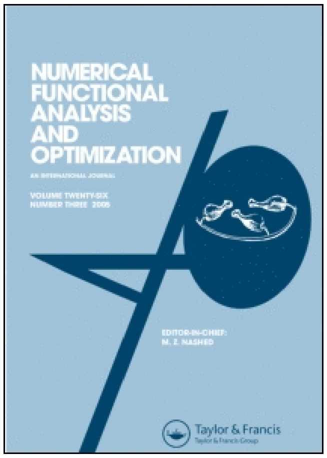

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WEAK CONVERGENCE OF APPROXIMATE SOLUTIONS OF STOCHASTIC EQUATIONS WITH APPLICATIONS TO RANDOM DIFFERENTIAL AND INTEGRAL EQUATIONS *

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Dedicated to the memory of
A.T.Bharucha-Reid

ABSTRACT:

In this paper, we considerably extend our earlier result about convergence in distribution of approximate solutions of random operator equations, where the stochastic inputs and the underlying deterministic equation are simultaneously approximated. As a by-product, we obtain convergence results for approximate solutions of equations between spaces of probability measures. We apply our results to random Fredholm integral equations of the second kind and to a random non-linear elliptic boundary value problem.

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1. INTRODUCTION AND PRELIMINARIES

In [12], we developed an approach for proving weak compactness and weak convergence of sequences of probability distributions of random variables which are approximate random solutions of random operator equations. This approach allows the simultaneous approximation of the underlying deterministic equation and of the stochastic inputs with respect to convergence in distribution. The proof of the main convergence result ([12, Theorem 2.11]) is essentially based on Prokhorov's Theorem (see [27, Theorem 1.12]). Another, but related approach for attacking this problem can be found in [5].
For concepts and general results concerning other modes of convergence of approximate random solutions (a.s.-convergence, convergence in probability), we refer e.g. to [23], [11].
In this paper, we considerably generalize the approach of [12]: Instead of compactness assumptions about the operators we need only more general regularity assumptions; the approximate equations are now allowed to be defined only on subspaces (and on different probability spaces); instead of convergence of subsequences, we will also be able to prove convergence (in distribution) of the whole sequence of approximate solutions. These extensions allow much more far-reaching applications than those considered in [12]. In order to achieve this, we have to develop an approximation concept for equations in spaces of probability measures ("stochastic equations"), which might also be of independent interest.
We will be concerned with a random equation

$$T(z(\omega),x) = y(\omega) \qquad (\omega \in \Omega) \tag{1.1}$$

and its approximations

$$T_n(z_n(\omega),x) = y_n(\omega) \qquad (\omega \in \Omega_n,\ n \in \mathbb{N}) \tag{1.2}$$

where T is a mapping from $Z \times X$ into Y, T_n from $Z \times X_n$ into Y, and X, X_n, Y, Z are metric spaces with $X_n \subseteq X$; z, y and z_n, y_n, $n \in \mathbb{N}$, are random variables (defined on possibly different probability spaces (Ω,A,P), (Ω_n, A_n, P_n), $n \in \mathbb{N}$).
We assume that the joint probability distributions $(D(y_n,z_n))$ converge weakly to $D(y,z)$ (in the space of probability measures on $Y \times Z$) and ask for conditions on T and (T_n) that imply weak convergence of $(D(x_n))$ to a "solution" of (1.1). Here (x_n) is a sequence of X_n-valued random variables (defined on (Ω_n, A_n, P_n)) that are almost surely solutions of (1.2). The basic convergence

results are Theorem 4.6 and its Corollaries. It turns out that the weak limit of $(D(x_n))$ is a probability measure defined on X which is a solution of (1.1) in a weak sense (see the definition of a weak solution in 4.1). This solution concept for (1.1) is motivated by Ershov's concepts (see e.g. [14]). The concept of a weak solution we use generalizes that of a "D-solution" used in [12] (for a further discussion we refer to Remark 4.2).
The approach for establishing the convergence results is based on the observation that the joint distribution $D(z_n, x_n)$ is a solution of the special "stochastic equation" (in the sense of [14])

$$\mu\tilde{T}_n^{-1} = D(y_n, z_n) \qquad (n \in \mathbb{N}) \tag{1.3}$$

with suitable mappings $\tilde{T}_n$, $n \in \mathbb{N}$ (see (4.5), (4.6)).
In Sections 2 and 3, we develop an approximation concept for general stochastic equations and establish an abstract convergence result (Theorem 3.1o) which is applied to the special equation (1.3) in Theorem 4.6. It turns out that well-known concepts from (deterministic) operator approximation theory such as "discrete convergence", "A-regularity" etc. (see e.g. [28], [29], [19], [2]) are also essential tools in this context.
In Sections 5 and 6, we apply our convergence results to approximations for random Fredholm integral equations of the second kind and to a Galerkin scheme for a nonlinear elliptic partial differential equation with random coefficients.
We dedicate this paper to the memory of our mentor and friend A.T.Bharucha-Reid; his work, especially [4], was a major guideline for research in the field of random equations for many people (including both authors of this paper). Furthermore, his steady interest in our work and his encouragement was essential for both authors' professional development. Moreover, as can be seen from the list of references, some of his work is of direct importance for this paper.

We now fix the terminology of this paper. For a metric space S we denote by $\mathcal{B}(S)$ the σ-algebra of Borel subsets of S and by $P(S)$ the set of all probability measures defined on $(S, \mathcal{B}(S))$ equipped with the topology of weak convergence (see [6, p. 236], [27]). It is known that $P(S)$ is metrizable (Polish) if S is separable (Polish) (see [6]).
Suppose that S_o is a Borel subset of S and that S_o is endowed with the relative topology Then $\mathcal{B}(S_o) = \{B \cap S_o | B \in \mathcal{B}(S)\}$. If μ is a probability measure on $(S_o, \mathcal{B}(S_o))$, let $\mu^e \in P(S)$ denote the

"canonical extension" of μ, defined by $\mu^e(B) := \mu(B \cap S_0)$, for all $B \in \mathcal{B}(S)$ (see [6, p.38]).

If x is an S-valued random variable defined on some probability space (Ω, A, P) (i.e., $x^{-1}(B) \in A$ for all $B \in \mathcal{B}(S)$), let $D(x) \in P(S)$ denote its "probability distribution", i.e.,

$$D(x)(B) := P(x^{-1}(B)), \quad B \in \mathcal{B}(S).$$

In the following, a sequence (s_n) in a certain metric space S' will be called "relatively compact" if it contains a convergent subsequence. (s_n) will be called "discretely compact" (see e.g. [28, p.288]) if every subsequence of (s_n) contains a further subsequence that is convergent in S'. The respective notions for weak convergence in $P(S)$ (where S is separable) will be called "relatively w-compact" and "discretely w-compact".

It is known from Prokhorov's Theorem ([27], [6, Theorem 6.1]) that a sequence (μ_n) in $P(S)$ with a separable metric space S is discretely w-compact if it is (uniformly) "tight", i.e., for all $\varepsilon > 0$ there exists a compact subset K_ε of S such that

$$\inf_{n \in \mathbb{N}} \mu_n(K_\varepsilon) \geq 1-\varepsilon.$$

The converse is true if S is Polish ([6, Theorem 6.2]).

A sequence (μ_n) in $P(S)$ will be called "w-bounded" iff for all $\varepsilon > 0$ there exists a bounded Borel set $B_\varepsilon \subseteq S$ such that

$$\inf_{n \in \mathbb{N}} \mu_n(B_\varepsilon) \geq 1-\varepsilon.$$

Note that a sequence (x_n) of S-valued random variables is D-bounded in the sense of [12] if $(D(x_n))$ is w-bounded.

If $\mu, \mu_n \in P(S)$, $n \in \mathbb{N}$, we denote by $\mu_n \xrightarrow{w} \mu$ the weak convergence of (μ_n) to μ. Note that if S is Polish, then every weakly convergent sequence (μ_n) in $P(S)$ is tight and, hence, w-bounded.

2. AN APPROXIMATION CONCEPT FOR EQUATIONS IN SPACES OF PROBABILITY MEASURES

Let X and Y be separable metric spaces and $\hat{T}$ be a mapping from $P(X)$ into $P(Y)$. In this Section, we consider the equation

$$\hat{T}(\mu) = \nu, \tag{2.1}$$

where $\nu \in P(Y)$ is a given probability measure, and its approximations

$$\hat{T}_n(\mu) = \nu_n \qquad (n \in \mathbb{N}), \tag{2.2}$$

where $\hat{T}_n: P(X_n) \to P(Y)$ are given mappings (converging to $\hat{T}$ in some sense, see Def. 2.2) and ν_n, $n \in \mathbb{N}$, are given probability measures in $P(Y)$ (converging weakly to ν); X_n $(n \in \mathbb{N})$ are Borel subsets of X.

In the following, we are interested in conditions guaranteeing "convergence" (see Notation 2.1) of a sequence (μ_n) of solutions of (2.2) to a solution μ of (2.1). Note that (2.1) and (2.2) (for $n \in \mathbb{N}$) can be viewed as equations in certain metric spaces (of probability measures). Hence, the well-developed approximation theory for operator equations in metric spaces (see e.g. [1], [2], [19], [26], [28], [29], [3o]) can be applied.

Notation 2.1: (see [21, Sect. 3])

Let $\mu \in P(X)$ and $\mu_n \in P(X_n)$, $n \in \mathbb{N}$. (μ_n) will be called "weakly convergent" to μ $(\mu_n \xrightarrow{w} \mu)$, if (μ_n^e) is weakly convergent to μ in $P(X)$ (see Sect. 1 for the notations). Analogously, (μ_n) is called "relatively w-compact", "discretely w-compact", "tight", "w-bounded", whenever (μ_n^e) has these properties in $P(X)$.
The following are straightforward adaptions of the corresponding approximation concepts in [2], [28], [3o] to the context of this Section.

Definition 2.2:

a) $\hat{T}$ and $(\hat{T}_n)$ are called "w-closed" iff for all $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and sequences (μ_{n_k}) with $\mu_{n_k} \in P(X_{n_k})$ for $k \in \mathbb{N}$, we have that $\mu_{n_k} \xrightarrow{w} \mu$ and $\hat{T}_{n_k}(\mu_{n_k}) \xrightarrow{w} \nu$ imply $\hat{T}(\mu) = \nu$.

b) $(\hat{T}_n)$ is called (asymptotically) "w-regular" iff for all $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and sequences (μ_{n_k}) with $\mu_{n_k} \in P(X_{n_k})$ for $k \in \mathbb{N}$, we have that (μ_{n_k}) is relatively w-compact if (μ_{n_k}) is w-bounded and $(\hat{T}_{n_k}(\mu_{n_k}))$ is relatively w-compact. (Note that it obviously suffices to prove relative w-compactness of (μ_{n_k}) only for those sequences (μ_{n_k}) for which $(\hat{T}_{n_k}(\mu_{n_k}))$ is weakly convergent.)

These notions turn out to be useful for the following convergence result which is closely in the spirit of [2], [19], [28], [30]. Its proof is straightforward and is included here only for convenience.

Theorem 2.3:

Let $\hat{T}$, $(\hat{T}_n)$ and ν, (ν_n) be as above. Assume that $(\hat{T}_n)$ is w-regular, $\hat{T}$ and $(\hat{T}_n)$ are w-closed and that $\nu_n \xrightarrow{w} \nu$. Then, every w-bounded sequence (μ_n) of solutions of (2.2) (for the index $n \in \mathbb{N}$) is discretely w-compact and every limit $\mu \in P(X)$ of a weakly convergent subsequence is a solution of (2.1).
If, furthermore, (2.1) is uniquely solvable, then (μ_n) converges weakly to the unique solution of (2.1).

Proof:

Let (μ_n) be a w-bounded sequence of solutions of (2.2):

$$\hat{T}_n(\mu_n) = \nu_n \qquad (n \in \mathbb{N}), \tag{2.3}$$

and let (μ_{n_k}) be an arbitrary subsequence. (2.3) implies that $(\hat{T}_{n_k}(\mu_{n_k}))$ is weakly convergent to ν. Hence, because of the w-regularity of $(\hat{T}_n)$, (μ_{n_k}) is relatively w-compact. Hence there are a further subsequence $(\mu_{n_{k_j}})$ and a probability measure $\mu \in P(X)$ such that $\mu_{n_{k_j}} \xrightarrow{w} \mu$. Since $\hat{T}$ and $(\hat{T}_n)$ are w-closed, this implies $\hat{T}_{n_{k_j}}(\mu_{n_{k_j}}) \xrightarrow{w} \nu = \hat{T}(\mu)$.

Now, let (2.1) be uniquely solvable. We have proven that every subsequence of (μ_n) contains a further subsequence that converges weakly to a solution of (2.1) and, hence, to the unique solution μ of (2.1). Thus, [6, Theorem 2.3] yields that $\mu_n \xrightarrow{w} \mu$.

$\square$

Remark 2.4.:

Note that every w-bounded sequence (μ_n) of solutions of (2.2) is already discretely w-compact if $(\hat{T}_n)$ is w-regular and (ν_n) is discretely w-compact.

