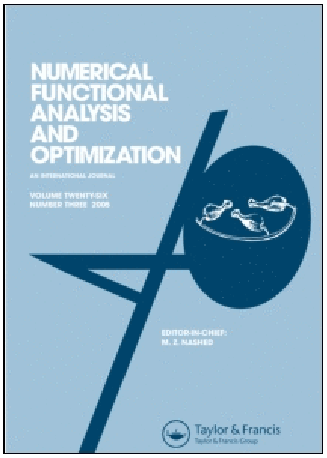

Numerical Functional Analysis and Optimization
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597287

Weak convergence of approximate solutions of random equations

Olga Fiedler [a]; Werner Römisch [a]
[a] Fachbereich Mathematik, Humboldt-Universität Berlin, Berlin, Germany

Online Publication Date: 01 January 1992
To cite this Article: Fiedler, Olga and Römisch, Werner (1992) 'Weak convergence of approximate solutions of random equations', Numerical Functional Analysis and Optimization, 13:5, 495 - 511
To link to this article: DOI: 10.1080/01630569208816495
URL: http://dx.doi.org/10.1080/01630569208816495

NUMER. FUNCT. ANAL. AND OPTIMIZ., 13(5&6), 495–511 (1992)

WEAK CONVERGENCE OF APPROXIMATE SOLUTIONS OF RANDOM EQUATIONS

Olga Fiedler and Werner Römisch

Fachbereich Mathematik
Humboldt-Universität Berlin
D-O-1086 Berlin, Germany

ABSTRACT

Approximations of random operator equations are considered where the stochastic inputs and the underlying deterministic equation are approximated simultaneously. The main convergence result asserts that, under reasonable and verifiable assumptions, a sequence of weak solutions of approximate random equations converges weakly to a weak solution of the original equation. It is shown that this theorem extends and unifies results already known. We apply our theory to approximations of random differential equations involving stochastic processes with discontinuous paths and to projection methods for nonlinear random Hammerstein integral equations in spaces of integrable functions.

1. INTRODUCTION

During the last fifteen years, approximation methods for solving random operator equations have been investigated by many authors (see e.g. [4, 5, 8–10, 14, 19]).

In this paper, we study approximations of a random equation

$$T(z(\omega), x) = 0 \qquad \omega \in \Omega \tag{1.1}$$

where T is a mapping from $Z \times X$ into Y, 0 a fixed element of Y, z a Z-valued random element on some probability space $(\Omega, \mathcal{Q}, P)$ and X, Y, Z metric spaces. We aim at extending and unifying the main result of [10] (Theorem 4.6) on the weak convergence of approximate (weak) solutions of (1.1) when approximating the stochastic input z (in the sense of weak convergence of the probability distribution) and the mapping T (in the sense of discrete convergence) simultaneously. The proof of our central convergence result (Theorem 3.1) is considerably shorter than that in [10] and does not rely on the methodology developed there. In our analysis we show that the crucial assumption of our convergence result is implied by two types of conditions: (i) The first one (Lemma 3.3) unifies a similar result from [10] and makes use of the well-known concept of A-regular (deterministic) operator approximations; (ii) the second one (Lemma 3.7) is formulated in terms of the uniform inverse Lipschitz continuity of the approximate operators. The latter condition combined with Theorem 3.1 leads to an extension of a result given in [14] on the tightness of distributions of approximate solutions. Our approach allows in addition to identify weak limits of approximate solutions as weak solutions of (1.1).

Our paper is organized as follows. In Section 2, we discuss the concept of a weak solution of random operator equations (1.1) in complete separable metric spaces and present an existence and uniqueness result. Section 3 contains the general theory on the weak convergence of approximate weak solutions. In Section 4, we apply the general results to nonlinear ordinary random differential equations where the stochastic inputs appearing in the right-hand side vary in spaces of cadlag functions equipped with the Skorokhod topology. Our Theorem 4.2 generalizes the corresponding result of [9] (Section 3) and allows applications to random differential equations driven by semimartingales. In Section 5, we establish a convergence result for Galerkin approximations of nonlinear random Hammerstein integral equations in spaces of integrable functions, thus, supplementing the results given in [9] and [10], where quadrature methods for integral equations were studied in spaces of continuous functions and generalizing results in [5] and [10] to the nonlinear case.

Now, let us fix the terminology of this paper. For a metric space X we denote the σ-algebra of Borel subsets of X by $\mathcal{B}(X)$ and the set of all probability measures defined on $(X, \mathcal{B}(X))$ by $\mathcal{P}(X)$. $\mathcal{P}(X)$ will usually be equipped with the topology of weak convergence. Weak convergence of a sequence in $\mathcal{P}(X)$ will be abbreviated by "$\xrightarrow{w}$" ([6]). For $u \in X$ let $\delta_u \in \mathcal{P}(X)$ denote the unit mass at u. If x is an X-valued random variable defined on a probability space $(\Omega, \mathcal{A}, P)$ (i.e. $x:\Omega \to X$ having the property $x^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}(X)$), we denote its probability distribution by $D(x) := P \circ x^{-1} \in \mathcal{P}(X)$.

2. RANDOM OPERATOR EQUATIONS: SOLUTIONS CONCEPTS, EXISTENCE AND UNIQUENESS RESULTS

Let X, Y, and Z be separable metric spaces, z a Z-valued random variable (defined on some probability space $(\Omega, \mathcal{A}, P)$) and T a mapping from $Z \times X$ into Y. We will consider the random operator equation

$$T(z(\omega), x) = 0 \qquad \omega \in \Omega \tag{2.1}$$

where 0 is some fixed element in Y.

A random variable $x:\Omega \to X$ is called a *random solution* of (2.1) iff $T(z(\omega), x(\omega)) = 0$ holds P-almost surely. This is a classical notion in probabilistic functional analysis (see [3]) enjoying a well-developed existence theory (e.g. [7, 20, 21]) which is based on measurable selection theorems (cf. [13, 18, 29]). This topic has found continuous interest up to now (see the more recent contributions [22, 26]). The first result we state is an immediate consequence of [20, Theorem 1].

Theorem 2.1: Let X be complete and $T: Z \times X \to Y$ Borel measurable, i.e.

measurable with respect to $\mathcal{B}(Z \times X)$ and $\mathcal{B}(Y)$. Assume that there exists a $B \in \mathcal{B}(Z)$ such that $D(z)(B) = 0$ and $T(\bar{z}, x) = 0$ is solvable for all $\bar{z} \in Z \backslash B$. Then (2.1) has a random solution.

Remark 2.2: An example in [20] shows that Theorem 2.1 would not remain true if T were Borel measurable only separately in each variable. Note that $T: Z \times X \to Y$ is Borel measurable if for each $\bar{z} \in Z$, $T(\bar{z}, \cdot)$ is continuous and for each $x \in X$, $T(\cdot, x)$ is Borel measurable ([13, Theorem 6.1]).

Definition 2.3: A probability measure $\mu_x \in \mathcal{P}(X)$ is called a *weak solution* of (2.1) iff there exists a $\mu \in \mathcal{P}(Z \times X)$ such that

$$\mu T^{-1} = \delta_0 \qquad D(z) = \mu p_Z^{-1} \qquad \text{and} \qquad \mu_x = \mu p_X^{-1}$$

where p_X and p_Z denote the coordinate projections from $Z \times X$ onto X and Z, respectively.

Lemma 2.4: Let $T: X \times X \to Y$ be Borel measurable.

(i) The probability distribution of each random solution of (2.1) is a weak solution.

(ii) There exists a probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$ such that for each weak solution μ_x of (2.1) there are random variables $\bar{x}: \bar{\Omega} \to X$ and $\bar{z}: \bar{\Omega} \to Z$ having the property that $D(\bar{z}) = D(z)$, $D(\bar{x}) = \mu_x$ and $T(\bar{z}(\omega), \bar{x}(\omega)) = 0$ holds $\bar{P}$-almost surely.

(iii) Let X and Z be complete and $(\Omega, \mathcal{A}, P)$ be nonatomic. Then for each weak solution of (2.1) the assertion of (ii) holds (even) for $(\Omega, \mathcal{A}, P)$ instead of $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$.

Proof: (i) Let $x: \Omega \to X$ be a random solution of (2.1). Setting $\mu := D(z, x) = P \circ (z(\cdot), x(\cdot))^{-1}$ we obtain, for each $B \in \mathcal{B}(Y)$, $\mu(T^{-1}(B)) = P(\{\omega \in \Omega: T(z(\omega), x(\omega)) \in B\}) = \delta_0(B)$. Hence, $\mu_x := \mu p_X^{-1}$ is a weak solution of (2.1).

(ii) We choose $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P})$ as the universal probability space of Theorem 2.5.1 in [23]. Now, if μ_x is a weak solution of (2.1), for the corresponding measure $\mu \in \mathcal{P}(Z \times X)$ there exists a random variable $(\bar{z}(\cdot), \bar{x}(\cdot)): \Omega \to Z \times X$ with $D(\bar{z}, \bar{x}) = \mu$ (Theorem 2.5.1 in [23]). Hence, we have $P(\{\omega \in \Omega: T(\bar{z}(\omega), \bar{x}(\omega)) = 0\}) = 1$.

