§3 The successor step

We are given $\langle \mathbb{B}_i \mid i \leq \mu \rangle$ satisfying (a)–(h) and wish to construct $\mathbb{B}_{\mu+1}$ s.t. (a)–(h) continue to hold. Let $\gamma = \gamma_{\mu+1}$. Then $\gamma = \omega_2^{V^{\mathbb{B}_\mu}}$, $2^{\gamma} = \gamma$, and $\gamma = \beta_\mu^+$ if β_μ exists. Otherwise $\gamma = \mu = \gamma_\mu$ is inaccessible in V. In either case we know that $\mathbb{B}_\mu \subset H_\gamma$.

§3.1 The first successor case

Suppose that $\gamma \in A_0$. Then GCH holds below κ and we wish to make γ ω_1-cofinal without collapsing $\beta_{\mu+1} = \gamma^+$. Thus we force with $\mathrm{coll}(\omega_1, \omega_2)$ over $V[G]$, where G is $\mathbb{B}_\mu$-generic. In other words we let $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$, where $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = \mathrm{coll}(\omega_1, \omega_2)$.

We verify (a)–(h). (a) is straightforward. (b) holds, since $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}}$ is subcomplete. (c) is immediate, since $\mu+1$ is not a limit point and (a)–(h) hold of $\langle \mathbb{B}_i \mid i \leq \mu \rangle$. Similarly for (d). (e) follows by §2 Lemma 4.1. (f) follows by §2 Lemma 4.2. We now sketch the proof of (f). Let B be $\mathbb{B}_h$-generic, where $h \leq \mu$.

Set $\tilde{\mathbb{B}}_i = \mathbb{B}_i / B$ for $h \le i \le \mu+1$. (Thus $\langle \tilde{\mathbb{B}}_{h+j} \mid j \le \mu+1-h \rangle$ is the new iteration in $V[B]$.) We first note that if B' is $\mathbb{B}_\mu$ - generic over V, then

$$\mathbb{B}_{\mu+1} / B' \simeq \mathring{\mathbb{B}}^{B'} = BA(\mathrm{coll}(\omega_1, \omega_2))$$

in $V[B']$. (Recall that $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$ where $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathrm{coll}(\omega_1, \omega_2))$.)

Now let $\tilde{B}$ be $\tilde{\mathbb{B}}_\mu$ - generic over $V[B]$. Then $\tilde{\mathbb{B}}_{\mu+1} / \tilde{B} = (\mathbb{B}_{\mu+1} / B) / \tilde{B} \simeq \mathbb{B}_{\mu+1} / B'$, where $B' = B * \tilde{B} = \{ b \in \mathbb{B}_\mu \mid b/B \in \tilde{B} \}$ is $\mathbb{B}_\mu$ - generic over V. Hence there is $\mathring{\sigma} \in V[B]^{\tilde{\mathbb{B}}_\mu}$ s.t.

$$\Vdash_{\tilde{\mathbb{B}}_\mu} \mathring{\sigma} : \tilde{\mathbb{B}}_{\mu+1} / \mathring{B} \overset{\sim}{\longleftrightarrow} BA(\mathrm{coll}(\omega_1, \omega_2))$$

in $V[B]$, ($\mathring{B}$ being the canonical generic name). Let

$$\Vdash_{\tilde{\mathbb{B}}_\mu} \mathring{\mathring{\mathbb{B}}} = BA(\mathrm{coll}(\omega_1, \omega_2))$$

Define $\sigma : \tilde{\mathbb{B}}_{\mu+1} \overset{\sim}{\longleftrightarrow} \tilde{\mathbb{B}}_\mu * \mathring{\mathring{\mathbb{B}}}$ in $V[B]$ by $\sigma(a) =$ that $\acute{a}'$ s.t. $\Vdash_{\tilde{\mathbb{B}}_\mu} a' = \mathring{\sigma}(\check{a}/\mathring{B})$

Then for $b \in \tilde{\mathbb{B}}_\mu$ we have:

$$\sigma(b) = b' \text{ where } \Vdash_{\tilde{\mathbb{B}}_\mu} b' = \begin{cases} 1 & \text{if } \check{b} \in \mathring{B} \\ 0 & \text{if } \check{b} \in \mathring{B} \end{cases}$$

- i.e $\sigma \upharpoonright \tilde{\mathbb{B}}_\mu$ is the natural injection.

Thus $\tilde{\mathbb{B}}_{\mu+1}$ satisfies precisely those conditions which we had placed upon $\mathbb{B}_{\mu+1}$ in V. Thus we can carry out all of our proofs in $V[B]$ with $\langle \tilde{\mathbb{B}}_{h+j} \mid j \leq \mu+1-h \rangle$ in place of $\langle \mathbb{B}_i \mid i \leq \mu+1 \rangle$.

Finally, we note that, since GCH holds below κ, elementary consideration give us: $\mathbb{B}_\mu * \mathring{\mathbb{B}}$ has cardinality $\leq \gamma^+$. Hence we can choose $\mathbb{B}_{\mu+1}$ s.t. $\mathbb{B}_{\mu+1} \subset H_{\gamma^+}$.

This completes the first successor case. The second will be much harder, since we shall need $\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \mathring{\mathbb{B}}$ for a $\mathring{\mathbb{B}}$ which has yet to be defined.

The second successor case

Now suppose $\gamma \notin A_0$, where $\gamma = \gamma_{\mu+1} = \omega_2^{V^{\mathbb{B}_\mu}}$. Then γ must acquire cofinality ω at the next stage. But then all regular cardinals in $[\gamma, \beta_{\mu+1})$ must become ω-cofinal. Recall that $\beta_{\mu+1}$ is the least β' s.t. either $cf(\beta') = \omega_1$ and $2^{\beta'} = \beta'$, or $\beta' \in A_0$ is regular. In the latter case $\beta' = \beta^+$ is a successor cardinal, $\beta \geq \gamma$, and $2^{\beta} = \beta$, since GCH holds below κ if $A \neq \emptyset$. From now on let β be defined by:

$$\beta = \beta_{\mu+1} \quad \text{if } cf(\beta_{\mu+1}) = \omega_1$$

$$\beta^+ = \beta_{\mu+1} \quad \text{if not,}$$

where β is a cardinal.

Now let B be $\mathbb{B}_\mu$-generic. We work in $V[B]$ to define a set of conditions $\mathbb{P} = \mathbb{P}_B$ which collapses all regular $\tau \in [\gamma, \beta]$ to ω. If $cf(\beta) = \omega_1$, then β^+ becomes ω_2 in $V[B]^{\mathbb{P}}$. Otherwise β^{++} becomes ω_2. We then take:

$\mathbb{B}_{\mu+1} \simeq \mathbb{B}_\mu * \dot{\mathbb{B}}$ where $\Vdash_{\mathbb{B}_\mu} \dot{\mathbb{B}} = BA(\mathbb{P}_{\dot{B}})$, $\dot{B}$ being the canonical generic name.

*/

Let $A \in V$ s.t. $A \subset H_\eta$ and $H_\eta^V = L_\eta^A$ whenever $\eta \leq \beta$ s.t. $2^\xi = \eta$. Set:

<u>Def.</u> $M = L_\beta^{A, \mathbb{B}_\mu} =_{Df} \langle L_\beta[A, \mathbb{B}_\mu], A, \mathbb{B}_\mu \rangle$.

$N = \langle H_{\beta^+}^V, M, <, m \rangle$

where $<$ is a well ordering of N.

$Q = H_\gamma^V$.

<u>Def</u> Working in $V[B]$ where B is $\mathbb{B}_\mu$-generic set:

$M^B = L_\beta^{A, \mathbb{B}_\mu, B}$, $N^B = \langle H_{\beta^+}^{V[B]}, M^B, < \rangle$

(<u>Note</u> M^B has the same sets as $H_\beta^{V[B]}$)

$Q^B = Q[B] =_{Df} \bigcup_{x \in Q} L_\gamma[x, B]$.

(<u>Note</u> $Q^B = H_\gamma^{V[B]}$ since $\gamma = \omega_2^{V[B]}$ is regular in $V[B]$. Hence if B, B' are $\mathbb{B}_\mu$-generic and $V[B] = V[B']$, then $Q^B = Q^{B'}$.)

Working in $V[B]$ we now define:

Γ_* = the collection of $\langle S, C \rangle$ s.t.