The reasons for introducing these concepts and stating Theorem 2.3 will become clear in the following Sections. Here, we indicate that (natural) sufficient conditions for the "w-closedness" and "w-regularity" are given in Section 3 (Prop. 3.4 and 3.9) for the case that the mappings $\hat{T}$ and $\hat{T}_n$, $n \in \mathbb{N}$, are induced by Borel measurable mappings $T: X \to Y$ and $T_n: X_n \to Y$, $n \in \mathbb{N}$, respectively.

These conditions are essentially based on Rubin's and Prokhorov's Theorems ([6, Theorem 5.5] and [27], [6, Theorems 6.1 and 6.2]).

For applying Theorem 2.3 one has to show that a sequence of solutions (μ_n) of (2.2) is w-bounded. The w-boundedness of (μ_n) can be proved in many concrete situations using additional properties of $(\hat{T}_n)$, as can be seen in [12, Sect. 3], [5] and the Sections 5 and 6 of this paper. Roughly spoken, the motivation for the w-boundedness condition for a sequence of probability measures is nearly the same as the boundedness contained in the concept of "A-properness" (see e.g. [26]) or in other compactness and regularity concepts (e.g. [2], [19], [3o]) for the approximate solution of operator equations: One needs some further information (e.g. a priori bounds) about the approximate solutions in order to be able to conclude their convergence.

If X is a real separable Banach space, then a result of de Acosta ([8, Theorem 2.3]; see also [5, Prop. 2.2]) implies that $(\hat{T}_n)$ is w-regular if and only if for all $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and sequences (μ_{n_k}) with $\mu_{n_k} \in P(X_{n_k})$ $(k \in \mathbb{N})$, we have that $(\mu^e_{n_k})$ is flatly concentrated (see [8]) if $(\hat{T}_{n_k}(\mu_{n_k}))$ is relatively w-compact and (μ_{n_k}) is w-bounded. An approach of this kind was used in [5] to prove tightness of the set of probability distributions of approximate random solutions of random operator equations. However, we do not pursue this line of research here.

3. WEAK CONVERGENCE OF APPROXIMATE SOLUTIONS TO STOCHASTIC EQUATIONS

In this Section we present a general framework for proving convergence of approximate solutions of stochastic equations, i.e., equations between spaces of probability measures. In the following Sections, we apply these results to more concrete equations (random operator equations, random integral and differential equations).

Let X and Y be separable metric spaces and $T: X \to Y$ be a (Borel) measurable mapping. We consider the associated mapping $\hat{T}: P(X) \to P(Y)$ defined by $\hat{T}(\mu)\ (B) := \mu(T^{-1}(B))$, $\mu \in P(X)$, $B \in \mathcal{B}(Y)$.
In this Section, we are concerned with "stochastic equations"

$$\hat{T}(\mu) = \nu \tag{3.1}$$

where ν is a given probability measure on Y.
Now, a probability measure μ on X is called a solution of (3.1) if $\mu(T^{-1}(B)) = \nu(B)$ holds for all $B \in \mathcal{B}(Y)$. Existence and uniqueness of solutions of stochastic equations of this type were first studied by Ershov (e.g. [14], [15]) (in a more general framework).

In the following, we are interested in the study of approximations for (3.1) using the concepts of the preceding Section.
Let, in addition, for all $n \in \mathbb{N}$ X_n be a Borel subset of X, $T_n: X_n \to Y$ be Borel measurable mappings and $\nu_n \in P(Y)$ be probability measures. As above, we consider the associated mappings $\hat{T}_n: P(X_n) \to P(Y)$, defined by $\hat{T}_n(\mu) := \mu \circ T_n^{-1}$, $\mu \in P(X_n)$, $n \in \mathbb{N}$, and the "approximate" stochastic equations

$$\hat{T}_n(\mu) = \nu_n \quad (n \in \mathbb{N}). \tag{3.2}$$

Now, we aim at establishing conditions on (T_n) and on T that imply uniqueness of solutions of (3.1), "w-closedness" and "w-regularity" (see Definition 2.2) of the sequence of the associated mappings $(\hat{T}_n)$ (as needed for the application of Theorem 2.3).

Proposition 3.1:

Let X and Y be Polish spaces and $T: X \to Y$ be measureable and injective. Then, for every $\nu \in P(Y)$, the solution of (3.1) is unique.

Proof:

Let $\mu_i \in P(X)$, $i = 1,2$, be solutions of (3.1) for some $\nu \in P(Y)$. Then, [25, Theorem 3.9] implies that for every $B \in \mathcal{B}(X)$, $T(B) \in \mathcal{B}(Y)$

and hence $\mu_1(B) = \mu_1(T^{-1}(T(B))) = \nu(T(B)) =$

$$= \mu_2(T^{-1}(T(B))) = \mu_2(B).$$

□

Definition 3.2:

Let S and S' be metric spaces, $S_n \subseteq S$ for $n \in \mathbb{N}$, and let

$A: S \to S'$, $A_n: S_n \to S'$ for $n \in \mathbb{N}$.

(A_n) is said to "converge discretely" to A (see e.g. [28], [29]) iff

(i) $d(s,S_n) := \inf\{d(s,\tilde{s}) \mid \tilde{s} \in S_n\} \to 0$, for all $s \in S$, where d is the metric in S, and

(ii) for all $s \in S$, $s_n \in S_n$, $n \in \mathbb{N}$, with $s_n \to s$ in S, we have $A_n s_n \to As$ in S'.

Remark 3.3:

If (A_n) converges discretely to A, then $A: S \to S'$ is continuous (see [29, Theorem 6.2, p.239]). For the relevance of "discrete convergence" of mappings we refer to the extensive literature on numerical functional analysis and "discretization methods" for operator equations (see e.g. [2], [19], [28], [29], [30]).

Proposition 3.4:

Let T and (T_n) be as above and $\hat{T}$, $(\hat{T}_n)$ be the associated mappings. Then, $\hat{T}$ and $(\hat{T}_n)$ are w-closed if (T_n) converges discretely to T.

Proof:

Using the same idea as in the proof of [21, Theorem 3.1], we define the mappings $\check{T}_n: X \to Y$, $n \in \mathbb{N}$, by

$$\check{T}_n x := \begin{cases} T_n x, & x \in X_n \\ Tx, & x \in X \setminus X_n. \end{cases}$$

Because of $\check{T}_n^{-1}(B) = T_n^{-1}(B) \cup (T^{-1}(B) \setminus X_n)$, for all $B \in \mathcal{B}(Y)$, the mappings $\check{T}_n$, $n \in \mathbb{N}$, are Borel measurable.

Now, let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and (μ_{n_k}) with $\mu_{n_k} \in P(X_{n_k})$, for $k \in \mathbb{N}$, $\mu_{n_k} \xrightarrow{w} \mu$ and $\hat{T}_{n_k}(\mu_{n_k}) \xrightarrow{w} \nu$ be chosen arbitrarily.

To prove that this implies $\hat{T}(\mu) = \nu$ (i.e.,"w-closedness"), we use Rubin's Theorem ([6, Theorem 5.5]) to show that

$\hat{T}_{n_k}(\mu_{n_k}) = \mu_{n_k} T_{n_k}^{-1} = \mu_{n_k} \tilde{T}_{n_k}^{-1} \xrightarrow{w} \mu T^{-1}$ and hence $\mu T^{-1} = \nu$, i.e., $\hat{T}(\mu) = \nu$. To this end, we first show that the sequence $(\tilde{T}_{n_k})$ satisfies the assumptions of that Theorem. Let x, $x_k \in X$, $k \in \mathbb{N}$, with $x_k \to x$ be arbitrary. We have to prove that

$$\tilde{T}_{n_k} x_k \to Tx. \tag{3.3}$$

If the set $K := \{k \in \mathbb{N} \mid x_k \in X_{n_k}\}$ is finite, then (3.3) holds because of the continuity of T (see Remark 3.3). Now, let K be infinite, i.e., $K = \{k_1 < k_2 < \ldots\}$. Since (T_n) converges discretely to T, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in X_n$ for all $n \in \mathbb{N}$, that converges in X to x. We define a further sequence by

$$\bar{x}_n := \begin{cases} x_{k_j}, & \text{if } n = n_{k_j} \text{ for some } j \in \mathbb{N}, \\ \tilde{x}_n, & \text{otherwise}, \end{cases} \qquad \in X,\ n \in \mathbb{N}.$$

Clearly, $(\bar{x}_n)$ converges to x and, hence, $T_n \bar{x}_n \to Tx$. This implies $T_{n_{k_j}} x_{k_j} \to Tx$. This, together with the continuity of T, yields that every subsequence of $(\tilde{T}_{n_k} x_k)$ converges to Tx. Hence, (3.3) is proved and we can apply [6, Theorem 5.5] and obtain:

$\mu_{n_k}^{e} \tilde{T}_{n_k}^{-1} = \mu_{n_k} T_{n_k}^{-1} = \hat{T}_{n_k}(\mu_{n_k}) \xrightarrow{w} \mu T^{-1} = \nu.$

This completes the proof. □

Remark 3.5:

Note that in the proof of Proposition 3.4 we did not use the assumption that $\hat{T}_{n_k}(\mu_{n_k})$ converges weakly; indeed, we proved that if $\mu_{n_k} \xrightarrow{w} \mu$ then $\hat{T}_{n_k}(\mu_{n_k}) \xrightarrow{w} \hat{T}\mu$. This implies that if $X_n = X$ for all

$n \in \mathbb{N}$, then we can even conclude in Proposition 3.4 that $(\hat{T}_n)$ converges discretely to $\hat{T}$ (see Definition 3.2).
If the X_n are not necessarily all equal, then one would have to show that for all $\mu \in P(X)$ there is a sequence $\mu_n \in P(X_n)$ with $\mu_n \xrightarrow{w} \mu$ in order to prove discrete convergence. This holds for all $\mu \in P(X)$ with the property that $\mu(X_n) \to 1$, as can be seen as follows: For sufficiently large $n \in \mathbb{N}$, let $\mu_n(A) := \frac{1}{\mu(X_n)} \cdot \mu(A \cap X_n)$ for $A \in B(X)$; we consider μ_n also as a probability measure on X_n. One can conclude from the Portmanteau Theorem ([6,Theorem 2.1]) that $\mu_n \xrightarrow{w} \mu$. However, it can happen that $\mu_n \xrightarrow{w} \mu$ also if $\mu(X_n) \not\to 1$, as the example $X := [0,1]$, $X_n := [\frac{1}{n}, 1]$, $\mu(\{0\}) = 1$, $\mu_n(\{\frac{1}{n}\}) = 1$ shows: Here $\mu_n \in P(X_n)$, $\mu_n \xrightarrow{w} \mu$, but $\mu(X_n) = 0$ for all $n \in \mathbb{N}$. Thus, we do not know if the conditions of Proposition 3.4 imply that $\hat{T}_n$ converges discretely to $\hat{T}$ in general.

Another natural question is if discrete convergence of (T_n) to T could be replaced by closedness (see [28]) of (T_n), T in the assumptions of Proposition 3.4. This is not the case, as the following example shows:
Let $X := \mathbb{R}_0^+$, Y be the metric space of compact subset of $[0,1]$, $T: X \to Y$ be defined by $T(x) := \{1\}$ for $x > 0$, $T(0) := [0,1]$, $X_n = X$, $T_n = T$ for all $n \in \mathbb{N}$. Then (T_n), T is closed. For all $n \in \mathbb{N}$, let $\mu_n \in P(X_n)$ be defined by $\mu_n(\{0\}) = \frac{1}{2}$, $\mu_n(\{\frac{1}{n}\}) = \frac{1}{2}$. Then $\mu_n \xrightarrow{w} \mu$ with $\mu(\{0\}) = 1$, but $(\hat{T}_n\mu_n)$ does not converge to $\hat{T}\mu$. Thus, $(\hat{T}_n)$, $\hat{T}$ is not w-closed.

After having made the observations 3.1 and 3.4, we now turn to sufficient conditions for the w-regularity of the sequence $(\hat{T}_n)$.

<u>Definition 3.6:</u>

Let S, S', $S_n \subseteq S$, $n \in \mathbb{N}$, and A, A_n, $n \in \mathbb{N}$, be as in 3.2.

a) A is called "regular" iff $A^{-1}(K) \cap B$ is relatively compact in S for each bounded $B \subseteq S$ and compact $K \subseteq S'$.

b) (A_n) is called "collectively regular" iff $\bigcup_{n \in \mathbb{N}} A_n^{-1}(K) \cap B$

is relatively compact in S for each bounded $B \subseteq S$ and compact $K \subseteq S'$.

c) (A_n) is called "asymptotically regular", in abbrevation "A-regular", iff for all $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and all $s_{n_k} \in S_{n_k}$, $k \in \mathbb{N}$, the sequence (s_{n_k}) is relatively compact in S if (s_{n_k}) is bounded in S and $(A_{n_k} s_{n_k})$ is relatively compact in S'. (Note that it obviously suffices to prove relative compactness of (s_{n_k}) only for those sequences where $(A_{n_k} s_{n_k})$ is convergent.)