(iii) This follows by the same argument as in (ii), but now we appeal to Lemma 2.5.1 in [23].

According to Lemma 2.4 a weak solution of (2.1) is the probability distribution of a random solution on *some* probability space (with given input distribution $D(z)$). Part (iii) of the lemma asserts the existence of a so-called D-solution of (2.1), a notion introduced and used in [9].

The following example shows that the set of weak solutions of (2.1) is, in general, larger than that of (distribution of) random solutions and that (iii) does not remain true if $(\Omega, \mathcal{A}, P)$ has atoms.

Example 2.5: Let $X := Y := \mathbb{R}$, $Z := \{a, b\}$ (equipped with the discrete metric), $(\Omega, \mathcal{A}) := (Z, \mathcal{B}(Z))$, let P be the discrete uniform distribution on $\mathcal{B}(Z)$ and z be the identity.

Let $T: Z \times X \to Y$ be defined by $T(a, x) := 1 - x$, $T(b, x) := 0$ for all $x \in X$. Clearly, T is Borel measurable.

We show that $\mu_x := \frac{1}{2}(\delta_1 + \nu)$ is a weak solution of (2.1) for arbitrary $\nu \in$

$\mathcal{P}(\mathbb{R})$. To this end we define $\mu:\mathcal{B}(Z \times X) \to [0, 1]$ by $\mu(\emptyset) := 0$, $\mu(\{a\} \times B) := \frac{1}{2}\delta_1(B)$, $\mu(\{b\} \times B) := \frac{1}{2}\nu(B)$ and $\mu(Z \times B) := \frac{1}{2}(\delta_1(B) + \nu(B))$ for all $B \in \mathcal{B}(X)$. We have $\mu \in \mathcal{P}(Z \times X)$ $\mu p_X^{-1} = \mu_x$ and $\mu p_Z^{-1} = D(z)$ immediately. It remains to show that $\mu T^{-1} = \delta_0$. For the case of $B \in \mathcal{B}(Y)$, $0 \notin B$, we have $\mu T^{-1}(B) = \mu(\{a\} \times \{x \in X: 1 - x \in B\}) = \frac{1}{2}\delta_0(B) = 0$. If $B \in \mathcal{B}(Y)$ and $0 \in \mathcal{B}$ we obtain

$$\begin{aligned}\mu T^{-1}(B) &= \mu(\{b\} \times X \cup \{a\} \times \{x \in X: 1 - x \in B\}) \\ &\geq \mu(\{b\} \times X) + \mu(\{(a, 1)\}) = \tfrac{1}{2} + \tfrac{1}{2} = 1\end{aligned}$$

Thus $\mu T^{-1} = \delta_0$ and, therefore, μ_x is a weak solution of (2.1). However, μ_x is *not* the distribution of a random variable on $(\Omega, \mathcal{Q}, P)$ in general.

The following result provides a characterization of weak solutions; moreover, it turns out to be useful for the proof of our existence and uniqueness result.

Lemma 2.6: Let $T: Z \times X \to Y$ be Borel measurable and the mapping $\tilde{T}: Z \times X \to Y \times Z$ be defined by $\tilde{T}(\bar{z}, \bar{x}) := (T(\bar{z}, \bar{x}), \bar{z})$ for all $(\bar{z}, \bar{x}) \in Z \times X$. Then μ_x is a weak solution of (2.1) iff there exists a $\mu \in \mathcal{P}(Z \times X)$ such that $\mu\tilde{T}^{-1} = \delta_0 \times D(z)$ and $\mu p_X^{-1} = \mu_x$, where $\delta_0 \times D(z)$ is the usual product measure on $\mathcal{B}(Y \times Z)$.

Proof: It is easy to check that $\tilde{T}$ is Borel measurable ([10, Prop. 4.3]). In view of Definition 2.3 it suffices to show that $\mu\tilde{T}^{-1} = \delta_0 \times D(z)$ holds iff $\mu T^{-1} = \delta_0$ and $\mu p_Z^{-1} = D(z)$.

The first direction is immediate, since $\mu\tilde{T}^{-1} = \delta_0 \times D(z)$ implies e.g. $D(z) = \mu(p_Z\tilde{T})^{-1} = \mu p_Z^{-1}$. For the converse, let $\mu T^{-1} = \delta_0$ and $\mu p_Z^{-1} = D(z)$. It is sufficient to show that $\mu\tilde{T}^{-1}(B_1 \times B_2) = \delta_0(B_1)D(z)(B_2)$ holds for all $B_1 \in \mathcal{B}(Y)$, $B_2 \in \mathcal{B}(Z)$. For $B_1 \in \mathcal{B}(Y)$ and $B_2 \in \mathcal{B}(Z)$ we obtain $\mu\tilde{T}^{-1}(B_1 \times B_2) = \mu(T^{-1}(B_1) \cap B_2 \times X)$. If $0 \in B_1$ we have $\mu(T^{-1}(B_1)) = 1$ and, hence, $\mu T^{-1}(B_1 \times B_2) = \mu(B_2 \times X) = \mu p_Z^{-1}(B_2) = \delta_0(B_1)D(z)(B_2)$. If $0 \notin B_1$, then $\mu(T^{-1}(B_1)) = 0$ and $\mu T^{-1}(B_1 \times B_2) = 0\ \delta_0(B_1)D(z)(B_2)$.

Lemma 2.6 suggests that existence and uniqueness results for weak solutions can be derived by studying the stochastic equation (in the sense of [11]) $\mu T^{-1} = \delta_0 \times D(z)$. In the following we provide a direct approach.

Theorem 2.7: Let X and Z be complete and $T: Z \times X \to Y$ Borel measurable. Suppose there exists a $B \in \mathcal{B}(Z)$ such that $D(z)(B) = 0$ and that the equation $T(\bar{z}, x) = 0$ has at most one solution for each $\bar{z} \in Z \backslash B$. Then (2.1) has at most one weak solution.

Proof: Let μ_x be a weak solution of (2.1). Then it follows from Lemma 2.6 that there exists a $\mu \in \mathcal{P}(Z \times X)$ such that $\mu\tilde{T}^{-1} = \delta_0 \times D(z)$ and $\mu_x = \mu p_X^{-1}$. Now, let $A \in \mathcal{B}(X)$ be chosen arbitrary. First we take $B \in \mathcal{B}(Z)$ according to the assumption and show that the mapping $\tilde{T}: (Z\backslash B) \times A \cap T^{-1}(\{0\}) \to Y \times Z$ is one-to-one. To see this, let $(z_i, x_i) \in (Z\backslash B) \times A \cap T^{-1}(\{0\})$, $i = 1, 2$, be such that $\tilde{T}(z_1, x_1) = \tilde{T}(z_2, x_2)$. Then $z_1 = z_2 \in Z\backslash B$ and $T(z_1, x_1) = T(z_2, x_2) = T(z_1, x_2) = 0$. Our assumption implies $x_1 = x_2$ and, consequently, $\tilde{T}$ restricted to $(Z\backslash B) \times A \cap T^{-1}(\{0\})$ is one-to-one. This observation together with the general assumptions on X, Y, Z and with the Borel measurability of $\tilde{T}$ implies that $\tilde{T}((Z\backslash B) \times A \cap T^{-1}(\{0\})) \in \mathcal{B}(Y \times Z)$ (see [17], Section 39). Finally, in view of $\mu T^{-1}(\{0\}) = 1$, we have

$$\begin{aligned}\mu_x(A) = \mu(Z \times A) = \mu((Z\backslash B) \times A) &= \mu((Z\backslash B) \times A \cap T^{-1}(\{0\}))\\ &= \mu\tilde{T}^{-1}(\tilde{T}((Z\backslash B) \times A \cap T^{-1}(\{0\})))\\ &= \delta_0 \times D(z)(\tilde{T}((Z\backslash B) \times A \cap T^{-1}(\{0\}))).\end{aligned}$$

Hence, μ_x is uniquely determined.

Corollary 2.8: Let X and Z be complete at $T: Z \times X \to Y$ be Borel measurable. Suppose there exists a $B \in \mathcal{B}(Z)$ such that $D(z)(B) = 0$ and that the equation $T(\bar{z}, x) = 0$ is uniquely solvable for each $\bar{z} \in Z\backslash B$. Then there exists a random solution $x: \Omega \to X$ of (2.1) and $\mu_x = D(x)$ is the unique weak solution.

Proof: The proof is an immediate consequence of the Theorems 2.1 and 2.7.