- S is a transitive set
- $S \models (ZFC^- + \omega_1$ is the largest cardinal)
- $C < S$ cofinally
- C is countable

(Recall that "$C \subset S$ cofinally" means $\bigcup C = S$.)

Def For $u = \langle s_u, C_u \rangle$, $v = \langle s_v, C_v \rangle \in \Gamma_*$ set:

$$\pi : u \lhd_* v \underset{Df}{\longleftrightarrow} (\pi : s_u \prec s_v \wedge \pi'' C_u = C_v)$$

Def $u \lhd_* v \underset{Df}{\longleftrightarrow} \bigvee \pi \;\; \pi : u \lhd_* v$

Def For $u = \langle s_u, C_u \rangle \in \Gamma_*$ set: $\alpha_u = \alpha_{s_u} = \omega_1^{s_u}$.

The following facts are readily verified and will be stated here without proof:

Fact 1 Let $\langle s, C \rangle \in \Gamma_*$, $\alpha = \alpha_s$. Then
$s = \{ f(\nu) \mid f \in C \wedge \nu < \alpha \}$

Fact 2 If $\pi : u \lhd_* v$, then $\alpha_u \leq \alpha_v$ and
$\mathrm{rng}(\pi) = \{ f(\nu) \mid f \in C_v \wedge \nu < \alpha_u \}$

Hence:

Fact 3 For any $\alpha \leq \alpha_v$ there is at most one pair $\langle u, \pi \rangle$ s.t.
$\pi : u \lhd_* v$ and $\alpha_u = \alpha$.

Hence:

Fact 4 Let $u \lhd_* v$. There is exactly one π s.t. $\pi : u \lhd_* v$.

Def $\pi_{uv} \simeq_{Df}$ that π s.t. $\pi : u \lhd_* v$.

Fact 5 $\langle \pi_{uv} \mid u \triangleleft_* v \rangle$ is a continuous commutative system.

Note "continuous" means that if $u_i \triangleleft_* u_j \triangleleft v$ for $i \le j < \lambda$, then the transitivized direct limit $\langle u, \langle \pi_{u_i u} \mid i < \lambda \rangle \rangle$ of $\langle \langle u_i \mid i < \lambda \rangle, \langle \pi_{u_i u_j} \mid i \le j < \lambda \rangle \rangle$ exists and There is $\pi : u \triangleleft_* v$ defined by

$$\pi \pi_{u_i u} = \pi_{u_i v} \quad (i < \lambda).$$

Hence:

Fact 6 $\{d_u \mid u \triangleleft_* v \wedge u \neq v\}$ is closed in d_v.

We now define:

Def R is a smooth model iff

- $R = L_\beta^{\vec{A}}$ for some $A_1, \ldots, A_n, \beta$
- R models ZFC^- or Zermelo set theory
- $L_\eta[\vec{A}] = H_\eta^R$ whenever $\aleph_\xi = \eta$ in R.

Def Γ = the set of $\langle R, C \rangle$ s.t.

- R is a smooth model
- $\langle Q, C \rangle \in \Gamma_*$ where $Q = H_{\omega_2}^R$.

We also write $Q_R = Q_{\langle R, C \rangle} = H_{\omega_2}^R$.

Def Let $u = \langle R_u, C_u \rangle$, $v = \langle R_v, C_v \rangle \in \Gamma$.

$\pi : u \lhd v$ iff

- $\pi : R_u \prec R_v$
- $\pi \restriction Q_u : \langle Q_u, C_u \rangle \lhd_* \langle Q_v, C_v \rangle$
- There is R_{uv} s.t. $\langle R_{uv}, \pi \rangle$ is the liftup of $\langle R_u, \pi \restriction Q_u \rangle$

(Hence the map π is wholly determined by $\pi \restriction Q_u$.)

Def $u \lhd v$ iff $\bigvee \pi \; \pi : u \lhd v$.

It follows easily that:

Fact 7 Let $u \lhd v$. There is exactly one π s.t. $\pi : u \lhd v$.

Def $\pi_{uv} \simeq$ that π s.t. $\pi : u \lhd v$

Fact 8 $\langle \pi_{uv} \mid u \lhd v \rangle$ is a continuous commutative system.

However, the analogue of Fact 3 does not hold for Γ, since $\{u \mid u \lhd v\}$ need not be linearly ordered by $\lhd$.

Nonetheless we do have:

<u>Fact 9</u> Let $u, w \triangleleft v$, $\mathrm{rng}(u,v) \subset \mathrm{rng}(w,v)$. Then $u \triangleleft w$ and $\pi_{wv}\pi_{uw} = \pi_{uv}$.

The following fact will often be used tacitly:

<u>Fact 10</u> Let $\pi : \langle R, C\rangle \triangleleft \langle R', C'\rangle$. Let $\gamma \in R$ s.t. $2^{\tau} = \gamma$ in R and $\gamma \geq \omega_2^R$. Let $\pi(\gamma) = \gamma'$, $R = L_\beta^{\vec{A}}$, $R' = L_{\beta'}^{\vec{A}'}$. Set: $\bar{R} = L_\gamma^{\vec{A}}$, $\bar{R}' = L_{\gamma'}^{\vec{A}'}$, $\bar{\pi} = \pi \restriction \bar{R}$,

Then $\bar{\pi} : \langle \bar{R}, C\rangle \triangleleft \langle \bar{R}', C\rangle$.

proof.

Clearly $\bar{\pi} : \bar{R} \prec \bar{R}'$ and $\bar{R}$ models ZFC^- or Zermelo. Moreover $\bar{\pi}''C = C'$ and $Q_{\bar{R}} = Q_R$, $Q_{\bar{R}'} = Q_{R'}$. Hence $\bar{\pi} \restriction Q_{\bar{R}} : \langle Q_{\bar{R}}, C\rangle \triangleleft_* \langle Q_{\bar{R}'}, C'\rangle$.

<u>Claim</u> Let $\bar{\pi} : \bar{R} \to \tilde{R}$ cofinally. Then $\langle \tilde{R}, \bar{\pi}\rangle$ is the liftup of $\langle \bar{R}, \pi \restriction Q_R\rangle$.

prf.

We must show that $\bar{\pi} : \bar{R} \to \tilde{R}$ is $\omega_2^{\bar{R}}$-cofinal. Let $x \in \tilde{R}$. Then $x \in \pi(w)$ where $w \in \bar{R}$, $\bar{\bar{w}} < \omega_2$ in $\bar{R}$. Let $x \in L_{\pi(\nu)}^{\vec{A}'}$, where $\nu \in \bar{R}$. Set $z = w \cap L_\nu^{\vec{A}}$. Then $z \in \bar{R}$, $\bar{\bar{z}} < \omega_2$ in $\bar{R}$ and $x \in \bar{\pi}(z) = \pi(w) \cap L_{\pi(\nu)}^{\vec{A}'}$.

QED (Fact 10)

We return now to Q^B, M^B, N^B as defined above. We shall use an infinitary language $\mathcal{L}_B$ on N^B to define an $\mathcal{L}$-forcing $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$ in $V[B]$. $\mathbb{P}_B$ is intended to add a C s.t. $\langle Q^B, C\rangle \in \Gamma_*$ without adding new reals. However, C should make not only γ ω-cofinal, but every regular $\tau \leq \beta$.

$\mathcal{L} = \mathcal{L}_B$ is the infinitary language on N^B with:

<u>Predicate</u> $\in$; <u>Constants</u> $\underline{x}$ $(x \in N^B)$, $\mathring{c}$

<u>Axioms</u> : ZFC^-, $\bigwedge v\, (v \in \underline{x} \leftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in N^B$, $H_{\omega_1} = \underline{H_{\omega_1}}$, $\mathring{c} \prec \underline{M^B}$, and

$(*)$ $\bigwedge x \in \underline{M}^B \; \bigvee u \in \underline{H_{\omega_1}} \; \bigvee \pi\, (\pi : u \lhd \langle \underline{M}^B, \mathring{c}\rangle \wedge$
$\wedge\; x \in \mathrm{rng}(\pi) \wedge \Psi(\pi))$ $\underset{\smile}{*}$

(This says, in particular, that every $x \in \underline{M}^B$ can be found in the liftup of a countable $\bar{M}$ by a π' s.t. $\pi' : \langle Q_{\bar{M}}, \bar{C}\rangle \lhd_* \langle M^B, \mathring{c}\rangle$.)