The notions of regularity and A-regularity are known to be important for approximations of operator equations (see e.g. [2], [19], [28], [3o]). In the sequel, the "collective regularity" of the sequence (T_n) turns out to be useful for the study of approximations of stochastic equations (compare Prop.3.1o). The next result provides a link between the introduced regularity concepts.

Lemma 3.7:

Let S, S', $S_n \subseteq S$, $n \in \mathbb{N}$, and A, A_n, $n \in \mathbb{N}$, be as in 3.2. (A_n) is collectively regular if and only if (A_n) is A-regular and A_n is regular for each $n \in \mathbb{N}$.

Proof:

Let (A_n) be collectively regular. Since the definitions imply that A_n is regular for each $n \in \mathbb{N}$, it remains to be shown that (A_n) is A-regular. To this end, let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$, $s_{n_k} \in S_{n_k}$, $k \in \mathbb{N}$, be such that (s_{n_k}) is bounded in S and $(A_{n_k} s_{n_k})$ is convergent. We put $B := \{s_{n_k} | k \in \mathbb{N}\}$, define K as the closure of $\{A_{n_k} s_{n_k} | k \in \mathbb{N}\}$ and conclude that $\bigcup_{n \in \mathbb{N}} A_n^{-1}(K) \cap B$ is relatively compact in S. Because of

$s_{n_k} \in A_{n_k}^{-1}(K) \cap B \subseteq \bigcup_{n \in \mathbb{N}} A_n^{-1}(K) \cap B$, for all $k \in \mathbb{N}$, (s_{n_k}) is relatively compact and, hence, (A_n) is A-regular. For the converse, assume that (A_n) is A-regular and that A_n is regular for each $n \in \mathbb{N}$.

Let $B \subseteq S$ be bounded and $K \subseteq S'$ be compact and let (w_k) be an arbitrary sequence in $\bigcup_{n \in \mathbb{N}} A_n^{-1}(K) \cap B$. It suffices to prove that (w_k) has a convergent subsequence. To this end, we distinguish two cases.

<u>Case 1:</u> There exists $n_o \in \mathbb{N}$ such that the set $(A_{n_o}^{-1}(K) \cap B) \cap \{w_k | k \in \mathbb{N}\}$ is infinite. Then the regularity of A_{n_o} completes the proof.

<u>Case 2:</u> For each $n \in \mathbb{N}$ the set $(A_n^{-1}(K) \cap B) \cap \{w_k | k \in \mathbb{N}\}$ is finite. Now, we construct a subsequence of (w_k) by induction as follows. Let $n_1 \in \mathbb{N}$ be such that $w_1 \in A_{n_1}^{-1}(K) \cap B$ and $k_1 := 1$.
Assume that $k_1 < k_2 < \dots < k_{i-1}$, $n_1 < n_2 < \dots, n_{i-1} \in \mathbb{N}$ are already defined with $w_{k_{j-1}} \in A_{n_{j-1}}^{-1}(K) \cap B$ for $1 \leq j \leq i$.
Then we have that $\{n \in \mathbb{N} | (A_n^{-1}(K) \cap B) \cap \{w_k | k > k_{i-1}\} \neq \emptyset, n > n_{i-1}\}$ is nonempty (since the contrary would imply case 1). Hence, there are $k_i, n_i \in \mathbb{N}$ such that $k_{i-1} < k_i$, $n_{i-1} < n_i$ and

$$w_{k_i} \in A_{n_i}^{-1}(K) \cap B. \tag{3.4}$$

This inductive procedure defines a subsequence (w_{k_i}) of (w_k).
Now, we define a new sequence $s_{n_i} := w_{k_i}$, $i \in \mathbb{N}$, and have from (3.4) that $s_{n_i} \in S_{n_i}$, $i \in \mathbb{N}$, (s_{n_i}) is bounded and $(A_{n_i} s_{n_i})$ is a sequence in K, hence, is relatively compact.
Because of the A-regularity of (A_n), this implies that (s_{n_i}) is relatively compact, i.e., $(s_{n_i}) = (w_{k_i})$ has a convergent subsequence; thus, (w_k) has a convergent subsequence, which completes the proof.

□

<u>Remark 3.8:</u>

Because of Lemma 3.7, the well-developed concept of A-regular operator approximations (see e.g. [2], [19], [3o] and also [26] for the related concept of A-proper maps) also applies to collectively regular operator approximations. Note that in many applications the approximations A_n, $n \in \mathbb{N}$, are defined on finite-dimensional spaces and hence are regular, because sets of the form $A_n^{-1}(K) \cap B$ are then bounded subsets of finite-dimensional spaces and, hence, are relatively compact.

The next result shows that (as one would expect in the view of [2]) the collective regularity of a sequence of mappings (with values in a linear space) is invariant under additive "collectively compact perturbations":

Lemma 3.9:

Let S be a metric space, S_n, $n \in \mathbb{N}$, be subsets of S, S' be a linear metric space and A_n, C_n: $S_n \to S'$, $n \in \mathbb{N}$, be such that (A_n) is collectively regular and (C_n) is "collectively compact", i.e.,

$$\bigcup_{n \in \mathbb{N}} C_n(B) \text{ is relatively compact for each bounded } B \subseteq S. \tag{3.5}$$

Then $(A_n + C_n)$ is collectively regular.

Proof:

Let $B \subseteq S$ be bounded and $K \subseteq S'$ be compact, and let us consider

$$V := \bigcup_{n \in \mathbb{N}} (A_n + C_n)^{-1}(K) \cap B. \tag{3.6}$$

We show that there is a relatively compact subset of S which contains V. To this end, let $s \in V$ be arbitrary.

By (3.6) there exists $n_o \in \mathbb{N}$ such that

$$(A_{n_o} + C_{n_o})s \in K \text{ and } s \in B. \tag{3.7}$$

Let K' be the closure of $\bigcup_{n \in \mathbb{N}} C_n(B)$; we obtain from (3.7) that

$$s \in A_{n_o}^{-1}((-K')+K) \cap B \subset \bigcup_{n \in \mathbb{N}} A_n^{-1}((-K')+K) \cap B. \tag{3.8}$$

By (3.5), K' is compact in S' and, therefore, $(-K')+K$ is compact. This, together with the collective regularity of (A_n) and with (3.8), implies that V is contained in the relatively compact subset $\bigcup_{n \in \mathbb{N}} A_n^{-1}((-K')+K) \cap B$ of S. Since B and K were chosen arbitrarily, this implies that $(A_n + C_n)$ is collectively regular.

□

Proposition 3.1o:

Let Y be Polish, (T_n) be as above and $(\hat{T}_n)$ be the sequence of associated mappings.

Then $(\hat{T}_n)$ is w-regular if (T_n) is collectively regular.

Proof:

Let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$, $\mu_{n_k} \in P(X_{n_k})$, $k \in \mathbb{N}$, (μ_{n_k}) be w-bounded and $(\hat{T}_{n_k}(\mu_{n_k}))$ be weakly convergent in $P(Y)$. By Prokhorov's Theorem it suffices to prove that $(\mu^e_{n_k})$ is tight (cf. Def. 2.2).

Let $\varepsilon > 0$ be arbitrary, but fixed. By assumption, there exist a bounded $B_\varepsilon \in \mathcal{B}(X)$ and a compact $K_\varepsilon \subset Y$ such that

$$\inf_{k \in \mathbb{N}} \mu^e_{n_k}(B_\varepsilon) \geq 1 - \frac{\varepsilon}{2} \text{ and } \inf_{k \in \mathbb{N}} \mu_{n_k}(T^{-1}_{n_k}(K_\varepsilon)) \geq 1 - \frac{\varepsilon}{2} . \tag{3.9}$$

(The latter again follows from Prokhorov's Theorem [6, Theorem 6.2] since Y is Polish.)

Let $\tilde{K}_\varepsilon$ denote the closure of $\bigcup_{n \in \mathbb{N}} T^{-1}_n(K_\varepsilon) \cap B_\varepsilon$. Since (T_n) is collectively regular, $\tilde{K}_\varepsilon$ is compact in X and we obtain from (3.9) that

$$\begin{aligned} \mu^e_{n_k}(X \setminus \tilde{K}_\varepsilon) &\leq \mu^e_{n_k}(X \setminus (\bigcup_{n \in \mathbb{N}} T^{-1}_n(K_\varepsilon) \cap B_\varepsilon)) \leq \mu^e_{n_k}(X \setminus (T^{-1}_{n_k}(K_\varepsilon) \cap B_\varepsilon)) \\ &\leq \mu^e_{n_k}(X \setminus T^{-1}_{n_k}(K_\varepsilon)) + \mu^e_{n_k}(X \setminus B_\varepsilon) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for all } k \in \mathbb{N} . \end{aligned}$$

Hence $(\mu^e_{n_k})$ is tight and the proof is complete.

□

The main convergence result of this Section is now an immediate consequence of Theorem 2.3, the Propositions 3.1, 3.4 and 3.10, and of Lemma 3.7.

Theorem 3.11:

Let X and Y be Polish spaces, $T: X \to Y$ be measurable, X_n, $n \in \mathbb{N}$, be Borel subsets of X, $T_n: X_n \to Y$, $n \in \mathbb{N}$, be regular and measurable and $\hat{T}$, $\hat{T}_n$, $n \in \mathbb{N}$, be the associated mappings defined as above, and let ν, $\nu_n \in P(Y)$, $n \in \mathbb{N}$.

Assume that (T_n) is A-regular and discretely convergent to T and that $\nu_n \xrightarrow{w} \nu$.

Then, every w-bounded sequence (μ_n) of solutions of (3.2) (for the index $n \in \mathbb{N}$) is discretely w-compact and every limit of a weakly convergent subsequence is a solution of (3.1).
If, furthermore, T is injective, then (μ_n) even converges weakly to the unique solution of (3.1).

Remark 3.12:

Note that the assumptions of Theorem 3.11 imply that the mapping $T: X \to Y$ is continuous (see Remark 3.3) and regular. Indeed, it can be shown similarly as in [2, Theorem 4.1] that the regularity of T is implied by the discrete convergence of (T_n) to T and (already) by the A-regularity of (T_n).
For a discussion of various aspects of Theorem 3.11 we refer to the Remarks 2.4 and 4.11.

4. APPROXIMATIONS OF NONLINEAR RANDOM OPERATOR EQUATIONS: WEAK CONVERGENCE OF DISTRIBUTIONS OF APPROXIMATE RANDOM SOLUTIONS

Throughout this Section, let X,Y and Z be Polish spaces and y and z be Y-valued and Z-valued random variables (defined on some probability space (Ω, A, P)). Let T be a mapping from $Z \times X$ into Y that is Borel measurable, i.e., measurable with respect to $\mathcal{B}(Z \times X)$ and $\mathcal{B}(Y)$.
We will be concerned with the random operator equation

$$T(z(\omega), x) = y(\omega) \qquad (\omega \in \Omega). \tag{4.1}$$

In the sequel, we make use of the following solution concepts for equation (4.1) and, in particular, introduce the concept of a "weak solution" of (4.1) which turns out to be suitable in the context of "weak approximations":

Definition 4.1:

a) A mapping $x: \Omega \to X$ is called a "random solution" of (4.1) iff x is measurable and $T(z(\omega), x(\omega)) = y(\omega)$ holds P-almost surely.

b) A mapping $x: \Omega \to X$ is called a "D-solution" of (4.1) iff x is measurable and there exists a random variable $\bar{z}: \Omega \to Z$ with $D(\bar{z}) = D(z)$ such that

$$D(T(\bar{z}(.), x(.))) = D(y). \tag{4.2}$$

c) A probability measure $\mu_X \in P(X)$ is called a "weak solution" of (4.1) iff there exists a probability measure $\mu \in P(Z \times X)$ such that

$$\mu T^{-1} = D(y) \quad \text{and} \tag{4.3}$$

$$\mu_X = \mu p_X^{-1} \text{ , } D(z) = \mu p_Z^{-1} \tag{4.4}$$

where p_X and p_Z are the coordinate projections from $Z \times X$ onto X and Z, respectively.
(Note that the coordinate projections are Borel measurable and that μp_X^{-1} and μp_Z^{-1} are the so-called marginals of $\mu \in P(Z \times X)$.)

Remark 4.2:

The concept of random solutions is already classical in probabilistic functional analysis (see e.g. [4], [2o]). An existence theory for random solutions (of random operator equations) is well-developed (e.g. [9], [24]). The concept of a D-solution was introduced and discussed in [12]. (Note that Definition 4.1 b is an adaptation to the type of random equationswe consider in this paper.) Our motivation for introducing D-solutions stems from the fact that (as proved in [12]) a sequence of random solutions of approximate random operator equations (defined on the same probability space) converges in distribution to a D-solution of the original equation under reasonable conditions (see also Corollary 4.7). If the approximate random operator equations are considered on (possibly) different probability spaces, it turns out (Theorem 4.6) that a sequence of probability

distributions of approximate random solutions converges weakly to a weak solution. Of course, the concept of a weak solution of (4.1) reminds of and is motivated by Ershov's concept (e.g. [14]) for solutions of stochastic equations (see also Remark 4.4).
Finally, we add the following obvious observations: A random solution is also a D-solution of (4.1); the distribution of a D-solution is a weak solution of (4.1) (the latter follows from $D(T(\bar{z}(.),x(.))) = D(\bar{z},x)T^{-1}$ and the fact that $D(x)$ and $D(z) = D(\bar{z})$ are the marginals of $D(\bar{z},x)$). However, a weak solution need not be the distribution of a D-solution. Hence, each solution concept of Definition 4.1 is a strict generalization of the preceding one.