Remark 2.9: If the domain of the mapping T in (2.1) is $Z \times X_0$, where X_0 is a Borel subset of the metric space X, it is sometimes convenient to introduce a weak solution as a probability measure belonging to $\mathcal{P}(X)$ instead of $\mathcal{P}(X_0)$. This can be done in a natural way as follows: We take $\mu \in \mathcal{P}(Z \times X_0)$ from Definition 2.3 such that $\mu p_{\bar{X}_0}^{-1}$ is a weak solution of (2.1). Let $\eta \in \mathcal{P}(Z \times X)$ be the (canonical) extension of μ, i.e., $\eta(B) := \mu(B \cap ZX \times X_0)$ for all $B \in \mathcal{B}(Z \times X)$ ([6, p. 38]). Then $\mu_x := \eta p_{\bar{X}}^{-1} \in \mathcal{P}(X)$ is the desired "extended" weak solution.

3. APPROXIMATE SOLUTION OF RANDOM OPERATOR EQUATIONS

Throughout this section, let X, Y, and Z be complete separable metric spaces, X_n ($n \in \mathbb{N}$) Borel subsets of X, and $0 \in Y$ some fixed element. Let $T: Z \times X \to Y$ and $T_n: Z \times X_n \to Y$ ($n \in \mathbb{N}$) be Borel measurable mappings, z and z_n ($n \in \mathbb{N}$) Z-valued random variables defined on some probability space $(\Omega, \mathcal{A}, P)$. We consider the random operator equation

$$T(z(\omega), x) = 0 \qquad \omega \in \Omega \tag{3.1}$$

and its "approximations"

$$T_n(z_n(\omega), x) = 0 \qquad \omega \in \Omega;\ n \in \mathbb{N} \tag{3.2}$$

Now, our aim is to find sufficient conditions on T and T_n ($n \in \mathbb{N}$) that imply the weak convergence of weak solutions μ_n of (3.2) to a weak solution of (3.1) if $(D(z_n))$ converges weakly to $D(z)$.

(By the extension argument in Remark 2.9 we may tacitly assume for the following that each weak solution μ_n of (3.2) belongs to $\mathcal{P}(X)$.)

The following notion of "discrete convergence" (see [25, 27]) turns out to be essential in this context. (T_n) is called discretely convergent to T iff $\inf\{d_X(\bar{x}, y): y \in X_n\} \to 0$ for each $\bar{x} \in X$ (d_X denoting the metric in X) and for all $\bar{x} \in X$, $\bar{x}_n \in X_n$, $\bar{z}$, $\bar{z}_n \in Z$ ($n \in \mathbb{N}$) such that $\bar{x}_n \to \bar{x}$, $\bar{z}_n \to \bar{z}$ we have $T_n(\bar{z}_n, \bar{x}_n) \to T(\bar{z}, \bar{x})$.

Now, we are in the position to state the main convergence result of this section, which, in fact, is an extension of Theorem 4.6 in [10]. Although this result can be proved by using the same technique as in [10], we provide a more direct and shorter proof.

Theorem 3.1: Let T, (T_n), z and (z_n) be as above and assume that

(a) $\bigcup_{n\in\mathbb{N}} T_n^{-1}(\{0\}) \cap K \times B$ is relative compact in $Z \times X$ for each bounded $B \subseteq X$ and compact $K \subseteq Z$.

(b) (T_n) converges discretely to T.

(c) $(D(z_n))$ converges weakly to $D(z)$.

(d) for each $n \in \mathbb{N}$ there exists a weak solution μ_n of (3.2) and $\{\mu_n : n \in \mathbb{N}\}$ is stochastically bounded, i.e., for each $\varepsilon > 0$ there exists a bounded Borel set B_ε in X such that $\inf_{n\in\mathbb{N}} \mu_n(B_\varepsilon) \geq 1 - \varepsilon$.

Then $\{\mu_n : n \in \mathbb{N}\}$ is relatively compact with respect to the weak topology on $\mathcal{P}(X)$ and every weak limit of a subsequence of (μ_n) is a weak solution of (3.1). Moreover, if the weak solution of (3.1) is unique, (μ_n) is weakly convergent to this limit.

Proof: Let $\eta_n \in \mathcal{P}(Z \times X)$ be such that $\eta_n T_n^{-1} = \delta_0$ and $\mu_n = \eta_n p_X^{-1}$, $D(z_n) = \eta_n p_Z^{-1}$ for all $n \in \mathbb{N}$ (see Remark 2.9). In a first step we show that $\{\eta_n : n \in \mathbb{N}\}$ is tight in $\mathcal{P}(Z \times X)$ ([6, p. 37]).

Let $\varepsilon > 0$ be arbitrary. (c) together with Prokhorov's theorem ([6]) implies that there exists a compact subset K_ε of Z such that $\inf_{n\in\mathbb{N}} D(z_n)(K_\varepsilon) \geq 1 - \varepsilon/2$. (d) implies that there is a bounded Borel subset of B_ε of X such that

$$\inf_{n\in\mathbb{N}} \mu_n(B_\varepsilon) \geq 1 - \frac{\varepsilon}{2}$$

We define $K := cl\{\bigcup_{n\in\mathbb{N}} T_n^{-1}(\{0\}) \times K_\varepsilon \times B_\varepsilon\}$ and conclude from (a) that K is compact. We obtain for each $n \in \mathbb{N}$ that

$$\begin{aligned}\eta_n(Z \times X\backslash K) &\leq \eta_n(Z \times X\backslash[T_n^{-1}(\{0\}) \cap K_\varepsilon \times B_\varepsilon]) \\ &\leq \eta_n(Z \times X\backslash T_n^{-1}(\{0\})) + \eta_n(Z \times X\backslash K_\varepsilon \times B_\varepsilon) \\ &= \eta_n((Z\backslash K_\varepsilon) \times X \cup Z \times (X\backslash B_\varepsilon)) \\ &\leq D(z_n)(Z\backslash K_\varepsilon) + \mu_n(X\backslash B_\varepsilon) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Hence, we have $\inf_{n\in\mathbb{N}} \eta_n(K) \geq 1 - \varepsilon$, i.e. $\{\eta_n : n \in \mathbb{N}\}$ is tight. This implies that $\{\mu_n : n \in \mathbb{N}\} = \{\eta_n p_X^{-1} : n \in \mathbb{N}\}$ is tight and thus, by Prokhorov's Theorem, relatively compact w.r.t. the weak topology on $\mathcal{P}(X)$.

Let (μ_{n_k}) be a weakly convergent subsequence of (μ_n), i.e., $\mu_{n_k} \xrightarrow{w} \mu \in \mathcal{P}(X)$. Since $\{\eta_n : n \in \mathbb{N}\}$ is relatively compact w.r.t. the weak topology on $\mathcal{P}(Z \times X)$, we may assume w.l.o.g. that (η_{n_k}) converges weakly to some $\eta \in \mathcal{P}(Z \times X)$. Now, (b) together with Rubin's Theorem ([6, p. 34]) implies $\delta_0 = \eta_{n_k} T_{n_k}^{-1} \xrightarrow{w} \eta T^{-1}$ in $\mathcal{P}(Y)$. Hence, we have $\eta T^{-1} = \delta_0$. Furthermore, because of the continuity of the projections p_X and p_Z, we obtain from the continuous mapping theorem ([6, Theorem 5.1]) that $\eta_{n_k} p_Z^{-1} = D(z_{n_k}) \xrightarrow{w} \eta p_Z^{-1}$ and $\eta_{n_k} p_X^{-1} = \mu_{n_k} \xrightarrow{w} \eta p_X^{-1}$. Hence, $\eta p_Z^{-1} = D(z)$ and $\eta p_X^{-1} = \mu$, and μ is a weak solution of (3.1).

Finally, let us assume that the weak solution $\mu \in \mathcal{P}(X)$ of (3.1) is unique. We conclude from the preceding proof that each subsequence of (μ_n) contains a further subsequence that converges weakly to μ. This implies weak convergence of (μ_n) to μ ([6, Theorem 2.3]) and the proof is complete.

Before discussing sufficient and verifiable conditions for assumption (a) of Theorem 3.1, our next example shows that the assertion of the Theorem does not hold if (a) is violated.

Example 3.2: Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$

and norm $\|\cdot\|$. We define $X := H$ with the usual metric, $Z := \{\tilde{z} \in H : \|\tilde{z}\| \leq 1\}$ and $Y := \{y \in H : \|y\| \leq 2\}$ equipped with the topology of weak convergence in H, which is metrizable on Z and Y, respectively. Furthermore we define $T_n := T$, $T : Z \times X \to Y$ by $T(\tilde{z}, x) := x - \langle x, \tilde{z}\rangle \tilde{z}$ for all $(\tilde{z}, x) \in Z \times X$. Let (e_n) be a complete orthonormal basis in H and we choose $D(z) := \delta_0$ and $D(z_n) := \delta_{e_n}$ for all $n \in \mathbb{N}$.

Since $T(e_n, e_n) = 0$ holds for all $n \in \mathbb{N}$, δ_{e_n} is a weak solution of (3.2) for each $n \in \mathbb{N}$.