$\underset{\smile}{*}$ where $\Psi(\pi) = \begin{cases} \bigcup \mathrm{rng}(\pi) = \underline{M}^B & \text{if } \beta \text{ is regular} \\ \pi = \pi & \text{if not} \end{cases}$

Lemma 1 $\mathcal{L}$ is consistent.

proof.

Let $\sigma : \bar{N} \prec N^B$ where $\bar{N}$ is countable and transitive. Set $\bar{Q} = H_{\omega_2}^{\bar{N}}$. Let $\sigma : \bar{Q} \prec \tilde{Q}$ cofinally. Let $\langle \tilde{N}, \tilde{\sigma} \rangle$ be the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$. Let $k : \tilde{N} \prec N^B$ s.t. $k\tilde{\sigma} = \sigma$ and $k \upharpoonright \tilde{Q} = \mathrm{id}$.

Let $\tilde{\mathcal{L}}$ be defined on $\tilde{N}$ like $\mathcal{L}$ on N.

It suffices to show:

Claim $\tilde{\mathcal{L}}$ is consistent,

since this is a $\Pi_1(\tilde{N})$ statement.

We show that $\langle H_{\omega_2}, \tilde{C} \rangle$ models $\tilde{\mathcal{L}}$, where $\tilde{C} = \mathrm{rng}(\sigma \upharpoonright \bar{Q})$. (Note that $\tilde{N} \in H_{\omega_2}$.) All axioms other than (*) are trivial. We verify (*). Set:

$D =$ the set of $\alpha < \omega_1$ s.t. there is a $u_\alpha = \langle Q_\alpha, C_\alpha \rangle$ with $\langle Q_\alpha, C_\alpha \rangle \triangleleft_* \langle \tilde{Q}, \tilde{C} \rangle$

Then D is club in ω_1 and $\alpha_0 = \omega_1^{\bar{Q}}$ is minimal in D with $u_{\alpha_0} = \langle \bar{Q}, \bar{Q} \rangle$.

Since $\langle \tilde{N}, \tilde{\sigma} \rangle$ is the liftup of $\langle \bar{N}, \sigma \upharpoonright \bar{Q} \rangle$ and $\sigma \upharpoonright \bar{Q} = \pi_{u_{\alpha_0}, \langle \tilde{Q}, \tilde{C} \rangle}$, we see that $\langle \bar{N}, \pi_{u_{\alpha_0} u_\alpha} \rangle$ has a transitive liftup $\langle N_\alpha, \tilde{\sigma}_{\alpha_0 \alpha} \rangle$. Moreover, there is a map $\tilde{\sigma}_{\alpha, \omega_1} : N_\alpha \prec \tilde{N}$ defined by $\tilde{\sigma}_{\alpha, \omega_1}(\tilde{\sigma}_{\alpha_0 \alpha}(f)(\nu)) = \tilde{\sigma}(f)(\nu)$, where $\nu < \alpha$ and $f \in \bar{N}$ s.t. $f : \omega_1 \to \bar{N}$.

Set: $\tilde{\sigma}_{\alpha\beta} = (\tilde{\sigma}_{\beta, \omega_1})^{-1} \tilde{\sigma}_{\alpha \omega_1}$ for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \{\omega_1\}$. It is clear by these definitions that:

$\langle N_\beta, \tilde{\sigma}_{\alpha\beta} \rangle =$ the liftup of $\langle N_\alpha, \pi_{u_\alpha u_\beta} \rangle$

for $\alpha \leq \beta$, $\alpha, \beta \in D \cup \{\omega_1\}$ (with $u_{\omega_1} = \langle \tilde{Q}, \tilde{C} \rangle$).

Now set: $\bar{M} = \sigma^{-1}(M)$, $M_\alpha = \tilde{\sigma}_{\alpha_0 \alpha}(\bar{M})$ for $\alpha \in D \cup \{\omega_1\}$. Then $M_{\omega_1} = \tilde{M} = \tilde{\sigma}(\bar{M})$.

Set $\sigma'_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} \restriction M_\alpha$ $(\alpha \leq \beta,\ \alpha, \beta \in D \cup \{\omega_1\})$.

Clearly $\tilde{M} = \bigcup_{\alpha \in D} \operatorname{rng}(\sigma'_{\alpha, \omega_1})$, so it suffices to show:

<u>Claim</u> $\sigma'_{\alpha, \omega_1} : \langle M_\alpha, C_\alpha \rangle \lhd \langle \tilde{M}, \tilde{C} \rangle$.

Clearly $\sigma'_{\alpha, \omega_1} : M_\alpha \prec \tilde{M}$ and $\sigma'_{\alpha\omega_1}{}'' C_\alpha = \tilde{C}$. Now let:

$\sigma'_{\alpha, \omega_1} : M_\alpha \longrightarrow \tilde{M}_\alpha$ cofinally, (Then $\tilde{M}_\alpha = \tilde{M}$ if β is regular.)

But $\sigma'_{\alpha,\omega_1}$ is $\omega_2^{M_\alpha}$ - cofinal, since $\tilde{\sigma}_{\alpha,\omega_1} : N_\alpha \to \tilde{N}$ is $\omega_2^{N_\alpha}$ - cofinal. (To see this let $x \in \tilde{M}_\alpha$. Then $x \in \tilde{\sigma}_{\alpha,\omega_1}(a)$ where $a \in N_\alpha$, $\bar{\bar{a}} < \omega_2$ in N_α. Since $\sigma'_{\alpha,\omega_1} : M_\alpha \to \tilde{M}_\alpha$ cofinally, there is $b \in M_\alpha$ s.t. $x \in \sigma'_{\alpha,\omega_1}(b)$. But then $a \cap b \in M_\alpha$ + $x \in \sigma'_{\alpha,\omega_1}(a \cap b)$, where $\overline{\overline{a \cap b}} < \omega_2$ in M_α. QED (Lemma 1)

We shall make heavy use of the following lemma:

<u>Lemma 2</u> Let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $\langle A_n \mid n < \omega \rangle \in \mathfrak{A}$ s.t. $A_n \subset M^B$ for $n < \omega$. Then there is $u = \langle s, c \rangle \in \Gamma \cap H_{\omega_1}$ and $\pi \in \mathfrak{A}$ s.t.

- $\pi : \langle s, c \rangle \lhd \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$
- $\pi : \langle s, \bar{A}_n \rangle \prec \langle M^B, A_n \rangle$ for $n < \omega$

where $\bar{A}_n =_{df} \pi^{-1} {}'' A_n$.

proof of Lemma 2

Set $M^* = \langle M^B, A_1, A_2, \dots \rangle$. Then $M^* \in \mathfrak{A}$.

Working in $\mathfrak{A}$ we successively pick

$X_i \prec M^*$, $\pi_i : u_i \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ s.t. $u_i \in H_{\omega_1}$

as follows: Let $<$ wellorder H_{ω_1}.

$X_0 =$ the smallest $X \prec M^*$

Let $\langle x^j_n \mid n < \omega \rangle$ enumerate X_j.

$u_i =$ the $<$ - least $u \in H_{\omega_1}$ s.t.

$u \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ and $x^j_n \in \mathrm{rng}(\pi_i)$

for $j, n < i$, where $\pi_i = \pi_{u, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$.

$X_{i+1} =$ the smallest $X \prec M^*$ s.t.

$X_i \cup \mathrm{rng}(\pi_i) \subset X$.

Let $X = \bigcup_i X_i$. Then $X = \bigcup_i (\pi_i)$

Let $\pi^* : \overline{M}^* \overset{\sim}{\longleftrightarrow} X$ where $\overline{M}^*$ is transitive.

Hence $\overline{M}^* \in H_{\omega_1}$. Let $\overline{M}^* = \langle \overline{M}, \overline{A}_1, \overline{A}_2, \dots \rangle$.

It suffices to show:

<u>Claim</u> $\pi : \langle \overline{M}, \overline{C} \rangle \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$,

where $\pi = \pi^* \upharpoonright \overline{M}$, $\overline{C} = \pi^{-1} {}'' \dot{c}^{\mathfrak{A}}$.

proof

Clearly $\dot{c}^{\mathfrak{A}} \subset \mathrm{rng}(\pi_0) \subset X$. Hence

$\pi \upharpoonright \overline{Q} : \langle \overline{Q}, \overline{C} \rangle \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$ where $\pi^*(\overline{Q}) = Q$.

Now let $\pi : \overline{M} \to \tilde{M}$ cofinally.