For the treatment of (4.1) and of its approximations we now aim at using the concept and the results of the preceding Sections. However, we first have to transform (4.1) to a suitable form, since in (4.1) the "stochastic inputs" do not only appear in the right-hand side as in (3.1) (which is called a "standard stochastic equation" in [16]). To transform (4.1) into this standard form, we make use of an idea of Ershov ([16, p.606]) and consider the following "induced" stochastic equation:

$$\tilde{T}: Z \times X \to Y \times Z, \quad \tilde{T}(z,x) := (T(z,x),z), \ (z,x) \in Z \times X, \tag{4.5}$$

$$\mu\tilde{T}^{-1} = D(y,z). \tag{4.6}$$

Proposition 4.3:

a) The mapping $\tilde{T}$ is Borel measurable (from $\mathcal{B}(Z \times X)$ to $\mathcal{B}(Y \times Z)$).

b) If $\mu \in P(Z \times X)$ is a solution of (4.6), then μp_X^{-1} is a weak solution of (4.1).

c) $\tilde{T}$ is injective if the mapping $T(z,.):X \to Y$ is injective for each $z \in Z$.

Proof:

a) Since $\mathcal{B}(Y \times Z)$ is the smallest σ-algebra containing all sets of the form $B_1 \times B_2$, $B_1 \in \mathcal{B}(Y)$, $B_2 \in \mathcal{B}(Z)$ (see [25, p.6]), it suffices to note that for sets of this form we have

$$\tilde{T}^{-1}(B_1 \times B_2) = T^{-1}(B_1) \cap p_Z^{-1}(B_2) \in \mathcal{B}(Z \times X). \tag{4.7}$$

b) Let $\mu \in P(Z \times X)$ be a solution of (4.6). It follows from (4.6) and

(4.7) that $D(y)(B_1) = \mu\tilde{T}^{-1}(B_1 \times Z) = \mu T^{-1}(B_1)$, for all $B_1 \in \mathcal{B}(Y)$, and $D(z)(B_2) = \mu\tilde{T}^{-1}(Y \times B_2) = \mu p_Z^{-1}(B_2)$, for all $B_2 \in \mathcal{B}(Z)$. Hence (4.3) and (4.4) hold.

c) Let $(z_1,x_1), (z_2,x_2) \in Z \times X$ be such that $\tilde{T}(z_1,x_1) = \tilde{T}(z_2,x_2)$. This implies $z_1 = z_2$ and $T(z_1,x_1) = T(z_2,x_2) = T(z_1,x_2)$. Thus, by assumption, $x_1 = x_2$. This shows that $\tilde{T}$ is injective.

Remark 4.4:

Proposition 4.3 seems to motivate an alternative for defining a weak solution $\mu_X \in P(X)$ of (4.1), namely, by requiring the existence of a solution $\mu \in P(Z \times X)$ of (4.6) such that $\mu_X = \mu p_X^{-1}$. This notion is stronger than the original one (Prop. 4.3 b!) and seems to be advantageous in some respects, since existence and uniqueness results for solutions of (4.6) are well-known ([14], [15], Prop. 3.1). However, it seems to be a disadvantage that this notion depends on the choice of $\tilde{T}$ and, thus, on the particular way of transforming (4.1) into some "standard form" and not on the original equation alone. For this reason,we feel that the original definition of a weak solution of (4.1) is justified and more suitable than the above alternative.

Now, let additionally for all $n \in \mathbb{N}$ Borel subsets X_n of X, mappings $T_n: Z \times X_n \to Y$ that are Borel measurable from $\mathcal{B}(Z \times X_n)$ to $\mathcal{B}(Y)$ and random variables y_n and z_n with values in Y and Z, respectively (defined on some probability space $(\Omega_n,\mathcal{A}_n,P_n)$), be given. We consider the "approximate" random operator equations

$$T_n(z_n(\omega),x) = y_n(\omega) \quad (\omega \in \Omega_n;\ n \in \mathbb{N}), \tag{4.8}$$

and define mappings $\tilde{T}_n: Z \times X_n \to Y \times Z$, $n \in \mathbb{N}$, analogous to (4.5). In view of Proposition 3.1o, the next observation is important for the proof of our basic convergence result (Theorem 4.6).

Proposition 4.5:

Let T_n and $\tilde{T}_n$, $n \in \mathbb{N}$, be as above and assume that

$$\left.\begin{array}{l}\text{for all } z \in Z, \text{ each } T_n(z,.) \text{ is regular and} \\ (T_n(z,.)) \text{ is A-regular}\end{array}\right\} \tag{4.9}$$

and

$$\left.\begin{array}{l}\text{for all bounded } B \subset X \text{ and compact } K \subset Z, \\ \{T_n(.,x) \mid x \in B \cap X_n,\ n \in \mathbb{N}\} \text{ is equicontinuous on } K.\end{array}\right\} \tag{4.10}$$

Then $(\tilde{T}_n)$ is collectively regular.

Proof:

Let $\bar{B}$ be bounded in $Z \times X$ and $\bar{K}$ be compact in $Y \times Z$, and let $((z_k,x_k))$ be an arbitrary sequence in $\bigcup_{n \in \mathbb{N}} \tilde{T}_n^{-1}(\bar{K}) \cap \bar{B} := \tilde{K}$. It suffices to prove that $((z_k,x_k))$ contains a convergent subsequence. For all $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $(z_k,x_k) \in \tilde{T}_{n_k}^{-1}(\bar{K}) \cap \bar{B}$.

Hence, $B := \{x_k \mid k \in \mathbb{N}\}$ is bounded and

$$\tilde{T}_{n_k}(z_k,x_k) = (T_{n_k}(z_k,x_k),z_k) \in \bar{K} \text{ for all } k \in \mathbb{N}. \tag{4.11}$$

This implies that $(\tilde{T}_{n_k}(z_k,x_k))$ has a convergent subsequence, and we may assume w.l.o.g. that $(\tilde{T}_{n_k}(z_k,x_k))$ converges to some element, say $(y,z) \in \bar{K}$. Hence, we have

$$z_k \to z \text{ (in } Z) \text{ and } T_{n_k}(z_k,x_k) \to y \text{ (in } Y). \tag{4.12}$$

Because of (4.10), $\{T_{n_k}(.,x_k) \mid k \in \mathbb{N}\}$ (as a subset of $\{T_n(.,x) \mid x \in B \cap X_n,\ n \in \mathbb{N}\}$) is equicontinuous on the compact set $p_Z(\bar{K})$. This, together with (4.12), implies by standard arguments that $(T_{n_k}(z,x_k))$ also converges to y. Hence $K_o := \{T_{n_k}(z,x_k) \mid k \in \mathbb{N}\} \cup \{y\}$ is compact in Y and we have

$$\left.\begin{array}{l}x_k \in [T_{n_k}(z,.)]^{-1}(K_o) \cap B \subset \bigcup_{n \in \mathbb{N}} [T_n(z,.)]^{-1}(K_o) \cap B \\ \text{for all } k \in \mathbb{N}.\end{array}\right\} \tag{4.13}$$

Because of (4.9) and Lemma 3.7, the set on the r.h.s. of (4.13) is relatively compact. This implies that (x_k) has a convergent subsequence and hence, by (4.12), $((z_k,x_k))$ has a convergent subsequence. This completes the proof.

□

Assume for the following that random solutions $x_n: \Omega_n \to X_n$ of the approximate equations (4.8) are given for all $n \in \mathbb{N}$. We are now interested in sufficient conditions on T and (T_n) that imply weak convergence of the sequence $(D(x_n))$ to a weak solution of (4.1) if $(D(y_n,z_n))$ converges weakly to $D(y,z)$.
(Note that we do not distinguish whether $D(x_n)$ is viewed as an element of $P(X_n)$ of of $P(X)$. This is justified because the "extension" $D(x_n)^e \in P(X)$ of $D(x_n) \in P(X_n)$ is just the distribution of x_n on X).

Theorem 4.6:

Let T, (T_n), z, (z_n), y, (y_n) be as above and let for all $n \in \mathbb{N}$ x_n be a random solution of (4.8) for the index n. Assume that

(T_n) satisfies the conditions (4.9) and (4.1o), (4.14)

(T_n) converges discretely to T (jointly in both variables) (4.15)

$(D(y_n,z_n))$ converges weakly to $D(y,z)$ (4.16)

$(D(x_n))$ is w-bounded. (4.17)

Then $(D(x_n))$ is discretely w-compact, and every limit of a weakly convergent subsequence is a weak solution of (4.1).
If furthermore $T(\tilde{z},.)$ is injective for all $\tilde{z} \in Z$, then $(D(x_n))$ converges weakly to a weak solution of (4.1).

Proof:

Since x_n is a random solution of (4.8), we have for all $n \in \mathbb{N}$

that $\tilde{T}_n(z_n(\omega), x_n(\omega)) = (y_n(\omega), z_n(\omega))$ P_n-almost surely. This implies $D(z_n,x_n)\tilde{T}_n^{-1} = D(y_n,z_n)$ for all $n \in \mathbb{N}$.
This means that $\mu_n := D(z_n,x_n)$ is a solution of the stochastic equation $\mu\tilde{T}_n^{-1} = \nu_n := D(y_n,z_n)$ for all $n \in \mathbb{N}$.
Because of (4.14) and Proposition 4.5, $(\tilde{T}_n)$ is collectively regular.
Moreover, (4.15) implies that $(\tilde{T}_n)$ converges discretely to $\tilde{T}$.
Hence, we can apply Theorem 3.11 to the present situation noting that (μ_n) is w-bounded because of (4.17), (4.16) (which in particular implies that $(D(z_n))$ is w-bounded) and the simple inequality $D((z_n,x_n))(B_1 \times B_2) \geq D(z_n)(B_1) + D(x_n)(B_2)-1$. Then, Theorem 3.11 yields that $(\mu_n) = (D(z_n,x_n))$ is discretely w-compact and every limit μ of a weakly convergent subsequence of (μ_n) is a solution of (4.6), i.e., we have $\mu\tilde{T}^{-1} = D(y,z)$.
This implies that also $(\mu_n p_X^{-1}) = (D(x_n))$ is discretely w-compact and every limit of a weakly convergent subsequence has the form μp_X^{-1}, where μ is a solution of (4.6)(because of the continuity of p_X and [6, Theorem 5.1 and Lemma 1, p.38]). Because of Proposition 4.3 b, $\mu_X := \mu p_X^{-1}$ is a weak solution of (4.1).
Now, let $T(\tilde{z},.)$ be injective for all $\tilde{z} \in Z$. Because of Proposition 4.3 c, $\tilde{T}$ is injective. Hence, the second part of Theorem 3.11 implies that $(D(z_n,x_n))$ converges weakly to the unique solution μ of (4.6) and, thus, $(D(x_n))$ converges weakly to μp_X^{-1} ([6, Theorem 5.1]).
This completes the proof. $\square$

Corollary 4.7:

Let T and (T_n) be as above and let y,z,y_n,z_n, $n \in \mathbb{N}$, be defined on the same probability space (Ω,A,P). Let for all $n \in \mathbb{N}$ x_n be a random solution (on (Ω,A,P)) of (4.8) for the index n.
Assume that (4.14) - (4.17) are fulfilled.
Then every subsequence (x_{n_k}) of (x_n) has a further subsequence that converges in distribution to a D-solution x of (4.1).

Proof:

As in the proof of Theorem 4.6 we have that $(D(z_n,x_n))$ is discretely w-compact and every limit $\mu \in P(Z \times X)$ of a weakly convergent subsequence $(D(z_{n_k},x_{n_k}))$ is a solution of (4.6). Then the main result of [13] implies that there exists a $Z \times X$-valued random variable $(\bar{z},x)$ defined on the same probability space (Ω,A,P) such that $\mu = D(\bar{z},x)$. Hence, $(D(x_{n_k})) = (D(z_{n_k},x_{n_k})\, p_X^{-1})$ converges weakly to $D(\bar{z},x)p_X^{-1} = D(x)$, i.e., (x_{n_k}) converges in distribution to x.

Since $\mu = D(\bar{z},x)$ is a solution of (4.6), we have

$$D(\tilde{T}(\bar{z}(.),x(.))) = D(\bar{z},x)\tilde{T}^{-1} = D(y,z). \tag{4.18}$$

(4.18) implies that

$D(T(\bar{z}(.),x(.))) = D(\tilde{T}(\bar{z}(.),x(.)))p_Y^{-1} = D(y)$ and

$D(\bar{z}) = D(\tilde{T}(\bar{z}(.),x(.)))p_Z^{-1} = D(z)$.

(Here $p_Y: Y \times Z \to Y$ and $p_Z: Y \times Z \to Z$ are the coordinate projections defined on $Y \times Z$.)