Let us check the assumptions of Theorem 3.1: (b) holds since T is continuous; (c) is satisfied since (e_n) converges to 0 in Z, and (d) is evident by definition of Z. However, $\{\delta_{e_n} : n \in \mathbb{N}\}$ is not tight in X. In fact, (a) is violated since the set $T^{-1}(\{0\}) \cap Z \times \{x \in H : \|x\| \leq 1\}$ contains the sequence $((e_n, e_n))$ and, hence, is not relatively compact in $Z \times X$.

The following result gives a first sufficient condition for (a). Although its proof is an adaptation of that of Proposition 4.5 in [10], it is included for the reader's convenience.

Lemma 3.3: Let (T_n) be as above and assume that

(i) $\bigcup_{n\in\mathbb{N}} [T_n(\tilde{z}, \cdot)]^{-1}(\tilde{K}) \cap B$ is relatively compact in X for each $\tilde{z} \in Z$, bounded $B \subseteq X$ and compact $\tilde{K} \subseteq Y$.

(ii) $\{T_n(\cdot, x) : x \in B \cap X_n, n \in \mathbb{N}\}$ is equicontinuous on K for each bounded $B \subseteq X$ and compact $K \subseteq Z$.

Then (i) and (ii) imply condition (a) of Theorem 3.1.

Proof: Let B be a bounded subset of X, K a compact subset of Z and let $((\tilde{z}_k, \tilde{x}_k))$ be an arbitrary sequence contained in $\bigcup_{n\in\mathbb{N}} T_n^{-1}(\{0\}) \cap K \times B$. Since $(\tilde{z}_k)$ is contained in the compact set K, we may assume that $(\tilde{z}_k)$ is convergent to some $\tilde{z} \in K$. Hence it suffices to prove that $(\tilde{x}_k)$ contains a convergent subsequence.

For each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ such that $T_{n_k}(\tilde{z}_k, \tilde{x}_k) = 0$. This, together with (ii) and $\tilde{z}_k \to \tilde{z}$, implies that $(T_{n_k}(\tilde{z}, \tilde{x}_k))$ converges to $0 \in Y$. Now, we define $\tilde{K} := \{0, T_{n_k}(\tilde{z}, \tilde{x}_k) : k \in \mathbb{N}\}$. Because of (i) we obtain that $\bigcup_{n\in\mathbb{N}} [T_n(\tilde{z}, \cdot)]^{-1}(\tilde{K}) \cap B$ is relatively compact in X. But, the latter set contains the sequence (x_k) and the proof is complete.

Remark 3.4: Consider the situation that $X_n := X$, $T_n := T$ for each $n \in \mathbb{N}$. Then the following conditions

(a)′ $[T(\tilde{z}, \cdot)]^{-1}(\tilde{K}) \cap B$ is relatively compact in X and $\{T(\cdot, \tilde{x}) : \tilde{x} \in B\}$ is equicontinuous on K for each $\tilde{z} \in Z$, bounded subset B of X, compact sets $\tilde{K} \subseteq Y$ and $K \subseteq Z$, respectively.

(b)′ $T : Z \times X \to Y$ is continuous.

together with the conditions (c) and (d) imply the assertion of Theorem 3.1. In the terminology of [10] condition (i) of Lemma 3.3 means that $(T_n(\tilde{z}, \cdot))$ is collectively regular for each $\tilde{z} \in Z$. Lemma 3.7 in [10] gives a complete characterization of collectively regular sequences of mappings in terms of the well-known concept of A-regular operator approximations (see e.g. [2, 25, 27, 28]).

Furthermore, we note that the mapping T ($= T_n$ for each $n \in \mathbb{N}$) considered in Example 3.2 satisfies condition (ii) of Lemma 3.3. However, since $T(0, \cdot)$ is the identity (of H), condition (i) is violated.

The following example, which is due to Heinz W. Engl, shows that even the conditions (b) and (i) of Lemma 3.3 do not imply condition (a).

Example 3.5: We consider the Hilbert space ℓ_2 with the norm $\|\cdot\|$ and put $X := Y := \ell_2$, $Z := \{(\bar{z}_k) \in \ell_2 : |\bar{z}_k| \leq k^{-1}$ for each $k \in \mathbb{N}\}$. It is well-known that the "Hilbert cube" Z is compact in ℓ_2 (see e.g. [15]). We define the following sets

$$M := Z \times X \qquad A := \{(k^{-1}e_k, e_k) : k \in \mathbb{N}\} \qquad B := \{0\} \times X$$

where e_k is the kth unit vector in ℓ_2. A and B are closed subsets of the metric space M with $A \cap B = \varnothing$. Due to Urysohn's lemma there exists a continuous function f from M into $[0, 1]$ such that $f|_A = 1$ and $f|_B = 0$.

Now, we define mappings T, $T: Z \times X \to Y$ by

$$\hat{T}(\bar{z}, \bar{x}) := \begin{cases} f(\bar{z}, \bar{x}) \dfrac{\bar{z}}{\|\bar{z}\|} & \bar{z} \neq 0 \\ 0 & \bar{z} = 0 \end{cases} \qquad T(\bar{z}, \bar{x}) := \bar{x} - \hat{T}(\bar{z}, \bar{x})$$

for all $\bar{z}$, $\bar{x} \in Z \times X$. By definition, T is continuous.

Next we show that condition (i) (with $T_n \equiv T$) is satisfied. It suffices to prove that $\hat{T}(\bar{z}, B)$ is relatively compact in Y for each $\bar{z} \in Z$ and each bounded set $B \subset X$ (see [10, Lemma 3.9]). For each $\bar{z} \in Z$ and each bounded $B \subset X$, the set $\hat{T}(\bar{z}, B)$ is a bounded one-dimensional set and thus relatively compact. However, $\hat{K} := T^{-1}(\{0\}) \cap Z \times \{\bar{x} \in X : \|\bar{x}\| \leq 1\}$ is *not* relatively compact and, thus, (a) is not satisfied. This can be seen as follows: For each $n \in \mathbb{N}$ we have that

$$T(n^{-1}e_n, e_n) = e_n - f(n^{-1}e_n, e_n)e_n = 0$$

Hence, the set A is contained in $\hat{K}$ and A is not relatively compact.

Remark 3.6: Remark 4.12 in [8] contains an inaccuracy. But, by using the same mapping $\hat{T}$ as in the preceding example and with K being the Hilbert cube, we obtain that $\hat{T}(K \times \{x \in X : \|x\| \leq 1\})$ is not relatively compact. Thus the conclusion of that remark remains true.

The next result presents a second sufficient condition for (a).

Lemma 3.7: Let T and (T_n) be as above and assume that (T_n) converges discretely to T and that $\{x \in X : T(\bar{z}, x) = 0\}$ is nonempty for each $\bar{z} \in Z$. Furthermore, assume that for each compact $K \subseteq Z$ there exist a constant $C = C(K)$ and $n_0 = n_0(K) \in \mathbb{N}$ such that

$$d_X(x, \bar{x}) \leq C d_Y(T_n(\bar{z}, x), T_n(\bar{z}, \bar{x}))$$

holds for all $n \in \mathbb{N}$, $n \geq n_0$, $\bar{z} \in K$, $x, \bar{x} \in X_n$ (d_X and d_Y denoting the metrics in X and Y, respectively).

Then condition (a) of Theorem 3.1 is satisfied.

Proof: Let $B \subseteq X$ be bounded, $K \subseteq Z$ be compact and let $((\bar{z}_k, \bar{x}_k))$ be an arbitrary sequence in the set $\bigcup_{n \in \mathbb{N}} T_n^{-1}(\{0\}) \cap K \times B$. Again we may assume w.l.o.g. that $(\bar{z}_k)$ is convergent to some $\bar{z} \in K$ and it suffices to prove that $(\bar{x}_k)$ contains a convergent subsequence. For each $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be such that $T_{n_k}(\bar{z}_k, \bar{x}_k) = 0$. We may assume that (n_k) is monotonically increasing (otherwise we choose a suitable subsequence).

By assumption there exist $\bar{x} \in X$ such that $T(\bar{z}, \bar{x}) = 0$ and a sequence $\hat{x}_k \in X_{n_k}$ $(k \in \mathbb{N})$ converging to $\bar{x}$. This implies $T_{n_k}(\bar{z}_k, \hat{x}_k) \to T(\bar{z}, \bar{x}) = 0$. For $k \in \mathbb{N}$ such that $n_k \geq n_0$ we obtain

$$\begin{aligned} d_X(\hat{x}_k, \bar{x}_k) &\leq C d_Y(T_{n_k}(\bar{z}_k, \hat{x}_k), T_{n_k}(\bar{z}_k, \bar{x}_k)) \\ &= C d_Y(T_{n_k}(\bar{z}_k, \hat{x}_k), 0) \end{aligned}$$

Hence, $d_X(\hat{x}_k, \bar{x}_k) \to 0$ and the proof is complete.