It suffices to show:

<u>Claim</u> $\langle \tilde{M}, \tilde{\pi} \rangle$ is the liftup of $\langle \overline{M}, \pi \upharpoonright \overline{Q} \rangle$.

pf.
We must show that $\pi : \overline{M} \to \tilde{M}$ is $\omega_2^{\overline{M}}$–cofinal. Let $x \in \tilde{M}$. Then

$x \in \pi_i(a_i)$ for an $i < \omega$, where $\overline{\overline{a}}_i < \omega_1$ in M_{a_i}, since $\tilde{M} = \bigcup X$ and $X = \bigcup_i rng(\pi_i)$.
Hence $a = \pi_i(a_i) \in X$ and $\overline{\overline{a}} \leq \omega_1$ in M.
Hence $x \in a = \pi(\overline{a})$ for an $\overline{a} \in \overline{M}$ s.t. card$(\overline{a}) \leq \omega_1$ in $\overline{M}$. QED (Lemma 2)

" " " " "

We are now ready to define the set of conditions $\mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B}$.

We first set:

<u>Def</u> $\tilde{\mathbb{P}}$ = the set of $p = \langle p_0, p_1 \rangle$ s.t.

$p_0 = \langle M_p, C^p \rangle \in \Gamma \wedge H_{\omega_1}$

$p_1 = F^p$ is an at most countable set of pairs $\langle a, \overline{a} \rangle$ s.t. $\overline{a} \subset M_p$, $a \subset M^B$.

<u>Def</u> For $p \in \tilde{\mathbb{P}}$ let φ_p be the conjunction of:

- $p_0 \lhd \langle \underline{M}^B, \mathring{c} \rangle$ (Let $\pi_p =_{Df} \pi_{p_0, \langle \underline{M}^B, \mathring{c} \rangle}$)
- $\pi_p : \langle \underline{m}_p, \underline{a} \rangle \prec \langle \underline{M}^B, \underline{a} \rangle$ for all $\langle a, \overline{a} \rangle \in F^p$
- $\pi_p : \underline{M}_p \to \underline{M}^B$ cofinally if β is regular

<u>Def</u> $\mathcal{L}(p) = \mathcal{L} + \varphi_p$.

<u>Def</u> $R^p = \mathrm{rng}(F^p)$, $D^p = \mathrm{dom}(F^p)$

<u>Def</u> $\mathbb{P} = \mathbb{P}_B = \mathbb{P}_{\mathcal{L}_B} =_{Df} \{p \in \tilde{\mathbb{P}} \mid \mathcal{L}(p) \text{ is consistent}\}$

For $p, q \in \mathbb{P}$ set:

$p \leq q$ iff the following hold:

- $R^q \subset R^p$
- $q_0 \lhd p_0$
- $\pi_{q_0 p_0} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$ whenever $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

<u>Lemma 3.1</u> Let $p, q \in \mathbb{P}$. Then $p \leq q$ iff

- $R^q \subset R^p$
- $\mathcal{L}(p) \vdash (\mathcal{L}(q) \wedge \mathrm{rng}(\pi_q) \subset \mathrm{rng}(\pi_p))$.

prf.

($\rightarrow$) Let $p \leq q$. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. It follows easily that $\mathrm{rng}(\pi_q^{\mathfrak{A}}) \subset \mathrm{rng}(\pi_p^{\mathfrak{A}})$ and $\mathfrak{A} \models \mathcal{L}(q)$.

($\leftarrow$) Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Then $\mathrm{rng}(\pi_q^{\mathfrak{A}}) \subset \mathrm{rng}(\pi_p^{\mathfrak{A}})$. Hence by Fact 9, $q_0 \lhd p_0$ and $\pi_q^{\mathfrak{A}} = \pi_p^{\mathfrak{A}} \circ \pi_{q_0 p_0}$.

Since $\mathfrak{A} \models \mathcal{L}(q)$, we then have:

$$\pi_{q_0 p_0} = (\pi_p^{\mathfrak{A}})^{-1} \pi_q^{\mathfrak{A}} : \langle M_q, \bar{a} \rangle \prec \langle M_p, a' \rangle$$

for $\langle a, \bar{a} \rangle \in F^q$, $\langle a, a' \rangle \in F^p$.

QED (3.1)

We set: $\pi_{qp} =_{Df} \pi_{q_0 p_0}$ if $p \leq q$.

Exactly as in [LF] §0.1 – §0.3 we prove:

Lemma 3.2 Let $p \in \mathbb{P}$. Then

- $(F^p)^{-1}$ is a function
- If R^p is closed under set difference, then $F^p : D^p \leftrightarrow R^p$
- $\pi^p =_{Df} F^p \upharpoonright M_p$ is injective into M^B.

The following lemma expresses a strong form of "revisability" in the sense of [LF].

Lemma 3.3 Let $p \in \mathbb{P}$. Let $c \prec M_p$ cofinally. Then $p' \in \mathbb{P}$ where:

$p'_0 = \langle M_p, c \rangle$, $p'_1 = p_1$.

proof

Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$. Form $\mathfrak{A}'$ by replacing $\dot{c}^{\mathfrak{A}}$ with $c' = \pi_p^{\mathfrak{A}}{}'' c$.

Claim $\mathfrak{A}' \models \mathcal{L}(p')$

We first show: $\mathfrak{A} \models \mathcal{L}$.

Note that if

$u = \langle Q_u, C_u \rangle \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$ and $\alpha_u \geq \alpha_p$,

then $\langle Q_p, C_p \rangle \triangleleft_* u \triangleleft_* \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle$

and $\operatorname{rng}(\pi_{u, \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle}) \supset \operatorname{rng}(\pi_p^{\mathfrak{A}} \upharpoonright Q_p)$

(where, of course, $Q_p = H_{\omega_2}^{M_p}$.)

Set: $u' = \langle Q_u, C_{u'} \rangle$ where $C_{u'} = \pi^{-1}_{u,\langle Q^B, \dot{c}^{\mathfrak{A}} \rangle}{}''c'$ $= \pi_{\langle Q_p, C_p \rangle, u}{}''c$. Then $u' \triangleleft_\gamma \langle Q, c' \rangle$ and $\pi_{u', \langle Q^B, c' \rangle} = \pi_{u, \langle Q^B, \dot{c}^{\mathfrak{A}} \rangle}$, as is easily seen. But this means that if $v = \langle S_v, C_v \rangle \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ with $\alpha_v \geq \alpha_p$, then $v' = \langle S_v, C_{v'} \rangle \triangleleft \langle M^B, c' \rangle$ where $C_{v'} = \pi_{\langle Q_p, C_p \rangle, \langle Q_v, C_v \rangle}{}''c$,

with $\pi_{v', \langle M^B, c' \rangle} = \pi_{v, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$, since $\pi = \pi_{v', \langle M^B, c' \rangle}$ is uniquely determined by $\pi \restriction Q_v$. Thus (*) continues to hold in $\mathfrak{A}'$. The other axioms are trivial.

Our argument shows, in particular, that $\pi_p^{\mathfrak{A}} = \pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle}$.

Hence $\pi_{\langle M_p, c \rangle, \langle M^B, c' \rangle} : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^p = F^{p'}$.

Thus $\mathfrak{A} \models \mathcal{L}(p')$. QED (3.3)

We now prove the main lemma on extendability of conditions.

<u>Lemma 3.4</u> $\mathbb{P} \neq \emptyset$. Moreover, if $p, q \in \mathbb{P}$ and $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent, then there is an r s.t. $r \leq p, q$. If $R \subset \mathcal{P}(M^B)$ is any countable set we may, in fact, choose r s.t. $R \subset R^r$.

proof

To see $\mathbb{P} \neq \emptyset$, let $\mathfrak{A}$ be a solid model of $\mathcal{L}$. Let $u \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$, $u \in H_{\omega_1}$. Then $p \in \mathbb{P}$ where $p_0 = u$, $p_1 = \emptyset$.