Hence, x is a D-solution of (4.1) and the proof is complete. □

Our next result is a sharpening of the convergence part of Theorem 4.6 and turns out to be useful in applications (e.g. in Section 5).

Corollary 4.8:

Let the assumptions of Theorem 4.6 be fulfilled and assume that

$$\left.\begin{array}{l} E:=\{\tilde{z} \in Z \mid T(\tilde{z},.) \text{ is injective}\} \in \mathcal{B}(Z) \text{ and} \\ D(z)(E) = P(\{\omega \mid T(z(\omega),.) \text{ is injective}\}) = 1. \end{array}\right\} \tag{4.19}$$

Then $(D(x_n))$ converges weakly to a weak solution of (4.1).

If furthermore (4.1) has a random solution x, then (x_n) converges in distribution to x.

Proof:

A look at the proof of Theorems 4.6 and 2.3, respectively, shows that the sequence $(D(z_n,x_n))$ (and hence $D(x_n))$ converges weakly if

the solution of (4.6) is unique. Thus, it suffices to show that under our assumptions $\mu\tilde{T}^{-1} = D(y,z)$ implies that $\mu(B)$ is uniquely determined for all $B \in \mathcal{B}(Z \times X)$.
Let $B \in \mathcal{B}(Z \times X)$ and $\mu \in P(Z \times X)$ be such that $\mu\tilde{T}^{-1} = D(y,z)$.
Then (4.19) implies (with the coordinate projection $p_Z: Z \times X \to Z$)

$\mu p_Z^{-1}(E) = \mu(E \times X) = D(z)(E) = 1$, and hence

$$\mu(B) = \mu(B \cap E \times X). \tag{4.2o}$$

As in the proof of Proposition 4.3 c it follows that $\tilde{T}$ viewed as a map from $B \cap E \times X$ into $Y \times Z$ is one-to-one. Then, [25, Theorem 3.9] implies that $\tilde{T}(B \cap E \times X)$ is a Borel subset of $Y \times Z$. This, together with (4.2o), implies that

$\mu(B) = \mu(\tilde{T}^{-1}(\tilde{T}(B \cap E \times X))) = D(y,z)(\tilde{T}(B \cap E \times X))$,

i.e., $\mu(B)$ is uniquely determined.

Finally, let x be a random solution of (4.1).
Then, $\tilde{T}(z(\omega),x(\omega)) = (T(z(\omega),x(\omega)),z(\omega)) = (y(\omega),z(\omega))$ holds P-almost surely. Hence $D(z,x)$ is the unique solution of (4.6) and, thus, $(D(x_n))$ converges weakly to $D(z,x)p_X^{-1} = D(x)$. This completes the proof.

□

Now, let us consider two types of particular situations for which Theorem 4.6 and its Corollaries are applicable and which correspond to the applications we study in Section 5.

Example 4.9:

(i) Let $X_n = X$ and $T_n = T$ for all $n \in \mathbb{N}$. Then (4.14) and (4.15) are implied by the following condition:

$$\left.\begin{array}{l} T: Z \times X \to Y \text{ is continuous,} \\ T(\tilde{z},.) \text{ is regular for all } \tilde{z} \in Z, \text{ and} \\ \{T(.,x) \mid x \in B\} \text{ is equicontinuous on } K, \text{ for all} \\ \text{bounded } B \subset X \text{ and compact } K \subset Z. \end{array}\right\} \tag{4.21}$$

(ii) Let $X = Y$ be a separable Banach space, $C: Z \times X \to X$ and $C_n: Z \times X_n \to X$, $n \in \mathbb{N}$, be mappings such that

$$(C_n(\tilde{z},.)) \text{ is collectively compact for all } \tilde{z} \in Z, \tag{4.22}$$

$$\left.\begin{array}{l}\text{for all bounded } B \subset X \text{ and compact } K \subset Z,\\ \{C_n(.,x) \mid x \in B \cap X_n,\ n \in \mathbb{N}\} \text{ is equicontinuous on } K,\end{array}\right\} \qquad (4.23)$$

$$(C_n) \text{ converges discretely to } C. \qquad (4.24)$$

If we put $T(\tilde{z},.) = I - C(\tilde{z},.)$, $T_n(\tilde{z},.) = I - C_n(\tilde{z},.)$, $\tilde{z} \in Z$, $n \in \mathbb{N}$, (4.22), (4.23) and (4.24) imply (4.14) (because of Lemma 3.9) and (4.15), respectively.

Remark 4.1o:

In Theorem 4.6 and its Corollaries we need a sequence of random solutions (x_n) of (4.8). As a first remark addressed to this assumption, we note that all that is really needed in the proof of Theorem 4.6 is that the joint distribution $D(z_n,x_n)$ solves the stochastic equation $\mu\tilde{T}_n^{-1} = D(y_n,z_n)$ $(n \in \mathbb{N})$.
Our second remark shows that it suffices to know a solution of (4.8) only on a measurable subset of X_n whose probability tends to 1 as $n \to \infty$: Assume that for all $n \in \mathbb{N}$ there exist $A_n \in \mathcal{A}_n$ and a measurable mapping $x_n: A_n \to X$ such that

$$T_n(z_n(\omega),x_n(\omega)) = y_n(\omega) \text{ for } P_n\text{-almost all } \omega \in A_n,$$

and that $\lim\limits_{n\to\infty} P_n(A_n) = 1$.

Then we define for all $n \in \mathbb{N}$

$$\bar{x}_n(\omega) := \begin{cases} x_n(\omega), & \omega \in A_n \\ \tilde{x}_n(\omega), & \omega \notin A_n \end{cases}, \quad \bar{y}_n(\omega) := \begin{cases} y_n(\omega) & , \omega \in A_n \\ T_n(z_n(\omega),\tilde{x}_n(\omega)) & , \omega \notin A_n \end{cases}$$

where $\tilde{x}_n$ is some X-valued random variable on $(\Omega_n,\mathcal{A}_n,P_n)$, and have that $D(z_n,\bar{x}_n)\tilde{T}_n^{-1} = D(\bar{y}_n,z_n)$.

In order to apply Theorem 4.6 to the sequence $(\bar{x}_n)$ we note that $D(\bar{y}_n,z_n) \xrightarrow{w} D(y,z)$ if $D(y_n,z_n) \xrightarrow{w} D(y,z)$. To see this, let B be an arbitrary closed subset of $Y \times Z$. Then we have for $n \in \mathbb{N}$

$$\begin{aligned} D(\bar{y}_n,z_n)(B) &= P_n(\{\omega \in \Omega_n \mid (\bar{y}_n(\omega),z_n(\omega)) \in B\}) \\ &\le P_n(\{\omega \in A_n \mid (y_n(\omega),z_n(\omega)) \in B\}) + P_n(\Omega_n \setminus A_n) \\ &\le D(y_n,z_n)(B) + P_n(\Omega_n \setminus A_n) \end{aligned}$$

and, hence, $\limsup_{n\to\infty} D(\bar{y}_n, z_n)(B) \le \limsup_{n\to\infty} D(y_n, z_n)(B) \le D(y,z)(B)$ because of the Portmanteau Theorem, which then also yields the desired result.
Hence, we can apply Theorem 4.6 to the sequence $(\bar{x}_n)$ if $(D(\bar{x}_n))$ is w-bounded. This latter condition obviously only depends on the properties of (x_n) and, hence, is independent of the particular choice of the $\tilde{x}_n$. This argument is useful in applications and will be used in Remark 5.5.

Remark 4.11:

Theorem 4.6 states that under certain (deterministic) assumptions on T and (T_n), a sequence $(D(x_n))$ of approximate "solution measures" is discretely w-compact or (even) weakly convergent (to a weak solution of (4.1)) if $(D(x_n))$ is w-bounded and $D(y_n, z_n) \xrightarrow{w} D(y,z)$.
The assumptions (4.15) and (4.16) are natural in this approximation context, the "regularity assumption" (4.9) is known to be important also for deterministic operator approximation (see Sect. 3), and (4.1o) is not too restrictive: These assumptions can be checked in concrete situations, as can be seen in the Sections 5 and 6, and in [12, Sect.3].
Concerning the w-boundedness assumption (4.17), we feel that there exist various approaches to show this property. Some of them are used in the Sections 5 and 6. And after all, a-priori bounds are also frequently used in deterministic approximation theory.

Finally we note that in view of Example 4.9 (ii) and Corollary 4.7, Theorem 4.6 extends and refines [12, Theorem 2.11]. Theorem 4.6 can now be applied to more general equations and "approximations" and gives (together with Corollary 4.8) a criterion for weak convergence of $(D(x_n))$ which is easy to check.

5. APPROXIMATION OF SOLUTIONS OF RANDOM FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

In this Section, we are concerned with random Fredholm integral equations of the second kind (for an introduction see [4, Chapt.4]). We apply the results described in Section 4 to certain approximation procedures for such equations, namely, a method based on the approximation of the random kernel by random degenerate kernels and a numerical method based on quadrature approximations. For a description of the underlying numerical methods in the deterministic case we refer e.g. to [17, vol.4]. Selected references for the "stochastic" case are [5, Sect. 4], [7], [12, Sect. 3] and [23].

Formulating our setup in the framework of Section 4, we put $X = Y: = L_2([0,1])$, $Z: = L_2([0,1]^2)$ and define $T: Z \times X \to X$ by

$$T(z,x): = x - \int_0^1 z(.,s) \quad x(s)ds, \text{ for } x \in X, \ z \in Z. \tag{5.1}$$

Note that T is well-defined and is (jointly) continuous (and thus Borel measurable as needed in Sect. 4).

Let z, z_n, $n \in \mathbb{N}$ and y, y_n, $n \in \mathbb{N}$, be Z-valued and Y-valued random variables, respectively (on possibly different probability spaces (Ω, A, P) and (Ω_n, A_n, P_n), $n \in \mathbb{N}$). Consider the random Fredhom integral equation

$$T(z(\omega),x) = x - \int_0^1 z(\omega,.,s)x(s)ds = y(\omega) \quad (\omega \in \Omega) \tag{5.2}$$

and its "approximations"

$$T(z_n(\omega),x) = y_n(\omega) \quad (\omega \in \Omega_n, \ n \in \mathbb{N}). \tag{5.3}$$

We note that this framework already contains "kernel approximation methods" mentioned above (see e.g. [5, Sect.4]) if the "approximations" z_n and y_n ($n \in \mathbb{N}$) are required to belong to certain finite-dimensional subspaces of Z and Y, respectively.

Lemma 5.1:

The set $E: = \{\tilde{z} \in Z \mid T(\tilde{z},.) \text{ is injective}\}$ is Borel.

If $D(z)(E) \geq r$, for some $r \in [0,1]$, and if $D(z_n) \xrightarrow{w} D(z)$ then we have $\liminf_{n\to\infty} D(z_n)(E) \geq r$. Especially, if $D(z)(E) = 1$, then $\lim_{n\to\infty} D(z_n)(E) = 1$.

Proof:

Let $\tilde{z} \in E$ be arbitrary. Then $T(\tilde{z},.)$ is injective and Fredholm, hence, $T(\tilde{z},.): X \to X$ is continuously invertible. Let $w \in Z$ be such that $\|w-\tilde{z}\|_Z < \frac{1}{2} \|[T(\tilde{z},.)]^{-1}\|^{-1}$, where the latter is the operator norm and $\|.\|_Z$ denotes the norm on Z. We have

$$\|(T(w,.) - T(\tilde{z},.))\| \, \|[T(\tilde{z},.)]^{-1}\| \leq \|w-\tilde{z}\|_Z \, \|[T(\tilde{z},.)]^{-1}\| < \frac{1}{2} .$$

Then, the well-known perturbation lemma implies that $T(w,.): X \to X$ is continuously invertible, i.e., $w \in E$. Thus E is open and, hence, is Borel.

If $D(z)(E) \geq r$ and $D(z_n) \xrightarrow{w} D(z)$, [6, Theorem 2.1] implies $\liminf_{n\to\infty} D(z_n)(E) \geq D(z)(E) \geq r$, since E is open in Z. □

Note that, by the same arguments, this result is also true for the case that $Z: = C([0,1]^2)$ and $X: = C([0,1])$.

Theorem 5.2:

Let $(D(y_n,z_n))$ be weakly convergent to $D(y,z)$ and let for all $n \in \mathbb{N}$ x_n be a random solution of (5.3) (for the index $n \in \mathbb{N}$). Assume that $P(\{\omega \in \Omega | T(z(\omega),.) \text{ is injective}\}) = 1$. Then (x_n) converges in distribution to the (a.s.) unique random solution of (5.2).