Remark 3.8: Let $X := Y$ be a linear normed space (with norm $\|\cdot\|$). Let T_n be of the form $T_n(\bar{z}, \bar{x}) := \bar{x} - A_n(\bar{z}, \bar{x})$ $(n \in \mathbb{N})$ and for $A_n : Z \times X_n \to X$ $(n \in \mathbb{N})$ assume that for each compact $K \subseteq Z$ there exist a constant $\alpha = \alpha(K) \in (0, 1)$ and $n_0 = n_0(K) \in \mathbb{N}$ such that $\|A_n(\bar{z}, x) - A_n(\bar{z}, \bar{x})\| \leq \alpha\|x - \bar{x}\|$ holds for all $\bar{z} \in K$, x, $\bar{x} \in X_n$, $n \geq n_0$.

Then the (main) assumption of Lemma 3.7 is fulfilled. We mention that a combination of Lemma 3.7 (and of the above observation) with Theorem 3.1 leads to a result that is very similar to Theorem 3.1 in [14] (and Theorem 3.3 in [14], respectively). It is worth noting that our method of proving the results is different from that of [14] (which extends the approach of [5]) and that distinct from [14], our approach allows the identification of weak limits of approximate solutions as weak solutions of the original random equation.

We close this section with a result presenting a sufficient condition for a sequence of weak solutions of (3.2) to be stochastically bounded (condition (c) of Theorem 3.1).

Lemma 3.9: Let X be a linear normed space (with norm $\|\cdot\|$), (T_n) be as above and let $D(z_n) \xrightarrow{w} D(z)$. Assume that there exists a continuous function $g : Z \to \mathbb{R}_+$ such that for all $n \in \mathbb{N}$, $(\bar{z}, \bar{x}) \in Z \times X_n$, $T_n(\bar{z}, \bar{x}) = 0$ implies that $\|\bar{x}\| \leq g(\bar{z})$.

Then each set $\{\mu_n\}$ of weak solutions of (3.2) is stochastically bounded.

Proof: Let $\{\mu_n\}$ be a set of weak solutions of (3.2) and, for each $n \in \mathbb{N}$, let $\eta_n \in \mathcal{P}(Z \times X)$ be such that $\eta_n p_Z^{-1} = D(z_n)$ and $\eta_n p_X^{-1} = \mu_n$. Let $\varepsilon > 0$. By assumption, there exists a compact set $K_\varepsilon \subseteq Z$ such that $\inf_{n \in \mathbb{N}} D(z_n)(K_\varepsilon) \geq 1 - \varepsilon$. Then we have for each $n \in \mathbb{N}$.

$$\begin{aligned} 1 - \varepsilon \leq D(z_n)(K_\varepsilon) &= \eta_n(K_\varepsilon \times X) \\ &= \eta_n(\{(\bar{z}, \bar{x}) \in K_\varepsilon \times X : T_n(\bar{z}, \bar{x}) = 0\}) \\ &\leq \eta_n(\{(\bar{z}, \bar{x}) \in K_\varepsilon \times X : \|\bar{x}\| \leq g(\bar{z})\}) \\ &\leq \eta_n\left(\left\{(\bar{z}, \bar{x}) \in Z \times X : \|\bar{x}\| \leq \sup_{\bar{z} \in K_\varepsilon} g(\bar{z})\right\}\right) \\ &\leq \eta_n\left(\left\{\bar{x} \in X : \|\bar{x}\| \leq \sup_{\bar{z} \in K_\varepsilon} g(\bar{z})\right\}\right). \end{aligned}$$

Since g is continuous, $\sup_{\bar{z} \in K_\varepsilon} g(\bar{z})$ is finite and, thus, the assertion is proved.

4. APPLICATION TO PERTURBATIONS OF A RANDOM DIFFERENTIAL EQUATION

In this section, we consider the initial value problem for a nonlinear random ordinary differential equation

$$x'(t) = f(t, z(\omega, t), x(t)) \qquad t \in [0, 1] \qquad x(0) = a \tag{4.1}$$

where $f : [0, 1] \times \mathbb{R}^{r+m} \to \mathbb{R}^m$, $a \in \mathbb{R}^m$, and $z : \Omega \times [0, 1] \to \mathbb{R}^r$ is a stochastic process on a probability space $(\Omega, \mathcal{A}, P)$ with parameter set $[0, 1]$, state space $\mathbb{R}^r$

and with paths in the space $D := D([0, 1]; \mathbb{R}^r)$, i.e., the space of right-continuous functions from $[0, 1]$ to $\mathbb{R}^r$ with left limits (see [6, 12] for more information on the metric space $\mathfrak{D}$; see below). It is well-known that z can be identified with a $\mathfrak{D}$-valued random variable on $(\Omega, \mathcal{C}, P)$ (see e.g. [6, Section 15]).

The aim of this section is to study the behavior of (weak) solutions to (4.1) in case the distribution $D(z)$ of z is perturbed with respect to the topology of weak convergence on $\mathcal{P}(\mathfrak{D})$. Of course, our approach is the application of the machinery developed in the previous section. Let, for each $n \in \mathbb{N}$, z_n be a stochastic process on $(\Omega, \mathcal{C}, P)$ with paths in $\mathfrak{D}$ or, equivalently, a $\mathfrak{D}$-valued random variable. The perturbed (or approximate) random differential equation then reads

$$x'(t) = f(t, z_n(\omega, t), x(t)) \qquad t \in [0, 1] \qquad x(0) = a \tag{4.2}$$

Formulating our setup in the framework of Section 3, we put $X := Y := C([0, 1]; \mathbb{R}^m)$ with the usual uniform metric and $Z := \mathfrak{D} := D([0, 1]; \mathbb{R}^r)$ equipped with the following metric d_0:

$$d_0(y, \tilde{y}) := \inf_{\lambda \in \Lambda_0} \left\{ \sup_{t \in [0,1]} \|y(t) - \tilde{y}(\lambda(t))\| + \|\lambda\|_0 \right\} \qquad (y, \tilde{y} \in \mathfrak{D})$$

where Λ_0 is the set of all functions λ from $[0, 1]$ onto $[0, 1]$ that are continuous, strictly monotonically increasing and have the property that $\|\lambda\|_0 = \sup_{t \neq s} |\log \lambda(t) - \lambda(s)/t - s|$ is finite. It is known that d_0 generates the Skorokhod topology on $\mathfrak{D}$ and that (Z, d_0) is complete and separable (see [6, Section 14], [12, Sect. VI, part 5]).

Furthermore, we define the mapping $T: Z \times X \to X$ by

$$[T(\tilde{z}, x)](t) := x(t) - a - \int_0^t f(s, \tilde{z}(s), x(s))\, ds$$

$$\text{for all } t \in [0, 1] \quad \tilde{z} \in Z \quad x \in X$$

and we put $T_n := T$ for each $n \in \mathbb{N}$. As we will assume later on that f is continuous, T is well-defined and (4.1), (4.2) are equivalent to (3.1), (3.2). In addition, in Theorem 4.2 we show T to be continuous and, hence, Borel measurable.

The following auxiliary result provides a necessary condition for relatively compact subsets of $\mathfrak{D}$ and will be used subsequently.

Lemma 4.1: Let $K \subset \mathfrak{D}$ be relatively compact. Then we have

$$\lim_{\|\lambda\|_0 \to 0} \sup_{y \in K} \sup_{t \in [0,1]} \|y(t) - y(\lambda(t))\| = 0 \tag{4.3}$$

Proof: Let $\varepsilon > 0$ be chosen arbitrary. Then, according to [6, Theorem 14.3], there exist $\delta = \delta(\varepsilon) > 0$, $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_i - t_{i-1} > \delta$ for all $i = 1, \ldots, n$, and

$$\sup_{y \in K} \max_{i=1,\ldots,n} \sup_{t,s \in [t_{i-1}, t_i)} \|y(t) - y(s)\| < \varepsilon \tag{4.4}$$

We assume without loss of generality that $\delta < \frac{1}{2}$ and we choose $\delta_0 \in (0, \delta)$ such that $t_i - t_{i-1} > \delta + 2\delta_0$ for all $i = 1, \ldots, n$. Let $\lambda \in \Lambda_0$ with $\|\lambda\|_0 < \delta_0/2$. We show that

$$\sup_{t \in [0,1]} \|y(t) - y(\lambda(t))\| = \sup_{t \in [0,1]} \|y(\lambda^{-1}(t)) - y(t)\| \leq \varepsilon \tag{4.5}$$

holds for each $y \in K$ and, hence, (4.3) is proved. Let $y \in K$ and $t \in [0, 1]$. Since $\lambda(1) = 1$, we may assume $t \in [0, 1)$. Then there exists an $i \in \{1, \ldots, n\}$ with $t \in [t_{i-1}, t_i)$. Since $\|\lambda\|_0 < \delta_0/2$, it follows from [6, Section 14, (14.22)] that

$$\sup_{t\in[0,1]} |t - \lambda(t)| = \sup_{t\in[0,1]} |\lambda^{-1}(t) - t| < \delta_0 \tag{4.6}$$

Hence, we have $\lambda(t) \in (t - \delta_0, t + \delta_0) \subseteq (t_{i-1} - \delta_0, t_i + \delta_0)$. For the case of $\lambda(t) \in [t_{i-1}, t_i)$ the desired conclusion (4.5) follows from (4.4). Let us consider the remaining cases:

(i) For $\lambda(t) \in (t_{i-1} - \delta_0, t_{i-1})$ we have that $t < \lambda^{-1}(t)$, and (4.6) implies $t \in [t_{i-1}, t_{i-1} + \delta_0)$ and $\lambda^{-1}(t) \in [t_{i-1}, t_{i-1} + 2\delta_0)$. Because of the inequality $t_i - \delta > t_{i-1} + 2\delta_0 > \lambda^{-1}(t)$ we obtain $\lambda^{-1}(t) \in [t_{i-1}, t_i)$ and (4.5) follows again from (4.4).