Now let $\mathfrak{A} \models \mathcal{L}(p) \cup \mathcal{L}(q)$. Set:

$X = \text{rng}(\pi_p^{\mathfrak{A}}) \cup \text{rng}(\pi_q^{\mathfrak{A}}) \cup F^p \cup F^q \cup R$,

Then $X \in \mathfrak{A}$ is countable in $\mathfrak{A}$ with $X \subset \mathcal{P}(M)$. By Lemma 2 there is $\langle \bar{M}, \bar{c} \rangle \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ s.t. $\langle \bar{M}, \bar{c} \rangle \in H_{\omega_1}$ and $\pi : \langle \bar{M}, \bar{A} \rangle \prec \langle M, A \rangle$ for all $A \in X$, and $\bar{A} = \pi^{-1}{}''A$, where $\pi = \pi_{\langle \bar{M}, \bar{c} \rangle, \langle M^B, \dot{c}^{\mathfrak{A}} \rangle}$

Define r by: $r_0 = \langle \bar{M}, \bar{c} \rangle$,

$r_1 =$ the set of $\langle A, \bar{A} \rangle$ s.t. $A \in R^p \cup R^q \cup R$ and $\bar{A} = \pi^{-1}{}''A$.

Then $\mathfrak{A} \models \mathcal{L}(r)$. Hence $r \in \mathbb{P}$. But $\pi_p^{\mathfrak{A}} : p_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$, $\pi_r^{\mathfrak{A}} : r_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{A}} \rangle$ and $\text{rng}(\pi_p^{\mathfrak{A}}) \subset \text{rng}(\pi_p^{\mathfrak{A}})$, Hence $\pi : p_0 \lhd r_0$ where $\pi = \pi_r^{\mathfrak{A}} \circ (\pi_p^{\mathfrak{A}})^{-1}$, by Fact 9. But then, if $\langle a, \bar{a} \rangle \in F^p$, $\langle a, a' \rangle \in F^r$, we have:

$$\pi : \langle M_p, \bar{a} \rangle \prec \langle M_r, a' \rangle.$$

Hence $r \leq p$ with $\pi = \pi_{pr}$.

Similarly $r \leq q$. QED (3.4)

<u>Cor 3.5</u> p, q are compatible in $\mathbb{P}$ iff $\mathcal{L}(p) \cup \mathcal{L}(q)$ is consistent.

prf:

($\leftarrow$) by Lemma 3.4

($\rightarrow$) If $r \leq p, q$, then $\mathcal{L}(r) \vdash \mathcal{L}(p) \cup \mathcal{L}(q)$.

QED (3.5)

<u>Cor 3.6</u> Let $p \in \mathbb{P}$, $R \subset \mathbb{P}(M^B)$ where R is countable. There is $q \leq p$ s.t. $R \subset R^q$.

<u>Cor 3.7</u> Let $p \in \mathbb{P}$, $u \subset M^B$, where u is countable. There is $q \leq p$ s.t. $u \subset \mathrm{rng}(\pi^q)$

<u>Lemma 3.8</u> Let $p \in \mathbb{P}$, $u \subset M_p$, u finite. There is $q \leq p$ s.t. $q_0 = p_0$ and $u \subset \mathrm{dom}(\pi^q)$.

prf: Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(p)$.

Set: $q_0 = p_0$, $F^q = F^p \cup (\pi_p^{\mathfrak{A}} \upharpoonright u)$.

QED (3.8)

Using the extension lemma we get:

<u>Lemma 3.9</u> Let G be $\mathbb{P}$-generic. Then

(a) $\langle P_p \mid p \in G \rangle, \langle \pi_{pq} \mid q \leq p \text{ in } G \rangle$ is a directed system with the limit:

$\langle M^B, c^G \rangle, \langle \pi_p^G \mid p \in G \rangle$

(Moreover $\pi_p^G = \bigcup \{\pi^q \mid q_0 = p_0 \wedge q \leq p \wedge q \in G\}$.)

(b) $\pi_p^G : P_0 \lhd \langle M^B, c^G \rangle$ for $p \in G$

(c) $\pi_p^G : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^p$.

The proof is left to the reader.

, " " " , '

Using the "revisability" lemma 3.3 we get:

<u>Lemma 3.9</u> $\mathbb{P}$ adds no reals.

prf.

Let $\Vdash \mathring{f} : \check{\omega} \to 2$. It is enough to show:

<u>Claim</u> The set Δ of p s.t. $\bigvee f\ p \Vdash \mathring{f} = \check{f}$ is dense in $\mathbb{P}$.

Let $r \in \mathbb{P}$. We construct $q \leq r$ s.t. $q \in \Delta$. Let:

$N^* = \langle H_\theta, N, <, \mathring{f}, \mathbb{P}, r, \ldots \rangle$ where $\theta > 2^B$

and $<$ well orders N^*.

Let $p \in \mathbb{P}$ conform to N^* (as defined in [LF] §3). Set:

$\bar N^* = \bar N^*(p, N^*) = \langle H', N'^{B'}, <, \mathring f', \mathbb{P}', r', \ldots \rangle$

Pick $G' \ni r$ which is $\mathbb{P}'$-generic over $\bar N^*$.

Define q by: $q_0 = \langle M_p, c^{G'} \rangle$, $q_1 = p_1$

Then $q \in \mathbb{P}$ by the reviability lemma.

Let $f = \mathring f'^{G'}$. It suffices to show:

Claim $q \Vdash \mathring f = \check f$ and q is compatible with r.

We first show: $q \Vdash \mathring f = \check f$. Suppose not.
Then there is $q' \le q$ s.t. $q' \Vdash \mathring f(\check n) \ne f(\check n)$ for some n. Let $\mathfrak{A}$ be a solid model of $\mathcal{L}(q')$. Let $\pi^* \supset \pi_q^{\mathfrak{A}} \cup F^s$ s.t. $\pi^* : \bar N^* \prec N^*$. Let $s' \in G'$ s.t. $s' \Vdash_{\mathbb{P}'} \mathring f'(\check n) = f(\check n)$. Set $s = \pi^*(s')$.
Then $s \Vdash_{\mathbb{P}} \mathring f(\check n) = f(\check n)$. Hence s, q are incompatible. We obtain a contradiction by proving:

Claim $\mathfrak{A} \models \mathcal{L}(q') \cup \mathcal{L}(s)$.

pf.
$\mathfrak{A} \models \mathcal{L}(q')$ is trivial. We prove $\mathfrak{A} \models \mathcal{L}(s)$.
Note that $s_0 = s'_0$ and F^s = the set of $\langle a, \bar a \rangle$ s.t. $a = \pi^*(a')$ and $\langle a', \bar a \rangle \in F^{s'}$ for some a'. Clearly
$\pi^{G'}_{s'} : s_0 \lhd \langle M'^{B'}, c^{G'} \rangle$, since $s' \in G'$.

But $q_0 = \langle M'^{B'}, c^{G'} \rangle$ and $\pi_q^{\mathfrak{U}} : q_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{U}} \rangle$.
Set $\pi = \pi_q^{\mathfrak{U}} \circ \pi_{r'}^{G'} = \pi^* \circ \pi_{r'}^{G'}$. Then $\pi \in \mathfrak{U}$
and $\pi : r_0 \triangleleft \langle M^B, \dot{c}^{\mathfrak{U}} \rangle$. It remains only
to show:

<u>Claim</u> $\pi : \langle M_r, a \rangle \prec \langle M^B, a \rangle$ for $\langle a, \bar{a} \rangle \in F_r^r$,
since then $\mathfrak{U} \models \mathcal{L}(r)$ with $\pi = \pi_r^{\mathfrak{U}}$.
Let $a = \pi^*(a')$. Then $\langle a', \bar{a} \rangle \in F^{r'}$. Then
$\pi_{r'}^{G'} : \langle M_r, \bar{a} \rangle \prec \langle M_q, a' \rangle$ and
$\pi_q^{\mathfrak{U}} : \langle M_q, a' \rangle \prec \langle M^B, a \rangle$ since $\pi^*(\langle M_q, a' \rangle) = \langle M^B, a \rangle$

This proves: $q \Vdash \mathring{f} = \check{f}$. But the last part of the proof shows that for <u>every</u> $r' \in G'$, $r = \pi^*(r')$ is compatible with q' for any $q' \leq q$ s.t. $\mathfrak{U} \models \mathcal{L}(q')$
But $r' \in G'$ and $r = \pi^*(r')$ since $\pi^* : \bar{N}^* \prec N^*$. Hence r is compatible with q. QED (3.9)

An immediate corollary is:

<u>Cor 3.10</u> Let $\theta \geq 2^\beta$ be regular. If $G \ni p$ is $\mathbb{P}$-generic and $c = c^G$, then $\langle H_\theta^{V[B][G]}, c \rangle$ models $\mathcal{L}(p)$.

<u>proof</u>: The only problematical axiom was $H\omega_1 = \underline{H\omega_1}$, which is now seen to hold.