Proof:

We apply Theorem 4.6 and Corollary 4.8. To this end, we have to check the assumptions of that Theorem (since (4.19) is fulfilled by Lemma 5.1 and our assumption). Hence, it remains to be shown that (4.21) holds (cf. Example 4.9) and that $(D(x_n))$ is w-bounded. To begin with, we first note that $T: Z \times X \to X$ is continuous and $T(\tilde{z},.)$ is regular for all $\tilde{z} \in Z$, since $T(\tilde{z},.) = I-C(\tilde{z},.)$, where the mapping $C(\tilde{z},.): X \to X$ is compact for $\tilde{z} \in Z$ (see also Lemma 3.9). Furthermore, it holds for all $\bar{z}$, $\tilde{z} \in Z$ and $\bar{x} \in X$ that $\|T(\bar{z},\bar{x}) - T(\tilde{z},\bar{x})\|_X \leq \|\bar{z}-\tilde{z}\|_Z \, \|\bar{x}\|_X$. This implies that $\{T(.,\bar{x}) | \bar{x} \in B\}$ is uniformly equicontinuous (even) on Z, for all bounded $B \subset X$. Hence, (4.21) is fulfilled.

Now, we show that $(D(x_n))$ is w-bounded. Let $\epsilon > 0$ be arbitrary, but fixed. Since $T(z(\omega),.): X \to X$ is linear, continuous and bijective for P-almost all $\omega \in \Omega$, $\|[T(z(.),.)]^{-1}\|$ (possibly modified on a subset of Ω with probability zero) is a real-valued random variable (see [20, p.192]). Hence there exists $r_\varepsilon > 0$ such that

$$D(z)(F_\varepsilon) \geq 1 - \frac{\varepsilon}{3}\,, \tag{5.4}$$

where $F_\varepsilon := \{\tilde{z} \in Z \mid T(\tilde{z},.) \text{ is injective}, \|[T(\tilde{z},.)]^{-1}\| < r_\varepsilon\}$.
Let $\tilde{z} \in F_\varepsilon$ and $w \in Z$ be such that $\|\tilde{z}-w\|_Z < \alpha$. Then we have

$$\|T(w,.) - T(\tilde{z},.)\| \, \|[T(\tilde{z},.)]^{-1}\| < \alpha r_\varepsilon$$

and, by the perturbation lemma, $T(w,.)$ is continuously invertible with

$$\|[T(w,.)]^{-1}\| < \frac{\|[T(\tilde{z},.)]^{-1}\|}{1-\alpha r_\varepsilon}$$

if $\alpha r_\varepsilon < 1$.

Hence, there exists $\alpha_\varepsilon > 0$ such that $\|[T(w,.)]^{-1}\| < r_\varepsilon$ if $\|w-\tilde{z}\|_Z < \alpha_\varepsilon$. Thus, F_ε is an open subset of Z.
Then the Portmanteau Theorem ([6, Theorem 2.1]) and (5.4) imply $\liminf_{n\to\infty} D(z_n)(F_\varepsilon) \geq D(z)(F_\varepsilon) \geq 1 - \frac{\varepsilon}{3}$. Hence, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(z_n)(F_\varepsilon) \geq 1 - \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0. \tag{5.5}$$

Since $(D(y_n))$ is weakly convergent and hence tight, there exists $C_\varepsilon > 0$ such that

$$D(y_n)(\{\tilde{y} \in Y \mid \|\tilde{y}\|_Y \leq C_\varepsilon\}) \geq 1 - \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}\,. \tag{5.6}$$

Let $n \geq n_0$ be arbitrary, but fixed and define

$A_1 := \{\omega \in \Omega_n \mid \|y_n(\omega)\|_Y \leq C_\varepsilon\}$, $A_2 := \{\omega \in \Omega_n \mid z_n(\omega) \in F_\varepsilon\}$ and

$A_3 := \{\omega \in \Omega_n \mid \|x_n(\omega)\|_X < C_\varepsilon r_\varepsilon\}$.

Let $N \in \mathcal{A}_n$ with $P_n(N) = 0$ be such that for all $\omega \in \Omega_n \setminus N$, $T(z_n(\omega), x_n(\omega)) = y_n(\omega)$ holds; such an N exists, since x_n is a random solution of (5.3). Now, let $\omega \in (\Omega \setminus N) \cap A_1 \cap A_2$. Then we have $\|x_n(\omega)\| \leq \|[T(z_n(\omega),.)]^{-1}\| \, \|y_n(\omega)\| < r_\varepsilon C_\varepsilon$, and hence $\omega \in A_3$.

This implies that

$$\Omega_n \setminus A_3 \subset N \cup (\Omega_n \setminus A_1) \cup (\Omega_n \setminus A_2)$$

and hence, because of (5.5) and (5.6) and the definitions of A_1 and A_2, $P_n(\Omega_n \setminus A_3) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Since $n \geq n_o$ was arbitrary, this means

$$D(x_n)(\{\tilde{x} \in X \mid \|\tilde{x}\|_X < C_\varepsilon r_\varepsilon\}) \geq 1-\varepsilon \quad \text{for all } n \geq n_o. \tag{5.7}$$

Since $\{D(x_n) \mid 1 \leq n \leq n_o\}$ is tight by [6, Theorem 1.4] and $\varepsilon > 0$ was arbitrary, (5.7) implies that $(D(x_n))$ is w-bounded.

Finally, since $T(z(.),.): \Omega \times X \to X$ is a continuous random operator that is bijective P-almost surely, [24, Theorem 1] implies that (5.2) has a (P-almost surely) unique random solution x. Thus it follows from Corollary 4.8 that (x_n) converges in distribution to x.

□

Remark 5.3:

An inspection of the above proof shows that $D(z_n) \xrightarrow{w} D(z)$ and

$$P(\{\omega \in \Omega \mid T(z(\omega),.) \text{ is injective}\}) = 1 \tag{5.8}$$

imply that $(D(x_n))$ is w-bounded (tight) if $(D(y_n))$ is w-bounded (tight). This observation extends [5, Theorem 4.1] in the sense that instead of requiring the convergence of the sequence (z_n) of kernels (defined on the common probability space $(\Omega, \mathcal{A}, P)$) P-almost surely to z we need only its convergence in distribution.

If (5.8) is not fulfilled in the situation of Theorem 5.2, we can still conclude the following from Theorem 4.6 (see also [5, Theorem 4.2]):

If $D(y_n, z_n) \xrightarrow{w} D(y,z)$ and $(D(x_n))$ is w-bounded, then $(D(x_n))$ is discretely w-compact and every limit of a weakly convergent subsequence is a weak solution of (5.2).

Now, we consider a method for solving (5.2) based on quadrature approximations for the integral operator; simultaneously, the "stochastic input" (y,z) will be approximated. For a description of this method we consider the integral operator as acting on C([0,1]) and use the same setting as in [12, Sect. 3.2].

Let $X: = Y: = C([0,1])$, $Z: = C([0,1]^2)$ and define $T: Z \times X \to X$ to be the restriction of the mapping defined by (5.1) to spaces of continuous functions, and $T_n: Z \times X \to X$, $n \in \mathbb{N}$, by

$$T_n(z,x): = x - \sum_{j=0}^{n} \alpha_{nj}\, z(\cdot, s_{nj})\ \ x(s_{nj}),\ x \in X,\ z \in Z, \tag{5.9}$$

where for each $n \in \mathbb{N}$, $\alpha_{no},\dots,\alpha_{nn}$ are the weights of a quadrature formula with nodes $s_{no},\dots,s_{nn}$ (in $[0,1]$).

Note that T and T_n, $n \in \mathbb{N}$, are well-defined and (jointly) continuous.

Let z, z_n, $n \in \mathbb{N}$, and y, y_n, $n \in \mathbb{N}$, be Z-valued and X-valued random variables, respectively (on possibly different probability spaces (Ω,A,P) and (Ω_n,A_n,P_n), $n \in \mathbb{N}$). We again consider the equation (5.2) and its "approximations"

$$\left.\begin{array}{c} T_n(z_n(\omega),x) = x - \sum\limits_{j=0}^{n} \alpha_{nj} z_n(\omega,\cdot,s_{nj}) x(s_{nj}) = y_n(\omega) \\ (\omega \in \Omega_n,\ n \in \mathbb{N}). \end{array}\right\} \tag{5.10}$$

Theorem 5.4:

Let $(D(y_n,z_n))$ be weakly convergent to $D(y,z)$ and let for all $n \in \mathbb{N}$ x_n be a random solution of (5.10) (for the index n). Assume that

$$P(\{\omega \in \Omega / T(z(\omega),.)\ \text{is injective}\}) = 1 \tag{5.11}$$

and

$$\left.\begin{array}{l} \text{the quadrature rule is convergent, i.e., for each} \\ v \in C([0,1]),\ \lim\limits_{n\to\infty} \sum\limits_{j=0}^{n} \alpha_{nj} v(s_{nj}) = \int\limits_0^1 v(s)\, ds\ \text{holds.} \end{array}\right\} \tag{5.12}$$

Then (x_n) converges in distribution to the (a.s.) unique random solution x of (5.2).

Proof:

We proceed similar as in the proof of Theorem 5.2 and show that the conditions of Theorem 4.6 are fulfilled. To this end, we put

$C, C_n: Z \times X \to X,\ n \in \mathbb{N},\quad C(z,x) := \int_0^1 z(.,s)x(s)\,ds$

$$C_n(z,x) := \sum_{j=0}^{n} \alpha_{nj} z(.,s_{nj}) x(s_{nj}). \qquad (x \in X,\ z \in Z)$$

To prove (4.14) and (4.15) (for (T_n) and T), it remains to be shown that (4.22), (4.23) and (4.24) hold (see Example 4.9). Actually, this part of the proof was carried out in [12, Sect. 3.2] (even for a nonlinear integral operator) using essentially (5.12).
Hence, for the application of Theorem 4.6 and Corollary 4.8, respectively, we only have to prove that $(D(x_n))$ is w-bounded, since we can conclude from the proof of Theorem 5.2 that (5.2) has a P-almost surely unique random solution.
The proof is similar to that of Theorem 5.2: It suffices to show that for all $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $K_\varepsilon > 0$ such that

$$D(z_n)(\{\tilde{z} \in Z \mid \|[T_n(\tilde{z},.)]^{-1}\| < K_\varepsilon\}) \geq 1-\varepsilon \text{ for all } n \geq n_0, \qquad (5.13)$$

which corresponds to (5.5).

To this end, let $\varepsilon > 0$ be arbitrary, but fixed, and let $r_\varepsilon > 0$ and $F_\varepsilon \subset Z$ be defined as in (5.4). One can prove analoguosly to the proof of Theorem 5.2 that F_ε is open in $Z = C([0,1]^2)$ and that this implies the existence of an $n_1 \in \mathbb{N}$ such that

$$D(z_n)(F_\varepsilon) \geq 1 - \frac{\varepsilon}{2} \quad \text{for all } n \geq n_1. \qquad (5.14)$$

Since $(D(z_n))$ is w-bounded, there exists $R_\varepsilon > 0$ such that

$$D(z_n)(\{\tilde{z} \in Z \mid \|\tilde{z}\| \leq R_\varepsilon\}) \geq 1 - \frac{\varepsilon}{4} \text{ for all } n \in \mathbb{N}. \qquad (5.15)$$

It is well-known that (5.12) implies that

$$K := \sup_{n \in \mathbb{N}} \sum_{j=0}^{n} |\alpha_{nj}| < \infty \text{ holds.} \qquad (5.16)$$

Then we have for $\tilde{z} \in Z$ with $\|\tilde{z}\| \leq R_\varepsilon$ that

$$\|C_n(\tilde{z},.)\| \leq \sum_{j=0}^{n} |\alpha_{nj}|\, \|\tilde{z}\| \leq KR_\varepsilon \text{ for all } n \in \mathbb{N}.$$

This implies together with (5.15) that

$$D(z_n)(\{\tilde{z}\in Z \mid \|C_n(\tilde{z},.)\| \le KR_\varepsilon\}) \ge 1 - \frac{\varepsilon}{4} \quad \text{for all } n \in \mathbb{N}. \quad (5.17)$$

Now, we define for each $n \in \mathbb{N}$ $h_n: Z \to \mathbb{R}$ by

$$h_n(\tilde{z}) := \|(C_n(\tilde{z},.) - C(\tilde{z},.))C_n(\tilde{z},.)\|, \ \tilde{z} \in Z. \quad (5.18)$$

We note that h_n is Borel measurable for each $n \in \mathbb{N}$, since $C(\tilde{z},x)$, $C_n(\tilde{z},x)$, $n \in \mathbb{N}$, are continuous with respect to $\tilde{z}$ for all $x \in X$ and because of the separability of Z.