(ii) For $\lambda(t) \in [t_i, t_i + \delta_0)$ we have that $t < \lambda(t)$. Hence, (4.6) implies $\lambda^{-1}(t) \in (t_i - 2\delta_0, t_i) \subseteq [t_{i-1}, t_i)$. This together with (4.4) implies (4.5) and the proof is complete.

Theorem 4.2: Let $f: [0, 1] \times \mathbb{R}^{r+m} \to \mathbb{R}^m$ be continuous and assume that $D(z_n) \xrightarrow{w} D(z)$ in $\mathcal{P}(Z)$. Suppose that, for each $n \in \mathbb{N}$, there exists a weak solution μ_n of (4.2) and $\{\mu_n : n \in \mathbb{N}\}$ is stochastically bounded.

Then there exists a subsequence of (μ_n) which converges weakly to a weak solution of (4.1). If the weak solution of (4.1) is unique, the whole sequence (μ_n) converges weakly in $\mathcal{P}(X)$ to this limit.

Proof: To prove the assertion we apply Theorem 3.1. To this end, we have to verify the conditions (a)′ and (b)′ from Remark 3.4. Let us start with (b)′. For our convenience, we define the following mapping $C: Z \times X \to X$, $[C(y, x)](t) := a + \int_0^t f(s, y(s), x(s))\, ds$, for all $t \in [0, 1]$, $y \in Z$ and $x \in X$. Hence, $T = I - C$, where I is the identity mapping on X. Let (y_n) and (x_n) be sequences (in Z and X, respectively) converging to y and x, respectively. This implies $\sup_{t\in[0,1]} \|x_n(t) - x(t)\| \to 0$ and $y_n(t) \to y(t)$ for all $t \in [0, 1] \setminus M$, where M is the (at most countable) set of discontinuity points of y ([6, Section 14]). Since, in addition, convergence of (y_n) in Z, i.e. convergence with respect to the Skorokhod topology, implies that $\sup_{n\in\mathbb{N}} \sup_{t\in[0,1]} \|y_n(t)\| < \infty$ and, since f is continuous, we can apply Lebesgue's dominated convergence theorem and obtain:

$$\sup_{t\in[0,1]} \|[C(y_n, x_n)](t) - [C(y, x)](t)\|$$

$$\le \int_0^1 \|f(t, y_n(t), x_n(t)) - f(t, y(t), x(t))\|\, dt \xrightarrow[n\to\infty]{} 0.$$

Hence, T is continuous.

To verify (a)′, let B be a bounded subset of X and K a compact subset of Z. We have to show that $\{T(\cdot, x) : x \in B\}$ is equicontinuous on K. Let $\varepsilon > 0$ be chosen arbitrary. We put

$$\alpha := \left\{\sup_{t\in[0,1]} \|x(t)\| : x \in B\right\} \text{ and}$$

$$\beta := \sup\{d_0(y, 0) : y \in K\} = \sup\left\{\sup_{t\in[0,1]} \|y(t)\| : y \in K\right\}$$

Since f is continuous, there exists a $\delta_0 > 0$ such that $\|f(t, w, v) - f(t, \tilde{w}, v)\| <$

$\varepsilon/2$ whenever $t \in [0, 1]$, $v \in R^m$, with $\|v\| \leq \alpha$, w, $\tilde{w} \in \mathbb{R}^r$ with $\|w\| \leq \beta$, $\|\tilde{w}\| \leq \beta$, $\|w - \tilde{w}\| < \delta_0$. Lemma 4.1 implies that there exists a $\delta \in (0, \delta_0)$ such that

$$\sup_{y \in K} \sup_{t \in [0,1]} \|y(t) - y(\lambda(t))\| < \delta_0 \quad \text{if} \quad \lambda \in \Lambda_0,\ \|\lambda\|_0 < \delta.$$

Let $x \in B$, y, $\tilde{y} \in K$ with $d_0(y, \tilde{y}) < \delta$. Then there exists a $\lambda \in \Lambda_0$ such that $\sup_{t \in [0,1]} \|y(t) - \tilde{y}(\lambda(t))\| + \|\lambda\|_0 < \delta$. We obtain

$$\sup_{t \in [0,1]} \|[T(y, x)](t) - [T(\tilde{y}, x)](t)\|$$

$$\leq \int_0^1 \|f(t, y(t), x(t)) - f(t, \tilde{y}(\lambda(t)), x(t))\| \, dt$$

$$+ \int_0^1 \|f(t, \tilde{y}(\lambda(t)), x(t)) - f(t, \tilde{y}(t), x(t))\| \, dt \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $\{T(\cdot, x) : x \in B\}$ is equicontinuous on K.

To prove the remaining part of condition (a)′, let $y \in Z$, a compact subset $\tilde{K}$ of $Y = X$ and a bounded subset B of X be chosen arbitrary. Then $[T(y, \cdot)]^{-1}(\tilde{K}) \cap B$ is relatively compact in X if $C(y, \cdot)(B)$ is relatively compact in $Y = X$ (see Lemma 3.9 in [10]). The latter property can be shown by standard arguments using the Arzela-Ascoli-Theorem.

Finally, we note that the existence of a weak solution μ_n to (4.2) (for each $n \in \mathbb{N}$) follows from the P-almost sure pathwise existence of solutions to (4.2) (Theorem 2.1 and Lemma 2.4(i)). Under a linear growth condition on f (see (3.6) in [9]) it can be shown similar to [9, p. 71/72] and by applying Lemma 3.9 that $\{\mu_n : n \in \mathbb{N}\}$ is stochastically bounded.

5. APPLICATION TO A GALERKIN SCHEME FOR NONLINEAR RANDOM HAMMERSTEIN INTEGRAL EQUATIONS

We are concerned with the following nonlinear random Hammerstein integral equation in spaces of summable functions

$$x(t) + \int_0^1 z_1(\omega, t, s) f(s, z_2(\omega, s), x(s)) \, ds = z_3(\omega, t) \qquad t \in [0, 1] \quad (5.1)$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous,

$z_1 : \Omega \times [0, 1]^2 \to \mathbb{R}$ is an $\mathcal{A} \times \mathcal{B}([0, 1]^2)$-measurable stochastic process with paths in $L_p([0, 1]^2)$ $(p > 1)$, and

$z_2, z_3 : \Omega \times [0, 1] \to \mathbb{R}$ are $\mathcal{A} \times \mathcal{B}([0, 1])$-measurable stochastic processes with paths in $L_\infty([0, 1])$ and $L_p([0, 1])$, respectively (on a probability space $(\Omega, \mathcal{A}, P)$).

We are going to study a twofold approximation of (5.1): A finite-dimensional Galerkin approximation of the integral operator and perturbations of the stochastic inputs appearing on the nonlinearity f, as kernel, and as right-hand side in (5.1). Again we aim at applying Theorem 3.1. In order to utilize the abstract theory, we put

$$X := L_p([0, 1]], Z := Z_1 \times Z_2 \times Z_3, Z_1 := L_p([0, 1]^2)(p > 1)$$
$$Z_2 := L_\infty([0, 1]), Z_3 := Y := X$$

We also need the dual $X^* := L_q([0, 1])$ to X, where $1/p + 1/q = 1$. Next we define the kernel operator $\mathcal{K}: Z_1 \times X^* \times X$ by

$$[\mathcal{K}(\bar{z}_1, y)](t) := \int_0^1 \bar{z}_1(t, s)y(s)\, ds,\ t \in [0, 1],\ (\bar{z}_1, y) \in Z_1 \times X^*$$

and the superposition operator $F: Z_2 \times X \to X^*$ by

$$[F(\bar{z}_2, x)](t) := f(t, \bar{z}_2(t), x(t)),\ t \in [0, 1],\ (\bar{z}_2, x) \in Z_2 \times X$$

F is well-defined under the growth condition for f appearing in assumption (i) in Theorem 5.1.