QED (3.10)

Def Let $C \prec M^B$ be countable and cofinal, $G^C =_{Df}$ the set of $p \in \mathbb{P}$ s.t. $P_0 \lhd \langle M^B, C\rangle$ and, letting $\pi = \pi_{P_0, \langle M^B, C\rangle}$, we have:
$\pi : \langle M_p, \bar{a}\rangle \prec \langle M^B, a\rangle$ whenever $\langle a, \bar{a}\rangle \in F^p$.
and $\pi : M_p \to M^B$ is cofinal if β is regular.

Lemma 3.11 Let G be $\mathbb{P}$-generic. Then $G = G^C$ where $C = C^G$.

proof

$G \subset G^C$ is trivial. We prove $(\supset)$

Let $p \in G^C$. If $p \notin G$ there is $q \in G$ which is incompatible with p. But then
$\langle H_\theta^{V[G]}, C\rangle \models \mathcal{L}(p) \cup \mathcal{L}(q)$.
for regular $\theta \geq 2^\beta$. QED (3.11)

Lemma 3.12 Let G be $\mathbb{P}$-generic. Then $\bar{\bar{\beta}} \leq \omega_1$ in $V[B][G]$

proof

For each $\xi < \beta$ there is $\langle \bar{M}, \bar{C}, \bar{\xi}\rangle \in H_{\omega_1} \cap \Gamma$ s.t. $\langle \bar{M}, \bar{C}\rangle \lhd \langle M^B, C^G\rangle$ and $\pi(\bar{\xi}) = \xi$ where $\pi = \pi_{\langle \bar{M}, \bar{C}\rangle, \langle M^B, C^G\rangle}$. This maps a subset of H_{ω_1} onto β. QED (3.12)

Lemma 3.13 Let G be $\mathbb{P}$-generic. If $\omega_1 < \tau \leq \beta$ and τ is regular in $V[B]$, then $cf(\tau) = \omega$ in $V[B][G]$

proof.

If $\tau = \beta$, then for any $p \in G$ we have $\sup \pi_p^G{}''\beta_p = \beta$ where β_p is countable.

Now let $\tau < \beta$. Let $p \in G$ s.t. $\pi_p^G(\bar{\tau}) = \tau$.

Then each $\xi < \tau$ lies in $\pi_p^G(u)$ for a $u \in M_p$ s.t. $\bar{\bar{u}} \leq \omega_1$ in M_p. But the set U of such u is countable. Set:

$\mu_u = \sup u \cap \bar{\tau}$ for $u \in U$. Then $\mu_u < \bar{\tau}$ and $\{\pi_p^G(\mu_u) \mid u \in U\}$ is cofinal in τ.

QED (3.13)

Cor 3.14 If $\omega_1 < \delta \leq \beta$ and $cf(\delta) \neq \omega_1$,*) then $cf(\delta) = \omega$ in $V[B][G]$.

We now recall [LF] §4 Lemma 4.1 which says:

Fact 11 Let β be a cardinal in an inner model W s.t. $2^{\beta} = \beta$ in W. Let $\gamma = 2^{\beta}$ in W. Assume that in V we have: $2^{\omega} = \omega_1$, $\bar{\bar{\beta}} = \omega_1$, $cf(\beta) = \omega$. Then $\bar{\bar{\gamma}} \leq \omega_1$ in V.

*) We are working over $V[B]$, so statements like $cf(\delta) \neq \omega_1$ are understood to be in the sense of $V[B]$.

Hence:

Cor 3.14.1 If $cf(\beta) \neq \omega_1$, then $\overline{\overline{\mu}} = \omega_1$ in $V[B][G]$, where $\mu = 2^{\beta}$.

(μ^+ remains a cardinal, however, since $\overline{\overline{\mathbb{P}}} \leq \mu$. Hence if $2^{\mu} = \mu$, we can conclude $cf(\mu) = \omega_1$, since otherwise μ^+ would be collapsed by Fact 11. In particular, $\mu = \beta^+$ + $cf(\mu) = \omega_1$ in $V[G]$ if GCH holds in V.)

The case $cf(\beta) = \omega_1$ is quite different as shown by:

Lemma 3.15 Let $cf(\beta) = \omega_1$ in $V[B]$. Then β^+ remains a cardinal in $V[B][G]$ (Hence $\beta^+ = \omega_2$ in $V[B][G]$)

proof.

We imitate the proof of [LF] §4 Lemma 3.1 to show:

Sublemma 3.15.1 $BA(\mathbb{P})$ has a dense subset of size β.

prf. Work in $V[B]$.

Set $H = H_{(\beta)^+}$. Then $\langle H[G], c^G \rangle$ models $\mathcal{L}$ whenever G is $\mathbb{P}$-generic (interpreting $\underline{x}$ by x). Let $\Vdash_{\mathbb{P}} \dot{c} = c^{\dot{G}}$, where $\dot{G}$ is the canonical generic name. We can give every $\mathcal{L}$ sentence φ an interpretation $[\![\varphi]\!] \in \mathbb{B} = BA(\mathbb{P})$ in $H^{\mathbb{P}}$, interpreting $\dot{c}$ by $\ddot{c}$ and $\underline{x}$ by $\check{x}$.

We then have:

$\langle H[G], c^G \rangle \Vdash \varphi(x_1, \dots, x_n) \longleftrightarrow$

$\longleftrightarrow [\![\varphi(\underline{x}_1, \dots, \underline{x}_n)]\!] \cap G \neq \emptyset$

for $x_1, \dots, x_n \in N$ and G a $\mathbb{P}$-generic set.

Thus it suffices to prove:

Claim For each $p \in \mathbb{P}$ there is an $\mathcal{L}$-statement $\varphi \in M^B$ s.t. $[\![\varphi]\!] \neq 0$ and $[\![\varphi]\!] \subset [p]$. ($[p]$ being the smallest $b \in \mathbb{B}$ s.t. $p \in b$).

If G is $\mathbb{P}$-generic, we have;
$[p] \cap G \neq \emptyset \longleftrightarrow p \in G \longleftrightarrow \langle H[G], c^G \rangle \models \varphi_p$.
$\longleftrightarrow [\![\varphi_p]\!] \cap G \neq \emptyset$. Hence
$[p] = [\![\varphi_p]\!]$ and it suffices to show:

<u>Claim</u> $\Vdash_{\mathbb{P}} \psi \to \varphi_p$ for a $\psi \in M^\beta$ s.t. $[\![\psi]\!] \neq 0$

Set $N^* = \langle H, N^\beta, < \rangle$ where $<$ well orders H.
We may assume w.l.o.g. that
p conforms to N^*, since the set
of such p is dense in $\mathbb{P}$. Let G be
$\mathbb{P}$-generic with $p \in G$. Let $\tilde{\beta} = \sup \pi_p^G{}'' \beta_p$.
Then $\tilde{\beta} < \beta$ since $\tilde{\beta}$ is ω-cofinal.
Set $\tilde{M} = J_{\tilde{\beta}}^{A, B_u, B}$. For $a \in R^p$ set $\tilde{a} = a \cap \tilde{M}$.
Then $\pi_p^G : \langle \bar{M}, \bar{a} \rangle \to \langle \tilde{M}, \tilde{a} \rangle$ is cofinal
and Σ_0-preserving whenever $\langle a, \bar{a} \rangle \in F^p$.
But then

(1) $\tilde{a} = \bigcup_{z \in M_p} \pi_p^G(z \cap \bar{a})$.

Let $\langle a_i | i < \omega \rangle$ enumerate R^p in V.
Then $\langle \bar{a}_i | i < \omega \rangle \in H_{\omega_1}$, where $\langle a_i, \bar{a}_i \rangle \in F^p$.
Moreover $\langle \tilde{a}_i | i < \omega \rangle \in M$, since $\tilde{a}_i \in M$
and $cf(\beta) > \omega$. Let ψ be the sentence:

There are π, σ s.t. $\sigma : \langle \underline{Q}_p, \underline{c}^p \rangle \triangleleft_* \langle \underline{Q}^B, \dot{c} \rangle \wedge$
$\wedge\ \langle \underline{\tilde{M}}, \pi \rangle$ is the liftup of $\langle \underline{M}_p, \sigma \rangle\ \wedge$
$\wedge \bigwedge_{i<\omega} (\underline{\tilde{a}}_i = \bigcup_{z \in \underline{M}_p} \pi(z \cap \underline{\bar{a}}_i))$,

Clearly $\psi \in M^B$. Moreover,
(2) $[\![\psi]\!] \neq 0$, since $\langle H[G], c^G \rangle \models \psi$
(since then ψ holds with $\sigma = \pi_p^G \upharpoonright Q_p$, $\pi = \pi_p^G$).