Let $\tilde{z}$, $\tilde{z}_n \in Z$, $n \in \mathbb{N}$, with $\tilde{z}_n \to \tilde{z}$. Then, both $(C_n(\tilde{z}_n,x))$ and $(C(\tilde{z}_n,x))$ converge to $C(\tilde{z},x)$ for all $x \in X$ and $(C_n(\tilde{z}_n,.))$ is collectively compact (cf. [17, p.163 ff]). Then, it is a well-known consequence of collectively compact operator approximation theory (see [1, Cor.1.9]) that

$$h_n(\tilde{z}_n) = \|(C_n(\tilde{z}_n,.) - C(\tilde{z}_n,.))C_n(\tilde{z}_n,.)\| \xrightarrow[n \to \infty]{} 0.$$

Hence, [6, Theorem 5.5] implies that $(D(z_n)h_n^{-1})$ converges weakly to μ_0, where μ_0 is the unit mass at $0 \in \mathbb{R}$. Because of the Portmanteau Theorem [6, Theorem 2.1], this implies

$$\liminf_{n\to\infty} D(z_n)h_n^{-1}(\{r\in\mathbb{R} \mid |r| < \frac{1}{2r_\varepsilon}\}) \ge \mu_0(\{r\in\mathbb{R} \mid |r| < \frac{1}{2r_\varepsilon}\}) = 1.$$

Hence, there exists $n_0 \in \mathbb{N}$, $n_0 \ge n_1$, such that for all $n \ge n_0$

$$D(z_n)h_n^{-1}(\{r\in\mathbb{R} \mid |r| < \frac{1}{2r_\varepsilon}\}) = D(z_n)(\{\tilde{z}\in Z \mid h_n(\tilde{z}) < \frac{1}{2r_\varepsilon}\}) \ge 1 - \frac{\varepsilon}{4}. \quad (5.19)$$

Now, we fix $n \in \mathbb{N}$, $n \ge n_0$, and consider the following Borel sets (cf. the proof of Theorem 5.2) in Z:

$$F_1 := F_\varepsilon, \quad F_2 := \{\tilde{z} \in Z \mid \|C_n(\tilde{z},.)\| \le KR_\varepsilon\},$$

$$F_3 := \{\tilde{z} \in Z \mid h_n(\tilde{z}) < \frac{1}{2r_\varepsilon}\},$$

$$F_4 := \{\tilde{z} \in Z \mid T_n(\tilde{z},.) \text{ is continuously invertible and } \|[T_n(\tilde{z},.)]^{-1}\| < 2(1+r_\varepsilon KR_\varepsilon)\}.$$

Let $\tilde{z} \in \bigcap_{i=1}^{3} F_i$. Then a variant of the perturbation lemma [17,p.368, Satz 4] implies that $[T_n(\tilde{z},.)]^{-1}: X \to X$ exists and that

$$\|[T_n(\tilde{z},.)]^{-1}\| \leq \frac{1 + \|[T(\tilde{z},.)]^{-1}\| \, \|C_n(\tilde{z},.)\|}{1 - \|[T(\tilde{z},.)]^{-1}\| \, \|(C_n(\tilde{z},.) - C(\tilde{z},.))C_n(\tilde{z},.)\|}$$

holds.

Hence,

$$\|[T_n(\tilde{z},.)]^{-1}\| < \frac{1 + r_\varepsilon KR_\varepsilon}{1 - r_\varepsilon \frac{1}{2r_\varepsilon}} = 2(1 + r_\varepsilon KR_\varepsilon)$$

and thus $\tilde{z} \in F_4$. This implies that $Z \setminus F_4 \subset \bigcup_{i=1}^{3} (Z \setminus F_i)$ and hence by (5.14), (5.17) and (5.19)

$$D(z_n)(Z \setminus F_4) \leq \sum_{i=1}^{3} D(z_n)(Z \setminus F_i) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \qquad (5.2o)$$

Since $n \in \mathbb{N}$, $n \geq n_o$, was arbitrary, this implies (by definition of F_4) that (5.13) is fulfilled with $K_\varepsilon := 2(1 + r_\varepsilon KR_\varepsilon)$. This completes the proof. □

Note that the second approximation procedure for random Fredholm integral equations of the second kind and the convergence result presented above provide a theoretical basis for the computational analysis performed in [7].

Remark 5.5:

It follows from the proof of Theorem 5.4 that for all $\varepsilon > 0$ there exists $n_o = n_o(\varepsilon) \in \mathbb{N}$ such that

$$P_n(A_n) \geq 1 - \varepsilon \quad \text{for all } n \geq n_o, \qquad (5.21)$$

where $A_n := \{\omega \in \Omega_n \mid T_n(z_n(\omega),.) \text{ is continuously invertible}\} \in \mathcal{A}_n$,

and hence $\lim_{n\to\infty} P_n(A_n) = 1$.

Now, the argument of Remark 4.1o yields that the sequence $(\bar{x}_n)$, defined by

$$\bar{x}_n(\omega) := \begin{cases} [T_n(z_n(\omega),.)]^{-1} y_n(\omega), & \omega \in A_n \\ \tilde{x}_n(\omega), & \omega \notin A_n \end{cases}, \quad n \in \mathbb{N}, \qquad (5.22)$$

where $\tilde{x}_n$ is some X-valued random variable on (Ω_n, A_n, P_n), converges in distribution to the unique random solution of (5.1).
This conclusion (under the assumptions of Theorem 5.4) is valid in any case, even if (5.10) has no solution for some $\omega \in \Omega_n$.
One can apply this conclusion to study convergence in distribution of a sequence of "generalized solutions" of (5.10) (for $n \in \mathbb{N}$) connected with the use of certain kinds of (random) generalized inverses. We give an outline of such an application to the case of the "Drazin inverse" (see [22] for a systematic treatment of generalized inverses). First we note that for each $\tilde{z} \in Z$ and $n \in \mathbb{N}$ the mapping $T_n(\tilde{z},.) = I - C_n(\tilde{z},.)$ is a bounded linear operator from X into itself with finite ascent and descent (see e.g. [22,p.77]).
Hence, the Drazin inverse $[T_n(\tilde{z},.)]^d$ of $T_n(\tilde{z},.)$ exists ([22, p.99/100]) and is a bounded linear operator from X into itself. Now, we define

$$\bar{x}_n^d(\omega) := [T_n(z_n(\omega),.)]^d y_n(\omega), \ \omega \in \Omega_n, \ n \in \mathbb{N},$$

and note that this is a particular case of (5.22) (since the Drazin inverse of a mapping coincides with its inverse if the mapping is bijective) if $\bar{x}_n^d$ is measurable. Evidently, it suffices to prove that the mapping (from $\Omega_n \times X$ into X)

$$(\omega,x) \to [T_n(z_n(\omega),.)]^d x$$

is a random operator.
To this end, we use [10, Theorem 5.14] and, thus, we only have to show that for all $\omega \in \Omega_n$, the ascent $a_n(\omega)$ of $T_n(z_n(\omega),.)$ is bounded by an integer $k(n) \in \mathbb{N}$ (independent of ω).
To see this, one first observes that for all $\tilde{z} \in Z = C([0,1]^2)$ and $n \in \mathbb{N}$, $R(C_n(\tilde{z},.)) = \operatorname{span}(\{\tilde{z}(.,s_{nj}) \mid j = 0,\dots,n\})$ and, hence, $\dim R(C_n(\tilde{z},.)) \le n+1$. Since

$$[T_n(\tilde{z},.)]^k = [I-C_n(\tilde{z},.)]^k = I + \sum_{j=1}^{k} (-1)^j \binom{k}{j} [C_n(\tilde{z},.)]^j,$$

for all $k \in \mathbb{N}$, this implies that the nullspace $N([T_n(\tilde{z},.)]^k)$ is contained in $R(C_n(\tilde{z},.))$ and, thus, $\dim N([T_n(\tilde{z},.)]^k) \le n+1$ for all $k \in \mathbb{N}$ and $\tilde{z} \in Z$. Let $\tilde{z} \in Z$ be arbitrary.
Since $(N([T_n(\tilde{z},.)]^k))_{k\in\mathbb{N}}$ is increasing, this implies that

$$\text{ascent } (T_n(\tilde{z},.)) := \min\{k \in \mathbb{N} \mid N([T_n(\tilde{z},.)]^k) = N([T_n(\tilde{z},.)]^{k+1})\} \le n+1.$$

Hence, we obtain for all $n \in \mathbb{N}$, $\omega \in \Omega_n$, that

$a_n(\omega) := \text{ascent } (T_n(z_n(\omega),.)) \leq n+1.$

Now, [1o, Theorem 5.14] yields that $\bar{x}_n^d$ is measurable for all $n \in \mathbb{N}$. (Note that in [1o], the underlying probability space was assumed to be complete; if (Ω_n, A_n, P_n) is not necessarily complete, [1o, Theorem 5.14] yields the measurability of $\bar{x}_n^d$ with respect to the completion of (Ω_n, A_n, P_n) and thus the A_n-measurability of $\bar{x}_n^d$, modified on a suitable P_n-nullset.)

Summarizing these observations we conclude that

$(\bar{x}_n^d) = ([T_n(z_n(.),.)]^d y_n(.))$ converges in distribution to the unique random solution of (5.1) (under the assumptions of Theorem 5.4)!

We note that this result can also be proved without using our abstract setting if one proceeds as follows:

In a first step one shows that for all $\bar{z} \in E := \{\tilde{z} \in Z \mid T(\tilde{z},.)$ is continuously invertible$\}$, $\bar{y} \in X$, and all sequences $(\bar{z}_n)$ in Z, $(\bar{y}_n)$ in Y converging to $\bar{z}$ and $\bar{y}$, respectively,

$$[T_n(\bar{z}_n,.)]^d \, \bar{y}_n \to [T(\bar{z},.)]^{-1} \, \bar{y} \tag{5.23}$$

holds (e.g. using [1, Theorem 1.6] and the Banach-Steinhaus-Theorem). Then one shows that the mappings $f, f_n : Z \times X \to X$, $n \in \mathbb{N}$, defined by

$$f(\bar{z},\bar{y}) := \begin{cases} [T(\bar{z},.)]^{-1} \, \bar{y} & \text{if } \bar{z} \in E, \ \bar{y} \in X, \\ 0 & \text{otherwise,} \end{cases}$$

$f_n(\bar{z},\bar{y}) := [T_n(\bar{z},.)]^d \, \bar{y}$, $(\bar{z},\bar{y}) \in Z \times X$, are measurable (using [1o, Theorem 5.14]).

Since $D(z,y)((Z \setminus E) \times X) = 0$, it follows from (5.23), the definitions of f and f_n and Rubin's Theorem ([6, Theorem 5.5]) that

$$D(\bar{x}_n^d) = D(z_n,y_n) f_n^{-1} \xrightarrow{w} D(z,y) f^{-1} = D(x).$$

Instead of the Drazin inverse, one could also use other generalized inverses for which a measurability result is available (see [1o], [23]).

We stress that in this way we prove only the weak convergence of a special sequence of approximate solution, while Theorem 5.4 yields a much more general result.

6. APPROXIMATION OF SOLUTIONS OF A RANDOM NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

In this Section, we present an application of Theorem 4.6 to a method of Galerkin type for nonlinear random equations involving a monotone mapping. Later on we indicate that this result is relevant for a finite element method for solving a random nonlinear elliptic boundary value problem.

Let X be a real separable Hilbert space with inner product $(.,.)$ and norm $\|.\|$, respectively. Let $(X_n,Q_n)_{n\in\mathbb{N}}$ be a Galerkin scheme for X, i.e., for each $n \in \mathbb{N}$ $X_n \subset X$ is a finite-dimensional subspace and Q_n: $X \to X_n$ is the orthogonal projection onto X_n; the sequence (Q_n) is assumed to converge pointwise to the identity. Let Z be a Polish space, T: $Z \times X \to X$ be a Borel measurable mapping, z and (z_n) be Z-valued random variables, y be an X-valued and y_n be an X_n-valued random variable, $n \in \mathbb{N}$, respectively (defined on (Ω,A,P) and (Ω_n,A_n,P_n), $n \in \mathbb{N}$, respectively).
We define T_n: $Z \times X_n \to X$ by $T_n(\tilde{z},.) := Q_n T(\tilde{z},.)\big|_{X_n}$, for $\tilde{z} \in Z$, and consider the equations

$$T(z(\omega),x) = y(\omega) \qquad (\omega \in \Omega) \tag{6.1}$$

$$T_n(z_n(\omega),x) = y_n(\omega) \quad (\omega \in \Omega_n,\ n \in \mathbb{N}). \tag{6.2}$$

Theorem 6.1:

Let T, (T_n), z, (z_n), y and (y_n) be as above and let for all $n \in \mathbb{N}$ x_n be a random solution of (6.2) (for the index $n \in \mathbb{N}$). Assume that

$$T: Z \times X \to X \text{ is continuous,} \tag{6.3}$$

$$\left.\begin{array}{l}\{T(.,x)\,|\,x \in B\} \text{ is equicontinuous on } K \text{ for all}\\ \text{bounded } B \subset X \text{ and compact } K \subset Z,\end{array}\right\} \tag{6.4}$$

$$\left.\begin{array}{l}T(\tilde{z},.) \text{ is strongly monotone uniformly w.r.t. } \tilde{z} \in Z, \text{ i.e.,}\\ \text{there is a } \gamma > 0 \text{ such that for all } \tilde{z} \in Z,\ \bar{x},\ \tilde{x} \in X,\\ (T(\tilde{z},\bar{x}) - T(\tilde{z},\tilde{x}),\bar{x}-\tilde{x}) \geq \gamma\|\bar{x}-\tilde{x}\|^2,\end{array}\right\} \tag{6.5}$$

$$D(y_n,z_n) \xrightarrow{w} D(y,z). \tag{6.6}$$

Then $(D(x_n))$ converges weakly to a weak solution of (6.1).