Finally, we define the mapping $T: Z \times X \to X$ by $T(\bar{z}, x) := x + \mathcal{K}(\bar{z}_1, F(\bar{z}_2, x)) - \bar{z}_3$ for all $\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3) \in Z$, $x \in X$. Since [24] implies that, under the general assumptions, $z: \Omega \to Z$ defined by $z(\omega) := (z_1(\omega, \cdot, \cdot), z_2(\omega, \cdot), z_3(\omega, \cdot))$, for all $\omega \in \Omega$, can be viewed as a Z-valued random variable, equation (5.1) is equivalent to the abstract random Hammerstein equation

$$T(z(\omega), x) = 0 \qquad \omega \in \Omega \tag{5.2}$$

Let $(X_n, Q_n)_{n\in\mathbb{N}}$ be a Galerkin scheme for X, i.e. for each $n \in \mathbb{N}$, X_n is a finite-dimensional subspace of X and $Q_n: X \to X_n$ is a linear bounded projection onto X_n having the property that (Q_n) converges pointwise to the identity. For each $n \in \mathbb{N}$ we define

$$T_n: Z \times X_n \to X,\ T_n(\bar{z}, x) := Q_n T(\bar{z}, x),\ (\bar{z}, x) \in Z \times X_n$$

and consider the following approximations of (5.2)

$$T_n(z_n(\omega), x) = 0 \qquad \omega \in \Omega;\ n \in \mathbb{N} \tag{5.3}$$

where z_n is a Z-valued random variable on $(\Omega, \mathcal{Q}, P)$ for each $n \in \mathbb{N}$. The next theorem is the main convergence result of this section.

Theorem 5.1: In addition to the general assumptions we suppose that

(i) The set $\{f(t, \cdot, v): t \in [0, 1], v \in \mathbb{R}\}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ is equicontinuous on each compact subset of $\mathbb{R}$, and there exists a constant $\gamma > 0$ and for each $r > 0$ there exists a function $y_r \in X^*$ such that the growth condition $|f(t, u, v)| < y_r(t) + \gamma|v|^{p-1}$ holds for all $(t, u, v) \in [0, 1] \times \mathbb{R}^2$, $|u| \le r$.

(ii) $(D(z_n))$ converges weakly to $D(z)$.

(iii) There exists a weak solution μ_n of (5.3) for all $n \in N$, and $\{\mu_n : n \in \mathbb{N}\}$ is stochastically bounded.

Then there exists a subsequence of (μ_n) converging weakly to a weak solution of (5.2). If the weak solution of (5.2) is unique, the whole sequence (μ_n) converges weakly to this limit.

Proof: Assumption (i) implies that the superposition operator $F: Z_2 \times X \to X^*$ and, hence, $T: Z \times X \to X$ are well-defined (see e.g. [16]). In order to apply Theorem 3.1, we have to verify the assumptions (a) and (b) of that result. To show that (a) holds, we prove that the conditions (i) and (ii) of Lemma 3.3 are satisfied. As already mentioned in Remark 3.4, condition (i) can be checked by using Lemma 3.7 in [10]. According to that result it remains to be shown that

$(T_n(\bar{z}, \cdot))$ is A-regular for each $z \in Z$. To this end, let $\bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3) \in Z$, $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ and $x_{n_k} \in X_{n_k}$, $k \in \mathbb{N}$, be such that (x_{n_k}) is bounded in X and $(T_{n_k}(\bar{z}, \cdot))$ is convergent in X. Taking into account that the superposition operator $F(\bar{z}_2, \cdot): X \to X^*$ is bounded on bounded subsets of X (see [16, Section 17]), we conclude that $(F(\bar{z}_2, x_{n_k}))$ is bounded in X^*. Since the kernel operator $\mathbb{K}(\bar{z}_1, \cdot)$: $X^* \to X$ is compact (e.g. [30, Cor. 28.5]), the sequence $(\mathcal{K}(\bar{z}_1, F(\bar{z}_2, x_{n_k})))$ has a convergent subsequence. This observation together with the equation

$$x_{n_k} = T_{n_k}(\bar{z}, x_{n_k}) - Q_{n_k}\mathcal{K}(\bar{z}_1, F(\bar{z}_2, x_{n_k})) + \bar{z}_3 \qquad k \in \mathbb{N}$$

and the properties of (Q_{n_k}) imply that (x_{n_k}) also contains a convergent subsequence. Hence, $(T_n(\bar{z}, \cdot))$ is A-regular and condition (i) of Lemma 3.3 is proved. Next we prove the equicontinuity-condition (ii) of Lemma 3.3. Let B be a bounded subset of X and K be a compact subset of Z. Let $n \in \mathbb{N}$, $x \in B \cap X_n$ and $\bar{z}$, $\bar{\bar{z}} \in K$ be chosen arbitrarily. Then we obtain the following chain of inequalities by standard arguments.

$$\begin{aligned}
&\|T_n(\bar{z}, x) - T_n(\bar{\bar{z}}, x)\|_X \\
&\le \|Q_n\mathcal{K}(\bar{z}_1, F(\bar{z}_2, x)) - Q_n\mathcal{K}(\bar{\bar{z}}_1, F(\bar{\bar{z}}_2, x))\|_X + \|\bar{z}_3 - \bar{\bar{z}}_3\|_X \\
&\le C_1(\|\mathcal{K}(\bar{z}_1, F(\bar{z}_2, x)) - \mathcal{K}(\bar{\bar{z}}_1, F(\bar{z}_2, x))\|_X + \|\mathcal{K}(\bar{\bar{z}}_1, F(\bar{z}_2, x)) - \mathcal{K}(\bar{\bar{z}}_1, F(\bar{\bar{z}}_2, x))\|_X) \\
&\qquad + \|\bar{z}_3 - \bar{\bar{z}}_3\|_X \\
&\le C_1(\|\bar{z}_1 - \bar{\bar{z}}_1\|_{Z_1}\|F(\bar{z}_2, x)\|_{X^*} + \|\bar{\bar{z}}_1\|_{Z_1}\|F(\bar{z}_2, x) - F(\bar{\bar{z}}_2, x)\|_{X^*}) + \|\bar{z}_3 - \bar{\bar{z}}_3\|_X \\
&\le C_1(C_2\|F(\bar{z}_2, x) - F(\bar{\bar{z}}_2, x)\|_{X^*} + C_3\|\bar{z}_1 - \bar{\bar{z}}_1\|_{Z_1}) + \|\bar{z}_3 - \bar{\bar{z}}_3\|_{Z_3}
\end{aligned}$$

where the constants are chosen such that $\|Q_n\| \le C_1$ for all $n \in \mathbb{N}$ (according to the Banach-Steinhaus theorem), $C_2 := \sup\{\|\bar{z}_1\| : \bar{z} = (\bar{z}_1, \bar{z}_2, \bar{z}_3) \in K\}$ and $C_3 := \sup\{\|F(\bar{z}_2, x)\|_{X^*} : \bar{z} \in K, x \in B\}$. In view of the above inequality it remains to consider the term $\|F(\bar{z}_2, x) - F(\bar{\bar{z}}_2, x)\|_{X^*}$ where $x \in B$, $\bar{z}_2$, $\bar{\bar{z}}_2 \in K_2$ and K_2 denotes the projection of K onto Z_2. Let $C_4 := \sup\{\|\bar{z}_2\|_{Z_2} : \bar{z}_2 \in K_2\}$ and let $\varepsilon > 0$ be chosen arbitrarily.

Assumption (i) implies that there exists a $\delta > 0$ such that $|f(t, u, v) - f(t, \bar{u}, v)| < \varepsilon$ whenever $t \in [0, 1]$, $u, \bar{u} \in [-C_4, C_4]$ with $|u - \bar{u}| < \delta$, $v \in \mathbb{R}$. Hence, we obtain for all $\bar{z}_2, \bar{\bar{z}}_2 \in K_2$ with $\|\bar{z}_2 - \bar{\bar{z}}_2\|_{Z_2} < \delta$ and all $x \in B$ that $|f(s, \bar{z}_2(s), x(s)) - f(s, \bar{\bar{z}}_2(s), x(s))| < \varepsilon$ holds for almost all $s \in [0, 1]$, and, therefore, $\|F(\bar{z}_2, x) - F(\bar{\bar{z}}_2, x)\|_{X^*} \le \varepsilon$.

This proves condition (ii) in Lemma 3.3

Next we verify assumption (b) of Theorem 3.1. Let $\bar{x} \in X$, $\bar{x}_n \in X_n$, $\bar{z}$, $\bar{z}_n \in Z$ ($n \in \mathbb{N}$) such that $(\bar{z}_n, \bar{x}_n) \to (\bar{z}, \bar{x})$ as $n \to \infty$. We prove that $\lim_{n\to\infty} \|T_n(\bar{z}_n, \bar{x}_n) - T(\bar{z}, \bar{x})\|_X = 0$.

Let $B := \{\bar{x}_n : n \in \mathbb{N}\}$, $K := \{\bar{z}, \bar{z}_n : n \in \mathbb{N}\}$. The equicontinuity of $\{T_n(\cdot, x) : x \in B\}$ on K implies $\lim_{n\to\infty} \|T_n(\bar{z}_n, \bar{x}_n) - T_n(\bar{z}, \bar{x}_n)\|_X = 0$. It remains to mention that $\lim_{n\to\infty} \|T_n(\bar{z}, \bar{x}_n) - T(\bar{z}, \bar{x})\|_X = 0$ follows from the continuity of T and from the properties of (Q_n). An application of Theorem 3.1 completes the proof.