We show:
(3) $\langle H[G], c^G \rangle \models \psi \to \varphi_p$
whenever G is $\mathbb{P}$-generic.

Let $\langle H[G], c^G \rangle \models \psi$. Let $\sigma : \langle Q_p, c^p \rangle \triangleleft_* \langle Q, c^G \rangle$
and $\langle \tilde{M}, \pi \rangle =$ the liftup of $\langle M_p, \sigma \rangle$.
It remains only to show:
$\pi : \langle M_p, \bar{a} \rangle \prec \langle M^B, a \rangle$ whenever
$\langle a, \bar{a} \rangle \in F^p$, since then we have
$\pi : \langle M_p, C^p \rangle \triangleleft \langle M^B, C^G \rangle$ and
hence: $p \in G^{C^G} = G$ with $\pi = \pi_p^G$.
Let $b = \{\vec{z} \in M \mid \langle M, a \rangle \models \chi(\vec{z})\}$. Then $b \in R^p$
by the N^*-conformity of p. Let
$\langle b, \bar{b} \rangle \in F^p$. Then by N^*-conformity:
$\bar{b} = \{\vec{z} \in M_p \mid \langle M_p, \bar{a} \rangle \models \chi(\vec{z})\}$.
Hence: $\langle M_p, \bar{a} \rangle \models \chi(\vec{z}) \leftrightarrow \vec{z} \in \bar{b} \leftrightarrow$
$\leftrightarrow \pi(\vec{z}) \in \tilde{b} = b \cap \tilde{M} \leftrightarrow \langle M, a \rangle \models \chi(\pi(\vec{z}))$,
since $\tilde{b} = \bigcup_{u \in M_p} \pi(u \cap \bar{b})$. QED (3.15)

Note $cf(\beta) = \omega_1$ is the only case to consider if $A_0 = 0$.

We also note that we could have defined $\mathcal{L}$ (and hence $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$) somewhat differently: Let $\mathcal{L}'$ be like $\mathcal{L}$ except that in (*) we omit: $\mathrm{Urng}(\pi) = M$ if β is regular, and instead add the axiom:
(*) If β is regular, then whenever $u \in H_{\omega_1}$ and $\pi : u \lhd \langle \underline{M}, c \rangle$, we have: $\sup \mathrm{On} \cap \mathrm{rng}(\pi) < \beta$.
It turns out that $\mathcal{L}'$ is also consistent. If $\mathbb{P}' = \mathbb{P}_{\mathcal{L}'}$ and β is regular, we can modify the proof of Lemma 3.1 to get: $\mathbb{B}' = BA(\mathbb{P}')$ contains a dense subset of size β. Hence $\beta^+ = \omega_2$ and $cf(\beta) = \omega_1$ in $V[G']$, where G' is $\mathbb{P}'$-generic. We omit the proof, since this is not relevant to the present paper.

We are now ready to prove that $\mathbb{P}$ is subcomplete. Since we are working in $V[B]$ we shall again write V for $V[B]$ and - for the sake of simplicity - we also write Q, M, N for Q^B, M^B, N^B.

Lemma 4 $\mathbb{P}$ is subcomplete.

pf. (We work in $V[B]$)

Let $W = L_\tau^{A'}$ where $2^\beta < \theta < \tau$, τ is regular, and $H_\theta \subset W$. Let $\sigma : \overline{W} \prec W$ s.t. $\overline{W}$ is countable and full with:

$\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{M}, \overline{s}, \overline{\lambda}_i) = \theta, \mathbb{P}, M, s, \lambda_i \quad (i = 1, \ldots, n)$

where $\mathbb{P} \in H_{\lambda_i}$ (hence $N \in H_{\lambda_i}$), $\lambda_i < \theta$, and λ_i is regular for $i = 1, \ldots, n$. Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{W}$.

Claim There is $q \in \mathbb{P}$ s.t. whenever $G \ni q$ is $\mathbb{P}$-generic, then there is $\sigma_0 \in V[G]$ with:

(a) $\sigma_0 : \overline{W} \prec W$

(b) $\sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{M}, \overline{s}, \overline{\lambda}_i) = \theta, \mathbb{P}, M, s, \lambda_i \quad (i = 1, \ldots, n)$,

(c) $\sup \sigma_0 '' \overline{\lambda}_i = \sup \sigma '' \overline{\lambda}_i \quad (i = 0, \ldots, n)$

where $\overline{\lambda}_0 = On \cap \overline{W}$

(d) $\sigma_0 '' \overline{G} \subset G$.

We first show by standard methods:

Sublemma 4.1 Let δ be least s.t. $L_\delta(W)$ is admissible. The following language $\mathcal{L}^*$ on $L_\delta(W)$ is consistent:

Predicate $\in$, Constants $\underline{x}$ ($x \in L_\delta(W)$), $\dot{\sigma}$

Axioms: ZFC^-, $\bigwedge v\,(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in L_\delta(W)$,

$\dot{\sigma} : \underline{\overline{W}} \prec \underline{W}$, $\dot{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{P}}}, \underline{\overline{M}}, \underline{\overline{s}}, \underline{\overline{\lambda}_i}) = \underline{\theta}, \underline{\mathbb{P}}, \underline{M}, \underline{s}, \underline{\lambda_i} \quad (i = 1, \ldots, n)$,

$\sup \dot{\sigma} '' \underline{\overline{\lambda}_i} = \underline{\sup \sigma '' \overline{\lambda}_i} \quad (i = 0, \ldots, n)$, and:

$\dot{\sigma} \upharpoonright \underline{\overline{Q}} : \underline{\overline{Q}} \prec \underline{Q}$ cofinally (where $\sigma(\overline{Q}) = Q$),

$\langle \underline{N}, \dot{\sigma} \upharpoonright \underline{\overline{N}} \rangle$ is the liftup of $\langle \overline{N}, \dot{\sigma} \upharpoonright \underline{\overline{Q}} \rangle$

Note $\mathcal{L}^*$ does not posit that $H_{\omega_n} = H_{\omega_1}$,

prf. (sketch) of 4.1

Let $\mathcal{L}_0$ be like $\mathcal{L}^*$ except that the axiom

$$\sup \dot\sigma '' \bar\lambda_i = \sup \sigma '' \bar\lambda_i \quad (i = 0, \dots, n)$$

is replaced by:

$$\sup \dot\sigma '' \bar\lambda_i = \lambda_i \quad (i = 0, \dots, n) \quad (\text{where } \lambda_0 =_{\text{Df}} \tau)$$

Let $\sigma \upharpoonright \bar Q : \bar Q \prec \tilde Q$ cofinally $(\bar Q = H_{\omega_2}^{\bar W})$ and let $\tilde\sigma : \bar W \prec W$ be the liftup of $\bar W$ by $\sigma \upharpoonright \bar Q$.

Let $k : \tilde W \prec W$ s.t. $k \upharpoonright \tilde Q = \mathrm{id}$ and $k\tilde\sigma = \sigma$.

Let $\tilde{\mathcal{L}}_0$ be defined on $L_{\tilde\delta}(\tilde W)$ like $\mathcal{L}_0$ on $L_\delta(W)$ in the obvious sense, where $\tilde\delta$ is least s.t. $L_{\tilde\delta}(\tilde W)$ is admissible. (More precisely, $\mathcal{L}_0$ is defined in the parameter W and parameters $Q, M, N, \theta, s, \lambda_i, \dots \in W$. $\tilde{\mathcal{L}}_0$ has the same definition over $L_{\tilde\delta}(\tilde W)$ in the parameter $\tilde W$ and the parameters $k^{-1}(Q), k^{-1}(M), \dots, k^{-1}(s), k^{-1}(\lambda_i)$ $(i = 1, \dots, n)$.