Proof:

To show that Theorem 4.6 is applicable, we have to check the assumptions of that Theorem. It is easy to see that the properties of (Q_n) and (6.3), (6.4) imply that (4.1o) and (4.15) are fulfilled. Because of (6.5), $T(\tilde{z},.)$ is injective for all $\tilde{z} \in Z$. Hence, an application of Theorem 4.6 yields the assertion provided we prove that

$$(T_n(\tilde{z},.)) \text{ is collectively regular for all } \tilde{z} \in Z \tag{6.7}$$

and

$$(D(x_n)) \text{ is w-bounded.} \tag{6.8}$$

Since for all $n \in \mathbb{N}$ and $\tilde{z} \in Z$, $T_n(\tilde{z},.)$ is regular (consult Remark 3.7), (6.7) is implied by

$$(T_n(\tilde{z},.)) \text{ is A-regular for all } \tilde{z} \in Z \tag{6.9}$$

because of Lemma 3.7.

To prove (6.9) we proceed as usual in this context (cf. [31, p.2o1]): Let $\tilde{z} \in Z$ be arbitrary and let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and $w_{n_k} \in X_{n_k}$, $k \in \mathbb{N}$, be such that (w_{n_k}) is bounded (and hence discretely compact w.r.t. the weak topology in X) and $(T_{n_k}(\tilde{z},w_{n_k}))$ is relatively compact in X. We assume w.l.o.g. that (w_{n_k}) is weakly convergent to $w \in X$ and that $(T_{n_k}(\tilde{z},w_{n_k}))$ converges (in the norm on X) to some element, say $y \in X$.

Then $(w_{n_k} - Q_{n_k}w)$ converges weakly to zero and $(T_{n_k}(\tilde{z},w_{n_k}) - Q_{n_k}T(\tilde{z},Q_{n_k}w))$ converges in the norm to $y-T(\tilde{z},w)$.

Hence

$$\lim_{k\to\infty}(T_{n_k}(\tilde{z},w_{n_k})-Q_{n_k}T(\tilde{z},Q_{n_k}w),w_{n_k}-Q_{n_k}w) = 0. \tag{6.1o}$$

From (6.5) we obtain (note that $w_{n_k}-Q_{n_k}w = Q_{n_k}(w_{n_k}-Q_{n_k}w)$ and that Q_{n_k} is selfadjoint):

$$\gamma\|w_{n_k}-Q_{n_k}w\|^2 \le (T(\tilde{z},w_{n_k})-T(\tilde{z},Q_{n_k}w),w_{n_k}-Q_{n_k}w) =$$

$$= (Q_{n_k}T(\tilde{z},w_{n_k})-Q_{n_k}T(\tilde{z},Q_{n_k}w),\ w_{n_k}-Q_{n_k}w)$$

$$= (T_{n_k}(\tilde{z},w_{n_k})-Q_{n_k}T(\tilde{z},Q_{n_k}w),\ w_{n_k}-Q_{n_k}w).$$

Thus (6.10) implies that $\lim\limits_{k\to\infty}\|w_{n_k}-Q_{n_k}w\| = 0$, i.e., $w_{n_k}\to w$.
Hence, (6.9) holds, which implies (6.7).

Finally we prove (6.8). Since

$$T_n(z_n(\omega),x_n(\omega))-T_n(z_n(\omega),0) = y_n(\omega)-T_n(z_n(\omega),0)$$

holds almost surely, (6.5) implies for all $n \in \mathbb{N}$ that

$$\gamma\|x_n(\omega)\|^2 \le (y_n(\omega)-T_n(z_n(\omega),0),\ x_n(\omega))$$

$$\le (\|y_n(\omega)\| + \|T_n(z_n(\omega),0)\|)\ \|x_n(\omega)\| \quad \text{a.s.}$$

Hence we obtain for each $n \in \mathbb{N}$ and for P_n-almost all $\omega \in \Omega_n$ that

$$\|x_n(\omega)\| \le \frac{1}{\gamma}\ (\|y_n(\omega)\| + \|T(z_n(\omega),0)\|). \tag{6.11}$$

Let $\varepsilon > 0$ be arbitrary, but fixed. Since (because of (6.6)) $(D(y_n))$ is w-bounded and $(D(z_n))$ is tight, there exist a constant $C_\varepsilon > 0$ and a compact set $K_\varepsilon \subseteq Z$ such that

$$\begin{aligned}&\inf_{n\in\mathbb{N}} P_n(\{\omega \in \Omega_n \mid \|y_n(\omega)\| \le C_\varepsilon\}) \ge 1-\frac{\varepsilon}{2} \quad \text{and}\\ &\inf_{n\in\mathbb{N}}\ D(z_n)(K_\varepsilon) \ge 1-\frac{\varepsilon}{2}.\end{aligned} \tag{6.12}$$

Since T is continuous, $D_\varepsilon := \sup\{\|T(\tilde{z},0)\| \mid \tilde{z} \in K_\varepsilon\}$ is finite.
Let $n \in \mathbb{N}$ be arbitrary, but fixed and define

$$A_0 := \{\omega \in \Omega_n \mid (6.11) \text{ holds for } \omega\} \in A_n,\ P(A_0) = 1,$$

$$A_1 := \{\omega \in \Omega_n \mid \|y_n(\omega)\| \le C_\varepsilon,\ z_n(\omega) \in K_\varepsilon\} \in A_n,$$

$$A_2 := \{\omega \in \Omega_n \mid \|x_n(\omega)\| \le \frac{1}{\gamma}\ (C_\varepsilon + D_\varepsilon)\} \in A_n.$$

By definition, $A_0 \cap A_1 \subset A_2$; thus it follows from (6.12) and the fact that $P_n(A_o) = 1$ that $P_n(\Omega \setminus A_2) \le P_n(\Omega \setminus A_1) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Since $n \in \mathbb{N}$ was arbitrary, this means that

$$\inf_{n\in\mathbb{N}} P_n(\{\omega \in \Omega_n \mid \|x_n(\omega)\| \le \frac{1}{\gamma}(C_\varepsilon + D_\varepsilon)\}) \ge 1 - \varepsilon,$$

which implies (6.8).

□

Finally, we outline how Theorem 6.1 can be applied to Galerkin type methods for solving random nonlinear partial differential equations.

Let G be a bounded open subset of $\mathbb{R}^m$, b be a mapping from $G \times \mathbb{R}^r \times \mathbb{R}^m$ into $\mathbb{R}^m$, w and z be $L^2(G)$-valued and $L_r^\infty(G)$-valued random variables, respectively (defined on some probability space (Ω, A, P)). We consider the following random nonlinear p.d.e. in its variational form

$$\int_G \sum_{i=1}^m b_i(\xi, z(\omega,\xi), \frac{\partial x(\xi)}{\partial \xi_1}, \dots, \frac{\partial x(\xi)}{\partial \xi_m}) \frac{\partial h(\xi)}{\partial \xi_i}\, d\xi = \int_G w(\omega,\xi) h(\xi)\, d\xi, \quad \text{for all } h \in H_o^1(G) \ (\omega \in \Omega), \tag{6.13}$$

where $H_o^1(G)$ is the usual Sobolev space ([18], [31]), i.e., the closure of $C_o^\infty(G)$ in the Sobolov space $H^1(G)$ of real functions $u \in L^2(G)$ whose generalized derivatives of the first order also lie in $L^2(G)$.

First we show that (6.13) fits into the setting of this Section.

We define $X := H_o^1(G)$ with inner product $(x,y) := \int_G \sum_{i=1}^m \frac{\partial x}{\partial \xi_i} \frac{\partial y}{\partial \xi_i}\, d\xi$

and Z to be a closed subset of $L_r^\infty(G)$. We assume that the mapping $b: G \times \mathbb{R}^{r+m} \to \mathbb{R}^m$ satisfies the following conditions:

$$\left.\begin{array}{l} b \text{ is a Caratheodory map, i.e., measurable on } G \\ \text{and continuous on } \mathbb{R}^{r+m}; \end{array}\right\} \tag{6.14}$$

$$\left.\begin{array}{l} \text{there exist } M > 0 \text{ and } a \in L^2(G) \text{ such that} \\ |b(\xi,v,u)| \le a(\xi) + M|v||u|, \text{ for } \xi \in G,\ v \in \mathbb{R}^r,\ u \in \mathbb{R}^m, \\ |b(\xi,v,u) - b(\xi,\tilde{v},u)| \le M|v-\tilde{v}|(1+|u|),\ \tilde{v} \in \mathbb{R}^r; \end{array}\right\} \tag{6.15}$$

$$\left.\begin{array}{l}\text{there exists } \gamma > 0 \text{ such that}\\ \sum_{i=1}^{m} (b_i(\xi,v,u)-b_i(\xi,v,\tilde{u}))(u_i-\tilde{u}_i) \geq \gamma \sum_{i=1}^{m} (u_i-\tilde{u}_i)^2,\\ \qquad \text{for } \xi \in G,\ v \in \mathbb{R}^r,\ u,\tilde{u} \in \mathbb{R}^m.\end{array}\right\} \tag{6.16}$$

Then the mapping $B: Z \times L_m^2(G) \to L_m^2(G)$ given by

$B(\tilde{z},u)(\xi): = b(\xi,\tilde{z}(\xi),u(\xi))$, $\xi \in G$, $\tilde{z} \in Z$, $u \in L_m^2(G)$,

is well-defined and continuous (because of (6.14), (6.15)). Let L denote the mapping from $H_0^1(G)$ into $L_m^2(G)$ defined by

$$(Lx)(\xi): = \operatorname{grad} x(\xi): = \left(\frac{\partial x(\xi)}{\partial \xi_1},\ldots,\frac{\partial x(\xi)}{\partial \xi_m}\right),\ \xi \in G,\ x \in H_0^1(G).$$

Now, we define $T: Z \times X \to X$ (via the Riesz Representation Theorem) by

$$(T(\tilde{z},x),h): = (B(\tilde{z},Lx),Lh)_{L_m^2}$$
$$= \int_G \sum_{i=1}^{m} b_i(\xi,\tilde{z}(\xi),\operatorname{grad} x(\xi))\,\frac{\partial h(\xi)}{\partial \xi_i}\,d\xi,\ \tilde{z} \in Z,\ x,h \in X.$$

One can show, similar to [18,p.67 ff], that T is continuous and T satisfies (6.5) (because of (6.16)). Furthermore, (6.15) implies that for all $\tilde{z}$, $\bar{z} \in Z$, $x \in X$,

$\|T(\tilde{z},x)- T(\bar{z},x)\| \leq M(1+ \|x\|)\ \|\tilde{z}-\bar{z}\|_Z$ holds.

Hence, (6.4) is fulfilled; thus Theorem 6.1 is applicable to an equation of the kind (6.1) involving this mapping T.

Let y be the uniquely determined X-valued random variable (on (Ω,A,P)) such that $(y(\omega),h) = \int_G w(\omega,\xi)h(\xi)\,d\xi$, for all $\omega \in \Omega$ and $h \in X$; it follows from the Riesz Representation Theorem that y exists; since each $(y(.),h)$ is measurable, we can conclude the measurability of y (cf. [4]).

Now, (6.13) is equivalent to

$$(T(z(\omega),x),h) = (y(\omega),h),\ \text{for all } h \in X,\ \omega \in \Omega,$$

which is in turn equivalent to (6.1) with T defined as above. Let (w_n) and (z_n) be sequences of $L^2(G)$-valued and $L_r^\infty(G)$-valued random variables (defined on probability spaces (Ω_n,A_n,P_n)), respectively, such that $D(w_n,z_n) \to D(w,z)$, and let (X_n,Q_n) be a Galerkin scheme for $X = H_o^1(G)$ (e.g., a finite element scheme: cf. [31,p.234 ff.] for examples). For each $n \in \mathbb{N}$ we define y_n to be an X_n-valued random variable (see above) such that

$$(y_n(.),h) = \int_G w_n(.,\xi)h(\xi)d\xi, \quad \text{for all } h \in X_n.$$

Then (6.2) is equivalent to

$$\left.\begin{array}{l} \displaystyle\int_G \sum_{i=1}^m b_i(\xi,z_n(\omega,\xi),\operatorname{grad} x(\xi))\,\frac{\partial h(\xi)}{\partial \xi_i}\,d\xi = \int_G w_n(\omega,\xi)h(\xi)d\xi \\ \qquad \text{for all } h \in X_n \quad (n \in \mathbb{N},\ \omega \in \Omega_n). \end{array}\right\} \qquad (6.17)$$

If (x_n) is a sequence of random solutions of (6.17) (for $n \in \mathbb{N}$), we can apply Theorem 6.1 and conclude that $(D(x_n))$ converges weakly to a weak solution of (6.13).

We mention that the above general approach applies in particular to linear partial differential equations with random coefficients and random right-hand side, i.e.,

$b_i(\xi,v,u) := \sum_{k=1}^m a_{ik}(\xi,v)u_k$, $i = 1,\ldots,m$, with suitable coefficient functions a_{ik}, $i,k = 1,\ldots,m$.

Note that in this Section, it was essential that in our setup the approximate equations may be defined on subspaces X_n of X.

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