Sufficient conditions for the existence and uniqueness of weak solutions to (5.2) and (5.3) can be derived from the results of Section 2 together with the existence theory for solutions of deterministic Hammerstein integral equations (see e.g. [1], [30, Chapt. 28]). In the following example we finally demonstrate how our theory works for a special weakly singular nonlinear random integral equation. In particular, we show how it can be verified that a sequence of weak solutions of (5.3) is stochastically bounded.

Example 5.2: We consider the following weakly singular random integral equation

$$x(t) + \int_0^1 \frac{\alpha(\omega)}{|t - s|^\delta} f(x(s))\, ds = y(\omega, t) \qquad t \in [0, 1] \tag{5.4}$$

where α is a real-valued random variable and $y: \Omega \times [0, 1] \to \mathbb{R}$ is an $\mathcal{A} \times \mathcal{B}([0, 1])$-measurable stochastic process on a probability space $(\Omega, \mathcal{A}, P)$. We assume that $0 < \delta < 1$, f is continuous and satisfies the growth condition $|f(v)| \leq \gamma|v|^{p-1}$ $(v \in \mathbb{R})$ where p is chosen such that $1 < p < \min\{\delta^{-1}, 2\}$. We suppose, in addition, that the paths of y belong to $L_p([0, 1])$.

Then for $z_1: \Omega \times [0, 1]^2 \to \mathcal{R}$, $z_1(\omega, t, s) := \alpha(\omega)|t - s|^{-\delta}$, $\omega \in \Omega$, $t, s \in [0, 1]$, the general assumptions of this section as well as condition (i) for f are satisfied. Let $((\alpha_n, y_n))$ be a sequence of $\mathbb{R} \times L_p([0, 1])$-valued random variables such that $D(\alpha_n, y_n) \xrightarrow{w} D(\alpha, y)$ and let (X_n, Q_n) be a Galerkin scheme for X. Let μ_n be a weak solution of

$$x_n + Q_n \left(\int_0^1 \frac{\alpha_n(\omega)}{|\cdot - s|^\delta} f(x_n(s))\, ds \right) = y_n(\omega) \qquad x_n \in X_n$$

for each $n \in \mathbb{N}$. To show that $\{\mu_n : n \in \mathbb{N}\}$ is stochastically bounded, we use Lemma 3.9. Let $(\tilde{\alpha}, \tilde{y}, \tilde{x}) \in \mathbb{R} \times L_p([0, 1]) \times X_n$ such that

$$\tilde{x} = -Q_n \left(\int_0^1 \frac{\tilde{\alpha}}{|\cdot - s|^\delta} f(\tilde{x}(s))\, ds \right) + \tilde{y}$$

We obtain the following chain of inequalities by using Hölder's and Minkowski's inequalities:

$$\begin{aligned}
\|\tilde{x}\|_{L_p} &\leq \|Q_n\| \left(\int_0^1 \left| \int_0^1 \frac{\tilde{\alpha}}{|t - s|^\delta} f(\tilde{x}(s))\, ds \right|^p dt \right)^{1/p} + \|\tilde{y}\|_{L_p} \\
&\leq \|Q_n\|\, |\tilde{\alpha}| \gamma \left(\int_0^1 \left(\int_0^1 |t - s|^{-\delta} |\tilde{x}(s)|^{p-1}\, ds \right)^p dt \right)^{1/p} + \|\tilde{y}\|_{L_p} \\
&\leq \|Q_n\|\, |\tilde{\alpha}| \gamma \left(\int_0^1 \left(\int_0^1 |t - s|^{-\delta p}\, ds \right) \|\tilde{x}\|_{L_p}^{(p-1)p}\, dt \right)^{1/p} + \|\tilde{y}\|_{L_p} \\
&= \|Q_n\| \gamma |\tilde{\alpha}|\, \|\tilde{x}\|_{L_p}^{p-1} \left(\int_0^1 \int_0^1 |t - s|^{-\delta p}\, ds\, dt \right)^{1/p} + \|\tilde{y}\|_{L_p}
\end{aligned}$$

Hence, there exists a constant $C > 0$ such that

$$\|\tilde{x}\|_{L_p} \leq C|\tilde{\alpha}|\, \|\tilde{x}\|_{L_p}^{p-1} + \|\tilde{y}\|_{L_p}$$

This implies $\|\tilde{x}\|_{L_p} \leq \max\{1, (C|\tilde{\alpha}| + \|\tilde{y}\|_{L_p})^{1/2-p}\}$ and Lemma 3.9 yields the desired property.

ACKNOWLEDGMENT

The authors wish to thank Heinz W. Engl (Johannes Kepler Universität Linz, Austria) for valuable discussions during the work at this paper and, in particular, for providing Example 3.5.

REFERENCES

1. H. Amann, Zum Galerkin-Verfahren für die Hammersteinsche Gleichung. *Archive for Rational Mechanics and Analysis 35* (1969), 114–121.
2. P.M. Anselone and R. Ansorge, Compactness principles in nonlinear operator approximation theory. *Numerical Functional Analysis and Optimization 1* (1979), 589–618.
3. A.T. Bharucha-Reid, *Random Integral Equations*. Academic Press, New York, 1972.
4. A.T. Bharucha-Reid (ed.), *Approximate Solution of Random Equations*. North-Holland, New York-Oxford, 1979.
5. A.T. Bharucha-Reid and W. Römisch, Projective schemes for random operator equations. Weak compactness of approximate solution measures. *Journal of Integral Equations 8* (1985), 95–111.
6. P. Billingsley, *Convergence of Probability Measures*. Wiley, New York, 1968.
7. H.W. Engl, Random fixed point theorems, In *Nonlinear equations in abstract spaces* (V. Lakshmikantham, ed.), Academic Press, New York, 1978, 67–80.
8. H.W. Engl and W. Römisch, Convergence of approximate solutions of nonlinear random operator equations with non-unique solutions. *Stochastic Analysis and Applications 1* (1983), 239–268.
9. H.W. Engl and W. Römisch, Approximate solutions of nonlinear random operator equations: Convergence in distribution. *Pacific Journal of Mathematics 120* (1985), 55–77.
10. H.W. Engl and W. Römisch, Weak convergence of approximate solutions of stochastic equations with applications to random differential and integral equations. *Numerical Functional Analysis and Optimization 9* (1987), 61–104.
11. M.P. Ershov, Extension of measures and stochastic equations. *Theory of Probability and its Application 19* (1974), 431–444.
12. I.I. Gihman and A.V. Skorokhod, *The Theory of Stochastic Processes*, Vol. I, Springer, Berlin, 1979.
13. C.J. Himmelberg, Measurable relations. *Fundamenta Mathematicae 87* (1975), 53–72.
14. M.C. Joshi, Weak compactness of solution measures of nonlinear approximate random operator equations. *Stochastic Analysis and Applications 5* (1987), 151–166.
15. A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*. Prentice-Hall, Englewood Cliffs, NJ, 1970.
16. M.A. Krasnoselski et al., *Integral Operators in Spaces of Summable Functions*. Noordhoff, Leyden, 1976.
17. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
18. S.J. Leese, Multifunctions of Souslin type. *Bulletin Australian Mathematical Society 11* (1974), 395–411.
19. M.Z. Nashed and H.W. Engl, Random generalized inverses and approximate solutions of random operator equations, in [4], 149–210.
20. A. Nowak, Random solutions of equations, In Transactions Eighth Prague Conference on Information Theory, Statistical Decision Functions, Random Processes, Vol. B, Prague, 1978, 77–82.

21. A. Nowak, Random fixed points of multifunctions, Prace Naukowe Universytetu Ślaskiego Nr. 420, Prace Matematyczne, t.11, Katowice, 1981, 36–41.
22. N.S. Papageorgiou, On measurable multifunctions with stochastic domain. *Journal Australian Mathematical Society, Ser. A, 45* (1988), 204–216.
23. S.T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, Wiley, Chichester, 1991.
24. B.S. Rajput, Gaussian measures on L_p spaces, $1 \leq p < \infty$. *Journal of Multivariate Analysis 2* (1972), 382–403.
25. H.-J. Reinhardt, Analysis of Approximation Methods for Differential and Integral Equations. Springer, New York, 1985.
26. O.N. Ricceri, $\mathscr{C}$-fixed points of multi-valued contractions. *Journal of Mathematical Analysis and Applications 135* (1988), 406–418.
27. F. Stummel, Discrete convergence of mappings, In: *Topics in Numerical Analysis* (J.J.H. Miller, ed.), Academic Press, New York, 1973, 285–310.
28. G. Vainikko, Approximative methods for nonlinear equations. *Nonlinear Analysis 2* (1978), 647–687.
29. D.H. Wagner, Survey of measurable selection theorems. *SIAM Journal on Control and Optimization 15* (1977), 859–903.
30. E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. II/B, Springer, New York, 1990.

Received: February 1992
Accepted: May 1992