Then $\langle H_{\omega_2}, \tilde\sigma \rangle$ models $\tilde{\mathcal{L}}_0$. Assume w.l.o.g. $\lambda_0 > \dots > \lambda_n$ and let

$$\sigma \upharpoonright H^{\bar W}_{\bar\lambda_n} : H^{\bar W}_{\bar\lambda_n} \prec H' \text{ cofinally.}$$

(Here $H^{\bar W}_{\lambda_n} = \bar W$ if $n = 0$.) Let $\sigma' : \bar W \prec W'$ be the liftup of $\bar W$ by $\sigma \upharpoonright H^{\bar W}_{\bar\lambda_n}$. Let $k' : W' \prec W$ s.t. $k' \upharpoonright H' = \mathrm{id}$ and $k'\sigma' = \sigma$.

There is then $\tilde k : \tilde W \prec W'$ s.t.

$\tilde{k} \upharpoonright \tilde{Q} = id$ and $\hat{k}\tilde{\sigma} = \sigma'$. We then have $k'\tilde{k} = k$. Let δ' be least s.t. $L_{\delta'}(W')$ is admissible and let $\mathcal{L}_0'$ be defined over $L_{\delta'}(W')$ as $\mathcal{L}_0$ was defined over $L_\delta(W)$ (in the obvious sense). The statement that $\tilde{\mathcal{L}}_0$ is consistent is $\Pi_1(L_{\tilde{\delta}}(\tilde{W}))$ in the parameter $\tilde{W}$ and parameters $\vec{p} \in \tilde{W}$. The statement that $\mathcal{L}_0'$ is consistent is then $\Pi_1(L_{\delta'}(W'))$ in W' and $k'(\vec{p})$. Hence $\mathcal{L}_0'$ is consistent. Note that $k'(N) = N$. Let $\mathfrak{N}$ be a solid model of $\mathcal{L}'$ which lies in some generic extension $V[G]$ of V. Let $\mu > \overline{\tau}$ be regular in $V[G]$. Then $\langle H_\mu^{V[G]}, k'\circ\sigma'\rangle$ models $\mathcal{L}^*$, where $\sigma' = \dot{\sigma}^{\mathfrak{N}}$. QED (4.1)

Now let $N^* = \langle H_\delta, W, N, \sigma, \lambda_1, \ldots, \lambda_m, s, \mathbb{P}, \ldots\rangle$ where $\delta > On_W$. Let p conform to N^*. Set: $\overline{N}^* = \overline{N}^*(p, N^*) = \langle H', W', N', \sigma', \lambda_1', \ldots, \lambda_m', s', \mathbb{P}', \ldots\rangle$. Let $\overline{\mathcal{L}}^*$ be defined in $\overline{N}^*$ like $\mathcal{L}^*$ in N^*. Let $\mathfrak{N} \in H_{\omega_1}$ be a solid model of $\overline{\mathcal{L}}^*$. Set: $\sigma^* = \dot{\sigma}^{\mathfrak{N}}$. Set $\overline{C} = C^{\overline{G}}$, $C' = \sigma^{*}{}''\overline{C}$. Since $\sigma^* \upharpoonright \overline{Q} : \overline{Q} \prec Q'$ cofinally (where $Q' = Q_p$ is defined in $\overline{N}^*$ like Q in N^*), we have: $q \in \mathbb{P}$ where q is defined by:

$q_0 = \langle M_p, c' \rangle$, $q_1 = p_1$.

We show that this q satisfies the claim. Let $G \ni q$ be $\mathbb{P}$-generic. Note that, since $\langle N', \sigma^* \upharpoonright \bar{N} \rangle$ is the liftup of $\langle \bar{N}, \sigma^* \upharpoonright \bar{Q} \rangle$ and $\sigma^* \upharpoonright \bar{Q} : \bar{Q} \prec Q'$ with $\sigma^{*}{}'' \bar{c} = c' = c_q$, we have: $\langle \bar{M}, \bar{c} \rangle \lhd q_0 = \langle M_q, c' \rangle$ with $\pi_{\langle \bar{M}, \bar{c} \rangle, q_0} = \sigma^* \upharpoonright \bar{M}$. But $q_0 \lhd \langle M, c \rangle$ with $\pi_{q_0, \langle M, c \rangle} = \pi_q^G$. (Here $c = c^G$.) Thus $\langle \bar{M}, \bar{c} \rangle \lhd \langle M, c \rangle$ with $\pi_{\langle \bar{M}, \bar{c} \rangle, \langle M, c \rangle} = \pi_q^G \circ \sigma^* \upharpoonright \bar{M}$.

Now let $\pi^* \supset \pi_q^G \cup F^q$ s.t. $\pi^* : \bar{N}^* \prec N^*$. Set $\sigma_0 = \pi^* \sigma^*$. Then (a) – (c) are readily established. We show:

(d) $\sigma_0 {}'' \bar{G} \subset G$.

Let $\bar{r} \in \bar{G}$, $r = \sigma_0(\bar{r})$. Then $r_0 = \bar{r}_0$. But then $r_0 \lhd \langle \bar{M}, \bar{c} \rangle$ and $\pi_{r_0, \langle \bar{M}, \bar{c} \rangle} = \pi_{\bar{r}_0}^{\bar{G}}$, since $\bar{r} \in \bar{G}$. Hence $r_0 \lhd \langle M, c \rangle$ and $\pi_{r_0, \langle M, c \rangle} = \sigma_0 \circ \pi_{\bar{r}}^{\bar{G}}$ since $\pi_{\langle \bar{M}, \bar{c} \rangle, \langle M, c \rangle} = \sigma_0 \upharpoonright \bar{M}$ by the above.

It remains only to show:

<u>Claim</u> $\sigma_0 \pi_{\bar{r}_0}^{\bar{G}} : \langle M_r, \bar{a} \rangle \prec \langle M, a \rangle$ whenever $\langle a, \bar{a} \rangle \in F^r$.

prf.

Let $\langle a, \bar{a} \rangle = \sigma_0(\langle a', \bar{a} \rangle)$ where $\langle a', \bar{a} \rangle \in F^{\bar{r}}$.

Then $\pi_{\bar{r}}^{\bar{G}} : \langle M_r, \bar{a} \rangle \prec \langle \bar{M}, a' \rangle$. But $\sigma_0 \upharpoonright \bar{M} : \langle \bar{M}, a' \rangle \prec \langle M, a \rangle$ since $\sigma_0(\langle \bar{M}, a' \rangle) = \langle M, a \rangle$.

QED (Lemma 4)

Note that $\overline{\overline{BA(\mathbb{P}_{\mathbb{B}})}} \leq 2^{\beta_{\mu+1}}$, since either $\beta = \beta_{\mu+1}$, $cf(\beta) = \omega_1$, and $BA(\mathbb{P}_{\mathbb{B}})$ has a dense subset of size β, by Sublemma 3.15.1, or else $\beta_{\mu+1} = \beta^+$, $\overline{\overline{\mathbb{P}_{\mathbb{B}}}} = 2^{\beta} = \beta^+$ (since then GCH holds below κ). Now let $\Vdash_{\mathbb{B}_\mu} \mathring{\mathbb{B}} = BA(\mathbb{P}_{\mathring{\mathbb{B}}})$, $\mathring{\mathbb{B}}$ being the canonical generic name. We then form $\mathbb{B}_\mu * \mathring{\mathbb{B}}$, which also has cardinality $\leq 2^{\beta_{\mu+1}}$, since $\mathbb{B}_\mu$ has cardinality $\leq 2^{\beta_\mu} \leq \beta_{\mu+1}$, since $2^{\beta_{\mu+1}} = \beta_{\mu+1}$. Let $k : \mathbb{B}_\mu \to \mathbb{B}_\mu * \mathring{\mathbb{B}}$ be the natural injection. Choose $\mathbb{B}_{\mu+1} \supseteq \mathbb{B}_\mu$ s.t. there is an isomorphism l with:

$$\begin{tikzcd}
\mathbb{B}_{\mu+1} & & \mathbb{B}_\mu * \mathring{\mathbb{B}} \arrow[ll, "l"'] \\
& \mathbb{B}_\mu \arrow[ul, hook] \arrow[ur, "k"'] &
\end{tikzcd}$$

We ensure that $\mathbb{B}_{\mu+1} \subset H_{\beta^+_{\mu+1}}$. By Lemma 4 we know that $\mathbb{B}_{\mu+1}$ is subcomplete. However, we have found it necessary to devise another representation of $\mathbb{B}_{\mu+1}$ in order to elicit its deeper properties. Working in V define as before:

$$Q = H_\gamma, \quad M = L_\beta^{A, \mathbb{B}_\mu}, \quad N = \langle H_{\beta^+}, M, <, \ldots \rangle.$$

We then define a class Γ' of triples as follows: