_{b_h}} \setminus \rho$.

Thus $M \models \varphi(\pi_i(x)) \to (\mathfrak{M}_{b_n} | \pi_{i,b_n}(\xi)) \models \varphi(\pi(x))$

$\to (\mathfrak{M}_i | \xi) \models \varphi(x)$, since

$\pi^*_{i,b_n} \upharpoonright (\mathfrak{M}_i | \xi) : (\mathfrak{M}_i | \xi) \prec (\mathfrak{M}_{b_n} | \pi^*_{i,b_n}(\xi))$.

But this holds for all $\xi \in On_{\mathfrak{M}_i} \setminus \rho_i$.

Thus, if $\nu =$ the $(\mathfrak{M}_i | \xi)$ least ν

s.t. $(\mathfrak{M}_i | \xi) \models \varphi(x)$, then $\nu < \rho_i$.

Hence $\mathfrak{M}_i | \nu = M_i | \nu$ and $M_i \models \varphi(x)$.

QED (Claim)

Now let B be $\Sigma_1(M)$ in p where (w.l.o.g.) p is a grounding parameter for M. Let $p \in \mathrm{rng}(\pi_{i_m b_m}) \cap \mathrm{rng}(\pi_{i_{m+1}, b_{m+1}})$.

It follows exactly as before that if $\xi = h_M(i, \langle p, \mu\rangle) < \delta$, where $\mu < \kappa_{i_m}$, then $\xi < \kappa_{i_m}$. Set:

$X = h_M(\kappa_{i_m} \cup \{p\})$, $\sigma : \bar{M} \xrightarrow{\sim} X$.

Then $\kappa_{i_m} = \mathrm{crit}(\sigma)$, $\sigma(\kappa_{i_m}) = \delta$,

Then $\sigma : \bar{M} \to_{\Sigma_1} M$. Hence, if $\bar{B}$ is Σ_1 over $\bar{M}$ in $\bar{p} = \sigma^{-1}(p)$ by the same def. as B over M in p, then $B \cap (N|\varkappa_{i_m}) = \bar{B} \in N$, since $\bar{M} \in N$ and $\sigma(N|\varkappa_{i_m}) = N$. QED (Thm 2)